Statistical Analysis for Extreme Value with Applications to Hydrological Events

NIPADA PAPUKDEE

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Statistical Management Science at Mahasarakham University

August 2020

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The examining committee has unanimously approved this dissertation, submitted by Miss NIPADA PAPUKDEE, as a partial fulfillment of the requirements for the Doctor of Philosophy in Statistical Management Science at Mahasarakham University.

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NIPADA PAPUKDEE

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ABSTRACT

Four parameter kappa distribution (K4D) is a generalization of common three-parameter distributions and in particular of the Genelarized extreme value (GEV) distribution involved in several important real-life applications such as insurance, hydrological events, earth- quake and etc. Previous studies showed that maximum likelihood estimators (MLE) of parameters are unstable for small sample size. The method of L-moments estimation is an alternative method of estimation similar to a conventional method of moments. However, L-moment estimators are sometimes considered neither computable nor feasible. In this study, we proposed to use of maximum penalized likelihood estimation (MPLE) by adjusting the penalty function of Coles and Dixon (1999) and the penalty function of Martins and Stedinger (2000) for K4D. Monte-Carlo simulation was performed to illustrate the performance of the estimation methods, the maximum penalized likelihood estimation developed from the penalty function of Martins and Stedinger (2000) is MPLE.MS3 and MPLE.MSP3 are better than the MLE and L-moment method in terms RRMSE of all quantiles used K4D. To illustrate, its applicability including annual maximum rainfall data and annual maximum temperature was analyzed, and its fitness was compared with other estimation methods. Finally to study the r-largest order statistics for K4D with can be applied to hydrological data.

Keywords : Four parameter kappa distribution, Penalty function, Maximum penalized likelihood estimation, Monte Carlo simulation.

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LIST OF ABBREVIATIONS AND SYMBOL

cdf	Cumulative Distribution Function
GEV	Generalized Extreme Value
i.i.d.	Independent and Identically Distribution
K4D	Four parameter Kappa Distribution
LM	L-moment Estimator
MLE	Maximum Likelihood Estimator
MPLE	Maximum Penalized Likelihood Estimator
MPLE.CD1	Maximum Penalized Likelihood Estimator using Penalty Function of Coles and Dixon (1999)
MPLE.CD2	Maximum Penalized Likelihood Estimator Adjusting Penalty Function of Coles and Dixon (1999) using Constant Equal to 1.2
MPLE.CD3	Maximum Penalized Likelihood Estimator Adjusting Penalty Function of Coles and Dixon (1999) using Constant Equal to 1.5
MPLE.CD4	Maximum Penalized Likelihood Estimator Adjusting Penalty Function of Coles and Dixon (1999) using Two Constant Equal to 1.2
MPLE.CD5	Maximum Penalized Likelihood Estimator Adjusting Penalty Function of Coles and Dixon (1999) using Two Constant Equal to 1.5
MPLE.MS1	Maximum Penalized Likelihood Estimator Using Penalty Function of Martins and Stedinger (2000)
MPLE.MS2	Maximum Penalized Likelihood Estimator Adjusting Penalty Function of Martins and Stedinger (2000) using Constant Equal to 1.2
MPLE.MS3	Maximum Penalized Likelihood Estimator Adjusting Penalty Function of Martins and Stedinger (2000) using Constant Equal to 1.5

MPLE.MSP1	Maximum Penalized Likelihood Estimator Using Penalty Function of Martins and Stedinger (2000) and Parameters in Penalty function are Used According to Park(2005)
MPLE.MSP2	Maximum Penalized Likelihood Estimator Adjusting Penalty Function of Martins and Stedinger (2000) and Parameters in Penalty function are Used According to Park(2005) Using Constant Equal to 1.2
MPLE.MSP2	Maximum Penalized Likelihood Estimator Adjusting Penalty Function of Martins and Stedinger (2000) and Parameters in Penalty function are Used According to Park(2005) Using Constant Equal to 1.5
pdf	Probability Density Function
RBIAS	Relative Bias
RRMSE	Relative Root Mean Square Error
r-GEV	The r Rargest-Order Statistics Model in Generalized Extreme Value Distribution
r-K4D	The r Rargest-Order Statistics Model in Four parameter Kappa Distribution
x(F)	Quantile Function
μ	Location Parameter of Four parameter Kappa Distribution
σ	Scale Parameter of Four parameter Kappa Distribution
k	Shape1 Parameter of Four parameter Kappa Distribution
h	Shape2 Parameter of Four parameter Kappa Distribution
n	Sample Size
α, λ	A Constant in Penalty Function of Coles and Dixon (1999)
p, q	A Constant in Penalty Function of Martins and Stedinger (2000)
λ_1	The First L-moment
λ_2	The Second L-moment

τ_3	The Third L-Ratios or L- Skewness L-moment
τ_4	The Fourth L-Ratios or L- Kurtosis L-moment
M	Number of Successful Convergence Among 10,000 Trail
q_j^e	The Estimated Quantiles
q_j^t	The True Quantiles
$P(\theta)$	Penalty Function
$L(\mu, \sigma, \theta)$	Likelihood Function
$-l(\mu, \sigma, \theta)$	Negative Log Likelihood Function
$L_{pen}(\mu, \sigma, \theta)$	Penalized Likelihood Function
$l_{pen}(\mu, \sigma, \theta)$	Penalized Negative Log Likelihood Function

CHAPTER 1

INTRODUCTION

A model of hydrological data such as flood, wind storm or heavy rain is essential in the design of water related structures in agriculture, weather modification, climate changes monitor and floodplain management. Knowledge related to the distribution of extreme events is of great importance for the design of water related structures. The model that represents the cornerstone in Extreme Value Theory for the maximum values is the generalized extreme value (GEV) distribution of Jenkinson (1955) This has been widely used for modeling extreme events and fitted the GEV distribution to the data which sometimes yields inadequate results. One of the extreme value distribution with related GEV distribution is four parameter kappa distribution (K4D) introduced by Hosking (1994) That is, that four parameters include the location parameter (μ), scale parameter (σ) and two shape parameters $(k\ ;\ h)$. It can be regarded a generalization of the three-parameter kappa distributions (K3D), generalized pareto distribution (GPD), generalized logistic distribution (GLO), gumbel distribution, exponential distribution and logistics distribution. Figure 1.1 shows the relationship of a four parameter kappa distribution with other distributions.

Many parameter estimation methods have been proposed to fit statistical distribution to hydrological data. The method of moment can be obtained relatively easily. They are often ineffective estimators, particularly in a small sample size Hosking (1994). The current method of estimation employs a linear function of expected order statistics, namely L- moments. For the estimation of K4D, the method of L-moments estimation procedures has been used Hosking (1990)) and found to be more reliable than that of moments estimates, particularly in a small sample size. They are usually computationally more tractable than maximum likelihood estimation (MLE). Sometimes, L-moments estimation of the K4D is not always computable or feasible as seen in Hosking (1994) and Parida (1999) that using the K4D with the method of L-moments estimates is neither always computable nor feasible. The method of MLE estimates is hence seen as an alternative method of parameter estimation for K4D.

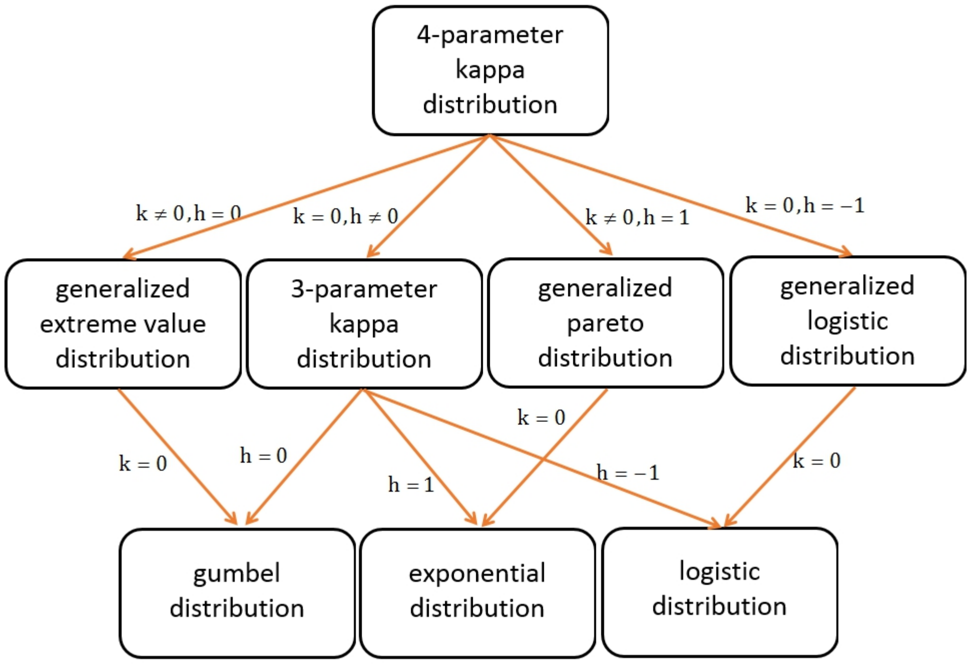


Figure 1.1: Relationship of a four-parameter kappa distribution (K4D) to other distributions, Jeong et al. (2014).

Dupuis and Winchester (2001) conducted a simulation studying a method of parameter estimation for MLE with L-moment estimation and applying to wind speeds from the tropical pacific data. A simulation result suggests that, in some areas of the parameter space, the MLE has lower bias and root mean square error (RMSE) than L-moments. For other areas, the bias and RMSE of the L-moments are smaller. Specifically, the bias and RMSE of the MLE tend to be lower in k negative while the bias and RMSE of L-moments is often smaller in k positive. It is indeed known that the MLE of estimation method performs badly for a small sample size and for a k positive value. This problem motivates the development of the method of MLE for K4D by adjusting a penalty function of Coles and Dixon (1999) considering exponential penalty function. Martins and Stedinger (2000) considered a beta probability density function on shape parameters that corresponds to the mode of the Beyesian posterior distribution of the parameter.

In this study, we aimed to model a four parameter kappa distribution with maximum penalized maximum likelihood estimator (MPLE), a generalization of the maximum likelihood estimation method (MLE). Further, no attempts have yet been made for the analytic expressions of MPLE using a K4D.

Furthermore, in Smith (1986) work, it was presented that a family of statistical distributions for extreme values, based on the joint distribution of the r-largest annual events. The method of estimation was numerical maximum likelihood. An and An and Pandey (2007) presented the statistical estimation of extreme wind speed using annually r-largest order statistics extracted from the time series of wind data. The method was based on a joint generalized extreme value distribution of GEV (r-GEV) derived from the theory of Poisson process and Murshed and Park (2007) used the generalized extreme value (r-GEV) distribution based on the r-largest order statistics. We do study the r-largest order statistics for four-parameter kappa distribution (r-K4D). It is well known the four parameter kappa (K4D) distribution, a generalization of common three-parameter distributions and in particular of the GEV distribution.

1.1 Objective of the research

1. To propose and compare the performance of maximum penalized likelihood estimator with maximum likelihood estimator and L-moments estimator for four-parameter kappa distribution.

2. To study the r-largest order statistics for four-parameter kappa distribution.

1.2 Scope of the study

The scope of the thesis will be presented in two parts following the research objectives.

1.2.1 Scope of study to performance of maximum penalized likelihood estimator with maximum likelihood estimator and L-moments estimator for four-parameter kappa distribution

1. The probability density function will be focused on four parameter kappa distribution.

2. The estimation methods are based on maximum likelihood estimation, L-moments estimation and maximum penalized likelihood estimation.

3. Construct parameter of K4D: location parameter, $\mu = 0$, scale parameter, $\sigma = 1$, shape1 parameter , $-0.4 \leq k \leq 0.4$ and shape2 parameter , $-1.2 \leq h \leq 1.2$.

4. The probability of quantile function, $x(F) = 0.90, 0.95, 0.99, 0.995, 0.999$.

5. The sample size (n) are $30, 50$ and 100.

6. Number of successful convergence among M = 10,000 trail.

7. Criteria to compare for simulation study, the relative bias (RBIAS) and relative root mean square error (RRMSE) of quantile estimators used as a criterion to compare the estimation method. For application with hydrology data, Anderson Darling (AD) goodness-of-fit test and modified from the expected prediction squared error (MPAE) are used a criteria to select the optimal model of hydrology data.

8. In the application with two hydrology data sets, maximum rainfall data was measured in Pattaya of Thailand and the annual maximum temperature data was measured in Surin province, Thailand.

9. Program R version 3.5.2 is used for data analysing.

1.2.2 Scope of study to propose the r-largest order statistics for four parameter kappa distribution (r-K4D)

The probability density function will be focused on r largest-order statistics model.

1.3 Expected outcome

The Expected outcome of the study are

1. Upon the completion of this work, the researcher can use penalized maximum likelihood estimator of K4D in hydrological events; and

2. This is to develop the new model of the r-largest order statistics for K4D for hydrological events.

CHAPTER 2

LITERATURE REVIEW

This chapter reviews the current research pertaining to the fundamental concept of statistics. It is important to study penalized maximum likelihood estimator (MPLE) for four-kappa distribution and basic concept of the r-largest order statistics model for developing an r-largest order statistics model in K4D (r-K4D).

2.1 Order statistics

Definition 2.1.1 The order statistics of a random sample $X_1, \ldots, X_n$ are the sample values placed in ascending order. They are denoted by $X_{(1)}, \ldots, X_{(n)}$. The order statistics are random variables that satisfy $X_{(1)}, \ldots, X_{(n)}$. In particular,

$$\begin{aligned} X_{(1)} &= \min_{1 \leq i \leq n} X_i, \\ X_{(2)} &= \text{second smallest } X_i, \\ &\vdots \\ X_{(n)} &= \max_{1 \leq i \leq n} X_i. \end{aligned}$$

Since they are random variable, we can discuss the probabilities that they take on various value. To calculate these probabilities, we need the pdfs or pmf of the order statistics of a random sample from a continuous population. We will mention some statistics that are easily defined in terms of the order statistics.

Theorem 2.1.1 Let $X_{(1)} \leq \cdots \leq X_{(n)}$ denote the order statistics of a random sample, $X_1, \ldots, X_n$, from a continuous population with cdf $F_X(x)$ and pdf $f_X(x)$. Then the pdf of $X_{(j)}$ is

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} \left[F_X(x)\right]^{j-1} \left[1 - F_X(x)\right]^{n-j} f_X(x).$$

Theorem 2.1.2 Let $X_{(1)} \leq \cdots \leq X_{(n)}$ denote the order statistics of a random sample, $X_1, \ldots, X_n$, from a continuous population with cdf $F_x(x)$ and pdf $f_x(x)$. Then the joint pdf of $X_{(i)}$ and $X_{(j)}$, l ≤i< $j \leq n$, is

$$f_{X_{(i)},X_{(j)}}(u,v) = \frac{n!}{(i-1)!(j-1-i)!(n-j)!}\left[F_X(u)\right]^{i-1}\left[F_X(v)-F_X(u)\right]^{j-1-i}f_X(u)f_X(v)$$
$$\text{for} -\infty < u < v < \infty.$$

The joint pdf of three or more order statistics could be derived using similar but event more involved arguments. Perhaps the other most useful pdf is $f_{X_{(1)},\ldots,X_{(n)}}(x_1,\ldots,x_n)$ the joint pdf of all the order statistics, which is given by

$$f_{X_{(1)},\ldots,X_{(n)}}(x_1,\ldots,x_n) = n!f_X(x_1)\cdots f_X(x_n) \quad ; -\infty < x_1 < \cdots < x_n < \infty.$$

2.2 Extreme value theory

Extreme Value Theory (EVT) has emerged as one of the most important statistical theories in applied sciences. EVT provides a firm theoretical foundation to establish a statistical model that describes extreme events. The feature that distinguishes extreme value analysis from other statistical analysis is the ability to quantify the behaviours of unusual, large values even when those values are scarce. One of the key results from EVT is the ability to estimate the distribution of maximum value, that usually called as maxima, using the asymptotic argument. In order to build such models, the Fisher-Tippett theorem which specifies the form of the limit distribution for transformed maxima will be greatly used. Furthermore, it can be shown that there are only three families of possible limit laws for distribution of maxima - Gumbel, Frechet, and Weibull distributions. These three distributions can be expressed in a single distribution function, called the generalized extreme value (GEV) distribution. The model focuses on the statistical behaviour of

$$M_n = \max(X_1, X_2, \ldots, X_n),$$

where $X_1, \ldots, X_n$, is a sequence of independent random variable having a common distribution function F. In applications, the X_i usually represents a value of a process measurement on a regular time-scale – perhaps a per-hour measurement of a sea level, or daily mean temperatures – so that M_n represents the maximum of the process over n time units of observation. If n is the number of observations in a year, then M_n corresponds to the annual maximum.

In theory, the distribution of M_n can be derived exactly for all value of n :

$$\begin{aligned} \Pr\{M_n \le z\} &= \Pr\{X_1 \le z, \dots, X_n \le z\}, \\ &= \Pr\{X_1 \le z\} \times \cdots \times \{X_n \le z\}, \\ &= \{F(z)\}^n. \end{aligned}$$

However, this is not immediately helpful in practice since the distribution function F is unknown and to look for approximate families of models for F^n. This is similar to the usual practice of approximating the distribution of sample means by the normal distribution, as justified by the central limit theorem. We proceed by looking at the behavior of F^n as $n \to \infty$. But this alone is not enough: for any $z, F^n(z) \to 0, n \to \infty$, so that the distribution of M_n degenerates to a point mass. This difficulty is avoided by allowing a linear renormalization of the variable $M_n : M_n^* = \dfrac{M_n - b_n}{a_n}$ for sequences of constants $a_n > 0$ and $b_n > 0$. Appropriate choices of the a_n and b_n stabilize the location and scale of M_n^* as n increases. The entire range of possible limit distribution for M_n^* is give by extremal type theorem.

Theorem 2.2.1 If there exist sequences of constants $a_n > 0$ and $b_n > 0$ such that

$$P\left(\frac{M_n - b_n}{a_n} \le z\right) \to G(z) \text{ as } n \to \infty,$$

where $G(z)$ is a non-degenerate distribution function, then G is must belong to one of the following three families:

$$I : G(z) = \exp\left\{-\exp\left[-\left(\frac{z-b}{a}\right)\right]\right\}, \quad -\infty < z < \infty;$$

$$II : G(z) = \begin{cases} 0 & \text{if } z \le b, \\ \left\{\exp -\left(\dfrac{z-b}{a}\right)^{-\alpha}\right\} & \text{if } z > b; \end{cases}$$

$$III : G(z) = \begin{cases} \exp\left\{-\left[-\left(\frac{z-b}{a}\right)^{-\alpha}\right]\right\} & \text{if } z < b, \\ 1 & \text{if } z \geq b; \end{cases}$$

where b is a location parameter, $a > 0$ a scale parameter and $\alpha > 0$ a shape parameter. These extreme value distributions are known as the Gumbel, Fréchet and Weibull families for subclasses I, II and III respectively. The sense that this is an analogy of the central limit theorem (CLT) comes from the fact that these three extreme value distributions are the only possible limits for the distributions of the M_n^*, regardless of the parent distribution F Coles (2001).

2.3 Probability density function

2.3.1 Generalized extreme value distribution

The generalized extreme value (GEV) distribution in corporate Gumbel's type I, Fréchet's type II, and the Weibull or type III distributions. The GEV distribution has cumulative distribution function follow Hosking (1994)

$$F(x) = \exp\left[-\left(1-k\left(\frac{x-\mu}{\sigma}\right)\right)^{1/k}\right]. \tag{2.1}$$

The probability density function (pdf)

$$f(x) = \sigma^{-1}(1-ky)^{(1/k)-1}\exp\left[-\left(1-k\left(\frac{x-\mu}{\sigma}\right)\right)^{1/k}\right],$$

where $y = \frac{x-\mu}{\sigma}$ and $1-k\left(\frac{x-\mu}{\sigma}\right) > 0$. Here, μ, σ, k are the location, scale and shape parameter, respectively. The quantile of the GEV distribution is given in terms of the parameters and the cumulative probability by

$$x(F) = \mu + \frac{\sigma}{k}(1-(-\log(F))^k).$$

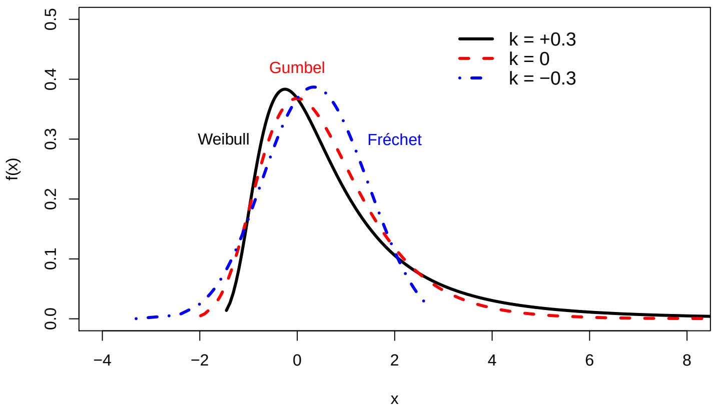


Figure 2.1: Probability density function of GEV distribution

2.3.2 Four parameter kappa distribution

The four parameter kappa distribution (K4D), introduced by Hosking (1994), is a generalized form of some commonly used distribution, such as, generalize logistics distribution (GLD), generalized extreme value distribution and generalized pareto distribution (GPD). The cumulative distribution function (cdf) is

$$F(x) = \left[1 - h\left[1 - k\left(\frac{x-\mu}{\sigma}\right)^{1/k}\right]\right]^{1/h},$$

and probability density function (pdf) of K4D is

$$f(x) = \sigma^{-1}\left[1 - k\left(\frac{x-\mu}{\sigma}\right)\right]^{(1/k)-1}\left\{\left[1 - h\left[1 - k\left(\frac{x-\mu}{\sigma}\right)^{1/k}\right]\right]^{1/h}\right\}^{1-h},$$

and quantile function (inverse cumulative distribution function) of K4D is

$$x(F) = \mu + \frac{\sigma}{k}\left[1 - \left(\frac{1-F^h}{h}\right)^k\right], \quad (2.2)$$

here, μ, σ, and k, h are the location, scale and two shape parameter respectively. In Figure 2.2 and 2.3 present some possible shapes (pdf) of K4D for various combinations of shape parameters (k, h) where location (μ) and scale (σ) parameter are fixed at 0 and 1. Apart from the aforementioned special cases, K4D yields an exponential distribution when $h = 1$ and $k = 0$, a Gumbel distribution when $h = 0$ and $k = 0$, a logistic distribution when $h = -1$ and $k = 0$, and a uniform distribution when $h = 1$ and $k = 1$ Hosking (1994). The bounds on the sample space are easily derived for case where $h = 0$, they may be calculated by noting that

$$\begin{aligned}
&\text{as} \quad F \to 1, \quad \frac{1-F^h}{h} \to 0, \qquad \text{for all} \quad h \neq 0 \\
&\text{as} \quad F \to 0, \quad \frac{1-F^h}{h} \to \infty, \qquad \text{for all} \quad h < 0 \\
&\text{as} \quad F \to 0, \quad \frac{1-F^h}{h} \to h^{-1}, \qquad \text{for all} \quad h > 0.
\end{aligned}$$

The bounds are as follow

$$\begin{array}{rcll}
\mu + \dfrac{\sigma(1-h^{-k})}{k} & \leq x \leq & \mu + \dfrac{\sigma}{k} & \text{if} \quad h > 0, k > 0, \\
\mu + \sigma \log h & \leq x < & \infty & \text{if} \quad h > 0, k = 0, \\
\mu + \dfrac{\sigma(1-h^{-k})}{k} & \leq x < & \infty & \text{if} \quad h > 0, k < 0, \\
-\infty & < x < & \mu + \dfrac{\sigma}{k} & \text{if} \quad h \leq 0, k > 0, \\
-\infty & < x < & \infty & \text{if} \quad h \leq 0, k = 0, \\
\mu + \dfrac{\sigma}{k} & \leq x < & \infty & \text{if} \quad h \leq 0, k < 0.
\end{array} \tag{2.3}$$

Knowledge of the bounds will be important in parameter estimation. When the parameters are estimated from a sample, each observation must satisfy the bounds in Equation 2.3 evaluated at the estimates. Parameter estimates for which this requirement is met are termed feasible, while estimates not satisfying the criteria are labeled non-feasible. The density of the K4D distribution, when graphed, may be fundamentally one of four shapes determined by the parameter value of k and h. In summary, the density may exhibit either a maximum, a minimum or one of two distinct shapes when there is no extremum. Specifically, the distribution has unique maximum if $h < 0$ and $1/h < k < 1$, or if $o \leq h < 1$ and $k < 1$; a single minimum if $h > 1$ and $k > 1$; otherwise $f(x)$ has no extremum Hosking (1994). Thus the graph of four shapes of a K4D distribution show in Figure 2.2. For Figure 2.3 fix k positive value and Figure 2.4 fix k negative value and variying h clearly affects the left tail and right tail probabilities.

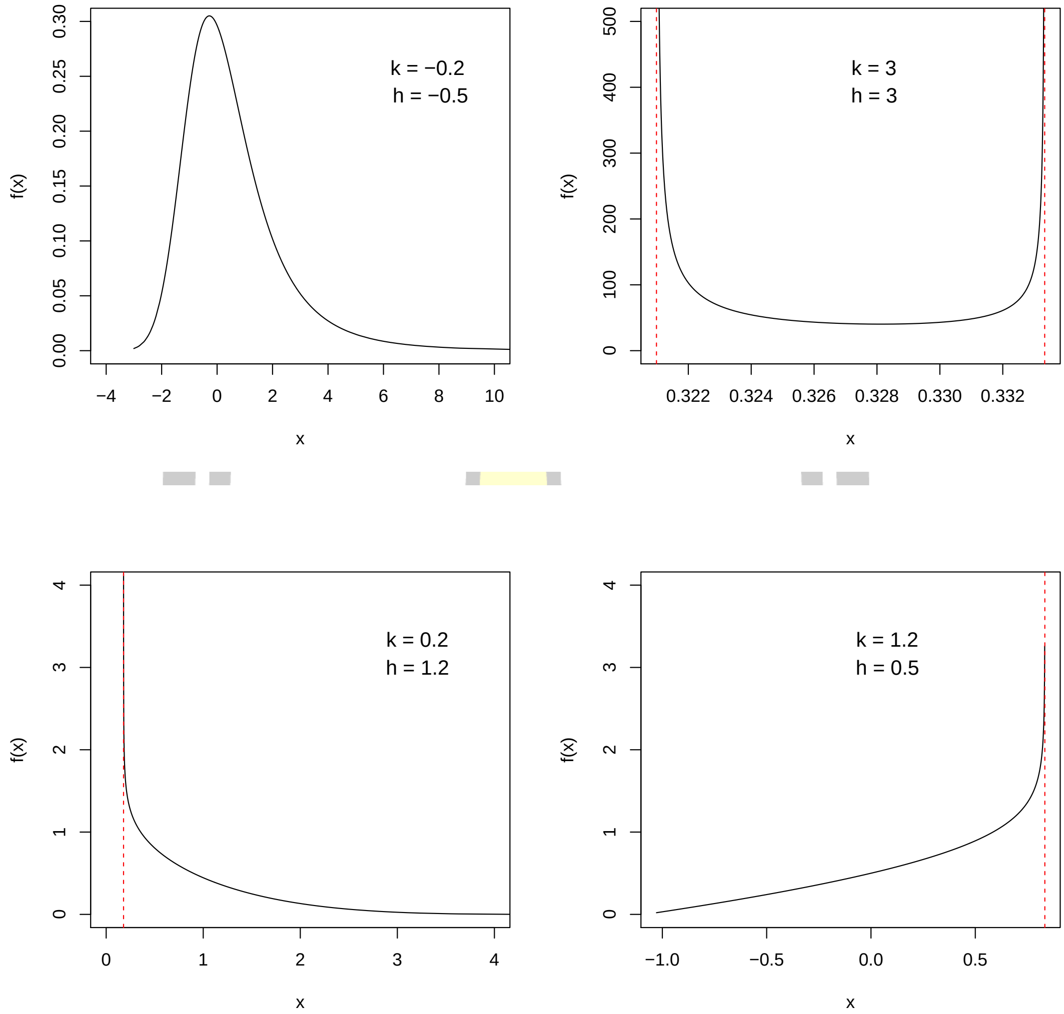


Figure 2.2: Examples of four shapes of a K4D distribution. Dotted lines indicate the bound of distribution at which the density becomes infinite.

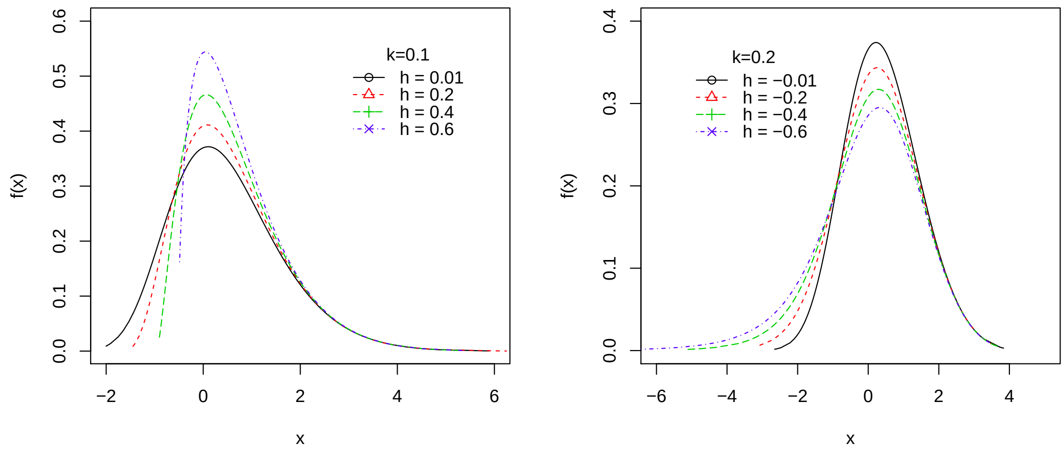


Figure 2.3: Shapes of the pdf of a K4D plotted for different combinations of shape parameters when k > 0

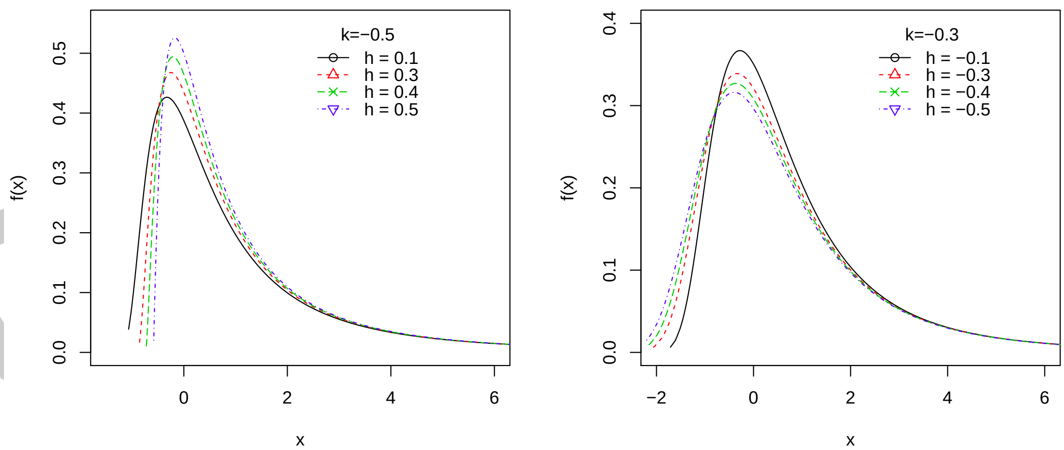


Figure 2.4: Shapes of the pdf of a K4D plotted for different combinations of shape parameters when k < 0

2.4 The r largest-order statistics model

Suppose $X_1, X_2, ..., X_n$ are independent and identically distribution variables, and aim to characterize the extremal behavior of the X_i. In section extreme value theory we obtained that the limiting distribution as $n \to \infty$ if M_n, suitably re-scaled, is GEV. We first extend this result to other extreme order statistics, by defining

$$M_n^{(j)} = j\text{th largest of } (X_1, ...X_n),$$

and identifying the limiting behavior of this variable, for fixed j, as $n \to \infty$.

$$P\left(\frac{M_n^{(j)} - b_n}{a_n} \leq z\right) \to G_j(z),$$

on $x : 1 + k(z - \mu)/\sigma > 0$, where

$$G_j(z) \;\; = \;\; \exp[-\tau(z)] \sum_{s=0}^{j-1} \frac{\tau(z)^s}{s!}, \qquad (2.4)$$

with $\tau(z) = \left[1 - k\left(\frac{z-\mu}{\sigma}\right)\right]^{1/k}$. If the jth largest order statistics in a block is normalized in exactly the same way as the maximum, then its limiting distribution is of the form give by Equation 2.4, the parameters of which correspond to the parameters of the limiting GEV distribution of the block maximum. However, a difficult in using Equation 2.4 as a model.

Smith (1986) presents a family of statistics distributions for extreme values base on a fixed number $r \geqslant 1$ of the largest events. The idea is as follows. Let $X_1, X_2, ...$ denote a sequence of independent and identically distribution random variables and let $M_n = \max(X_1, ...X_n)$. Suppose that normalizing constants a_n and b_n such that $(M_n - b_n)/a_n$ converges in distribution can be found. That is,

$$P\left(\frac{M_n - b_n}{a_n} \leq z\right) \to G(z) \text{ as } n \to \infty \qquad (2.5)$$

for some proper distribution G. Then, the only non-degenerate form for G is necessarily Equation 2.1. The basis of Smith's method is an extension of Equation 2.5 to the asymptotic distribution of the $r-$ largest value from a sample of size n $(r \text{ fixed}, n \to$

∞). Let $X_1, X_2, ...$ be i.i.d., as before, and suppose that, for each n, the value $X_1, ..., X_n$ are ordered as $X_{n,1} \geqslant ... \geqslant X_{n,n}$. Let a_n and b_n be as before and consider, for fixed r, the joint distribution of

$$\left(\frac{X_{n,1} - b_n}{a_n}, \frac{X_{n,2} - b_n}{a_n}, ..., \frac{X_{n,r} - b_n}{a_n}\right). \quad (2.6)$$

Then, under precisely the same condition to Equation 2.5, this joint distribution converges to that of a vector $(Z_1, ..., Z_n)$ for which an explicit formula is available. The joint density of$(X_1, ..., X_n)$ is

$$\begin{aligned} f(z_1, ..., z_r) &= \exp\left\{-\left[1 - k\left(\frac{z_r - \mu}{\sigma}\right)\right]^{1/k}\right\} \\ &\quad \times \prod_{j=1}^{r} \sigma^{-1}\left[1 - k\left(\frac{z_j - \mu}{\sigma}\right)\right]^{(1/k)-1}, \end{aligned} \quad (2.7)$$

where $-\infty < \mu < \infty, \sigma > 0$ and $-\infty < k < \infty$; $z_r \leq z_{r-1} \leq ... \leq z_1$ and $z_j : \left[1 - k\left(\frac{z_j - \mu}{\sigma}\right)\right] > 0$ for $j = 1, 2, ..., r$.

The likelihood for this model is obtained by absorbing the unknown scaling coefficients into location and scale parameters in the usual way, and by taking products across block.

$$\begin{aligned} L(\mu, \sigma, k) &= \prod_{i=1}^{n}\left(\exp\left\{-\left[1 - k\left(\frac{z_{ri} - \mu}{\sigma}\right)\right]^{1/k}\right\}\right. \\ &\quad \left.\times \prod_{j=1}^{r_i} \sigma^{-1}\left[1 - k\left(\frac{z_{ji} - \mu}{\sigma}\right)\right]^{(1/k)-1}\right). \end{aligned} \quad (2.8)$$

Provided $1 - k\left(\frac{z_j - \mu}{\sigma}\right) > 0, j = 1, ..., r_i,\ i = 1, ..., n$; otherwise the likelihood is zero. When $k = 0$

$$\begin{aligned} L(\mu, \sigma) &= \prod_{i=1}^{n}\left(\exp\left\{-\exp\left[-\left(\frac{z^{ri} - \mu}{\sigma}\right)\right]\right\}\right. \\ &\quad \left.\times \prod_{j=1}^{ri} \sigma^{-1}\exp\left[-\left(\frac{z_{ji} - \mu}{\sigma}\right)\right]\right). \end{aligned} \quad (2.9)$$

The likelihood Equation 2.8 and 2.9 or, more commonly, the corresponding log-likelihood , can be maximized numerically to obtain maximum likelihood estimates. In the spacial case of $r = 1$, the likelihood function reduces to the likelihood of the GEV model for block maxima. More generally, the r largest order statistics model gives a likelihood whose parameters correspond to those of the GEV distribution of block maxima, but which incorporates more of the observed extreme data. So, relative to a standard block maxima analysis, the interpretation of the parameters is unaltered, but precision should be improved due to the inclusion of extra information.

2.5 Estimation method

2.5.1 Maximum likelihood estimation

In statistics, maximum likelihood estimation (MLE) is a method of estimating the parameters of a statistical model, given observations. The method obtains the parameter estimates by finding the parameter values that maximize the likelihood function. The estimates are called maximum likelihood estimates, which is also abbreviated as MLE.

Let us suppose $X_1, X_2, ..., X_n$ are an iid sample from a population with pdf or pmf $f(x_i|\theta_1, \theta_2, ..., \theta_t)$, the likelihood function is defined by

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} f(x_i|\boldsymbol{\theta}). \tag{2.10}$$

The log likelihood function is

$$\log L(\boldsymbol{\theta}) = \prod_{i=1}^{n} \log f(x_i|\boldsymbol{\theta}). \tag{2.11}$$

The maximum likelihood estimator $\hat{\boldsymbol{\theta}}_0$ of $\boldsymbol{\theta}_0$ is defined as the value of θ that maximizes the appropriate likelihood function. Since the logarithm function is monotonic, the log likelihood takes its maximum at the same point as the likelihood function, so that the maximum likelihood estimator also maximizes the corresponding log likelihood function.

1. Maximum likelihood estimation for GEV distribution

Under the assumption that $X_1, X_2, ..., X_n$ are independent variables having the GEV distribution the likelihood for the GEV parameters when $k \neq 0$ is

$$L(\mu, \sigma, k) = \prod_{i=1}^{n}\left(\exp\left\{-\left[1-k\left(\frac{x_i-\mu}{\sigma}\right)\right]^{1/k}\right\} \times\sigma^{-1}\left[1-k\left(\frac{x_i-\mu}{\sigma}\right)\right]^{\left(\frac{1}{k}\right)-1}\right).$$

and the log-likelihood of GEV is

$$l(\mu, \sigma, k) = -n\log\sigma - \left(1-\frac{1}{k}\right)\sum_{i=1}^{n}\log\left[1-k\left(\frac{x_i-\mu}{\sigma}\right)\right] - \sum_{i=1}^{n}\left[1-k\left(\frac{x_i-\mu}{\sigma}\right)\right]^{1/k}, \quad (2.12)$$

provided

$$1-k\left(\frac{x-\mu}{\sigma}\right) > 0 \quad \text{for} \quad i = 1, ..., n. \quad (2.13)$$

At parameter combination for which Equation 2.13 is violated, corresponding to a configuration for which at least one of the observed data falls beyond and end-point of the distribution, the likelihood is zero and the log-likelihood equals -∞. Maximization Equation 2.12 with respect to the parameter vector (μ, σ, k) lead to the maximum likelihood estimate with respect to the entire GEV family. There is no analytical solution, but for any give data set the maximization is straightforward using standard numerical optimization algorithm.

2. Maximum likelihood estimation for K4D distribution

For given observation $X_1, X_2, ..., X_n$ assuming $k \neq 0$ and $h \neq 0$, the likelihood function is

$$L(\mu, \sigma, k, h) = \prod_{i=1}^{m}\left(\sigma^{-1}\left[1-k\left(\frac{x_i-\mu}{\sigma}\right)\right]^{(1/k)-1}\right.$$

$$\times\left\{\left[1-h\left[1-k\left(\frac{x_i-\mu}{\sigma}\right)^{1/k}\right]\right]^{1/h}\right\}^{1-h}\Bigg), \qquad (2.14)$$

and the log-likelihood function is

$$\begin{aligned} l(\mu,\sigma,k,h) &= -m\log\sigma+\left(\frac{1}{k}-1\right)\sum_{i=1}^{m}\log\left[1-k\left(\frac{x_i-\mu}{\sigma}\right)\right]+ \\ &\quad (1-h)\sum_{i=1}^{m}\log\left[1-h\left[1-k\left(\frac{x_i-\mu}{\sigma}\right)^{1/k}\right]\right]^{1/h}, \end{aligned}$$

the negative log-likelihood is

$$\tau(\mu,\sigma,k,h) = m\log\sigma-\left(\frac{1-k}{k}\right)\sum_{i=1}^{m}\log\, G_i-(1-h)\sum_{i=1}^{m}\log\, F(x_i). \qquad (2.15)$$

where $G_i=\left[1-k\left(\frac{x_i-\mu}{\sigma}\right)\right]$ and $F(x_i)=\left[1-h\left[1-k\left(\frac{x_i-\mu}{\sigma}\right)^{1/k}\right]\right]^{1/h}$.

The maximum likelihood estimates are obtained by minimizing Equation 2.15 with respect to the parameter vector (μ,σ,k,h). Since no explicit minimizer is possible, this function can be numerically minimized by Newton-type algorithm.

The first derivatives of $\tau(\mu,\sigma,k,h)$ with respect to (μ,σ,k,h) are

$$\frac{\partial\tau(\mu,\sigma,k,h)}{\partial\mu}=\sigma^{-1}\sum_{i=1}^{m}G_i^{-1}(k-1-H_i)=0$$

$$\frac{\partial\tau(\mu,\sigma,k,h)}{\partial\sigma}=\frac{m}{\sigma}+\sigma^{-2}\sum_{i=1}^{m}W_i(k-1-H_i)=0$$

$$\frac{\partial\tau(\mu,\sigma,k,h)}{\partial h}=h^{-1}\sum_{i=1}^{m}(\log\, F(x_i)-H_i)=0$$

$$\frac{\partial\tau(\mu,\sigma,k,h)}{\partial k}=k^{-1}\left[\sum_{i=1}^{m}H_i(k^{-1}\log\, G_i+\sigma^{-1}W_i)+\sum_{i=1}^{m}k^{-1}(\log\, G_i-(k-1)\sigma^{-1}W_i)\right]=0.$$

Where $H_i=(h-1)G_i^{1/k},\ W_i=G_i^{-1}(x_i-\mu)$.

2.5.2 L-moments estimation

L-moments estimation provides an alternative method of estimation analogous to a conventional moment, but with several advantages, including more robust estimates with less bias. Hosking (1990) defined the L-moments of probability distributions as follows:

$$\lambda_r = r^{-1}\sum_{k=0}^{r}(-1)^k\binom{r-1}{k}E[X_{r-k:r}]. \quad r = 1, 2, ...$$

That is λ_r is a linear function of the expected order statistics. Since,

$$E[X_{j:r}] = \frac{r!}{(j-1)!(r-j)!}\int x\{F(x)\}^{j-1}\{1-F(x)\}^{r-j}dF(x),$$

λ_r may be re-expressed as

$$\lambda_r = \int x(F)P^*_{r-1}(F)dF, \qquad r = 1, 2, ...$$

where

$$P^*_r(F) = \sum_{k=0}^{r} p^*_{r,k}F^k,$$

and

$$p^*_{r,k} = (-1)^{r-k}\binom{r}{k}\binom{r+k}{k}. \tag{2.16}$$

The first four L-moments are

$$\lambda_1 = E(X) = \int_0^1 x(F)dF,$$

$$\lambda_2 = \frac{1}{2}E(X_{2:2} - X_{1:2}) = \int_0^1 x(F)(2F-1)dF,$$

$$\lambda_3 = \frac{1}{3}E(X_{3:3} - 2X_{2:3} + X_{1:3}) = \int_0^1 x(F)(6F^2 - 6F + 1)dF, \quad \text{and}$$

$$\lambda_4 = \frac{1}{4}E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}) = \int_0^1 x(F)(20F^3 - 30F^2 + 12F - 1)dF.$$

Often the higher moments $\lambda, r \geq 3$, are standardized so that they are independent of the units of measurement of X. The L-moment ratio of X are defined

$$\tau_r = \lambda_r/\lambda_2, \quad r = 3, 4, ...$$

In addition, the L-CV is defined to be $\tau = \lambda_2/\lambda_1$, and is analogous to the coefficient of variation (CV). The following theorem, proved in Hosking (1989), gives the bounds on the L-moments ratios and the L-CV which are particularly useful when L-moments are describe probability distributions. L-moments and L-moments ratios are more convenient than probability weighted moments, because they are more easily interpretable as measures of distribution shape. In particular, λ_1 is the mean of the distribution, a measure of location; λ_2 is a measure of scale or dispersion of the random variable; and τ_3 , τ_4 are measure of skewness and kurtosis, respectively.

Greenwood defined probability weighted moments (PWMs) Greenwood et al. (1979) related to L-moments defined by as

$$M_{p,r,s} = E\{X^p F^r [1 - F]^s\} = \int_0^1 [x(F)]^p F^r (1 - F)^s dF,$$

where $F(x)$ is the cumulative distribution function and p, r and s are real numbers. Setting $p = 1$ and $s = 0$ yields the special cases

$$\beta_r = M_{1,r,0} = E\{X[F(X)]^r\} = \int_0^1 [x(F)] F^r dF, \quad r = 0, 1, ...$$

The PWMs can be expressed as linear combinations of L-moments

$$\lambda_{r+1} = \sum_{k=0}^{r} p^*_{r,k} \beta_k, \quad r = 0, 1, ...$$

where the $p^*_{r,k}$ are defined in Equation 2.16. Although the probability weighted moments are useful in bomputing the L-moments, use of the L-moment for inference instead of the β_r is more convenient since they are more easily interpreted as measures scale and shape of probability distribution than are the latter Hosking (1990). This study used the method L-moments (LM), are linear combinations of probability-weighted moments. the four parameter model. The L-moments of the K4D exist if and only if the mean of its distribution is finite. For the K4D, if $h \geq 0$ and $k > -1$, or if $h < 0$

and $-1 < k < -1/h$. The β_r are most conveniently evaliated by writing as

$$\begin{aligned}\beta_r &= \int_0^1 x(F)F^r dF \\ &= (r+1)^{-1}\left(\mu + \frac{\sigma}{k}\right) - \frac{\sigma}{k}\int_0^1 [h^{-1}(1-F^h)]^k F^r dF.\end{aligned}$$

For $k \neq 0$, the β_r are given by

$$r\beta_{r-1} = \begin{cases} \mu + \dfrac{\sigma}{k}\left[1 - \dfrac{r\Gamma(1+k)\Gamma(r/h)}{h^{1+k}\Gamma(1+k+r/h)}\right] & \text{if } h > 0, k > -1, \\ \mu + \dfrac{\sigma}{k}\left[1 - r^{-k}\Gamma(1+k)\right] & \text{if } h = 0, k > -1, \\ \mu + \dfrac{\sigma}{k}\left[1 - \dfrac{r\Gamma(1+k)\Gamma(-k-r/h)}{(-h)^{1+k}\Gamma(1-r/h)}\right] & \text{if } h < 0, \; -1 < k < -1/h. \end{cases} \tag{2.17}$$

If $k = 0$, we have

$$r\beta_{r-1} = \begin{cases} \mu + \sigma\left[\gamma + \log(h) + \psi(1+r/h)\right] & \text{if } h > 0, \\ \mu + \sigma\left[\gamma + \log(r)\right] & \text{if } h = 0, \\ \mu + \sigma\left[\gamma + \log(-h) + \psi(-r/h)\right] & \text{if } h < 0, \end{cases} \tag{2.18}$$

where $\gamma = 0.5772$ is Euler's constant and $\psi(x) = d[\log\Gamma(x)]/dx$. is the digamma function. Since given the β_r, the L-moments of K4D can be obtain from Equation 2.17 and 2.18. It is possible to express the λ_r as linear combination of $E[X_{k:k}], k = 1, ..., r$ as follows,

$$\lambda_r = \sum_{k=1}^{r} p^*_{r-1,k-1} k^{-1} E[X_{k:k}], \qquad r = 0, 1, ...,$$

where

$$E[X_{k:k}] = k\int x(F(X))^{k-1} dF(X), \qquad k = 1, ..., r.$$

Hosking (1990). With this reformulation, estimating the L-moments from a random sample drawn from an unknown distribution essentially reduces to estimating the $E[X_{k:k}]$

Let $x_1, x_2, ..., x_n$ be the sample and $x_{1:n} \leq x_{2:n} \leq ... \leq x_{n:n}$ be the ordered sample.

The U-statistics estimator of $E[X_{k:k}]$ may be expressed as a linear combination of order statistics by counting the occurrences in it of each $x_{i:n}$ and let

$$b_k = n^{-1} \sum_{i=1}^{n} \frac{(i-1)(i-2)...(i-k)}{(n-1)(n-2)...(n-k)} x_{i:n}, \qquad k = 0, 1, ..., n-1$$

So that the U-statistics estimator of $E[X_{k:k}]$ is kb_{k-1}. THE rth sample L-moments is therefore defined to be

$$l_{r+1} = \sum_{k=0}^{r} p^*_{r,k} b_k, \qquad r = 0, 1, ..., n-1,$$

An alternative method of estimating the L-moments is by plotting position estimators. Hosking (1990) define a plotting position to be a distribution free estimator of $F(x_{i:n})$. Common plotting position include $p_{i:n} \equiv (i+\gamma)/(n+\delta)$ for $\delta > \gamma > -1$. and λ_r may be estimated by

$$\hat{\lambda}_r = \sum_{i=1}^{n} P^*_{r-1}(p_{i:n}) x_{i:n}.$$

The estimator $\hat{\lambda}_r$ is consistrnt but not generally an unbiased estimator of λ_r. Asymptotically, $\hat{\lambda}_r$ and l_r are equivalent. While b_r and l_r are unbiased estimators of β_r ans λ_r respectively, the estimator $t_r = l_r/l_2$ of τ_r, called the rth sample L-moments ratio is consistent but not unbiased. Being linear functions of the data, the sample L-moments (for $r \geq 1$) are not as affected by extreme value in the data and sampling variability as are the conventional moments. Accordingly, the quantile l_1 and l_2 along with the sample L-skewness, t_3 and the sample L-kurtosis, t_4 are useful in summarizing a data set and in parameter estimation.

Parameter estimation by L-moments is performed much like parameter estimation by conventional moments where the sample quantiles are equated to the population quantiles and solved for the parameters of the distribution. For the K4D, $\lambda_1, \lambda_2, \tau_3$ and τ_4 may be obtained from the β_r in Equation 2.17 and 2.18. It can be shown that the L-moments ratio τ_3 and τ_4 are function only of the shape parameters h and k. The L-kurtosis,τ_4 versus the L-skewness, τ_3, for fixed h values and k varying are plotted in Figure 2.5. The L-moments ratio diagram show that some (τ_3, τ_4) points

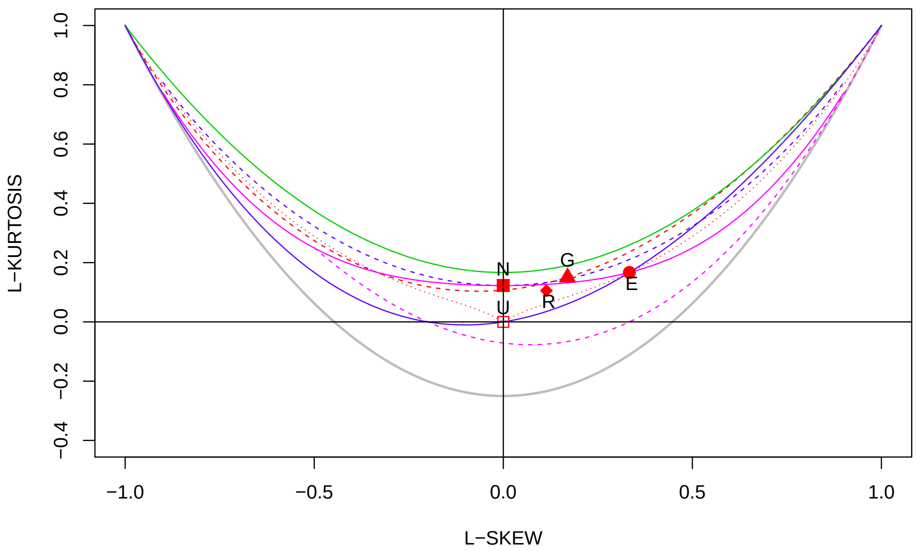


Figure 2.5: L-moments ratios τ_3 and τ_4 for the K4D. the graph shows τ_4 as a function of τ_3 as k varies, for fixed h. Solid lines indicate the value $h = 1, h = 0$ and $h = -1$. Dotted lines $h = -2, -3, -4, -5$. Light dashed lines, lower bound of τ_4 for all distributions obtained by setting $h \rightarrow \inf$. Speccific point marked by N, U, R, G and E represent Normal, Uniform, Rayleigh, Gumbel and Exponential distribution, respectively.

correspond to more than one set of (k, h) coordinates and that not all possible (τ_3, τ_4) points are generated from the K4D. To enable only one set of parameter estimates to be found, given the first four sample L-moments, the parameter space must be restricted. Hosking (1990) found that the maximum attainable values of τ_4 for several values of τ_3 are approximately related to k and h by $k + 0.725h = -1$ when $h > -1$. K4D for which $h > -1$ include all (τ_3, τ_4) values between the "generalized logistic line" (the line generate when h is fixed to -1 and k is allowed to vary) and the lower bound for all distribution (the line generated when $h \rightarrow \infty$). For most practical purposes the restriction $h > -1$ is reasonable. The "generalized logistic line" in Figure 2.5 satisfies $\tau_4 = (5\tau_3^2 + 1)/6$ and for all K4D distributions with $h > -1, \tau_4 \leq (5\tau_3^2 + 1)/6$. The lower bound on τ_4 is the line $\tau_4 = (5\tau_2^2 - 1)/4$ Hosking (1990). In summary, the following conditions are imposed on the parameters:

$$(a) \qquad k > -1$$

$$(b) \quad \text{if } h < 0, \text{ then } hk > -1$$
$$(c) \quad h > -1$$
$$(d) \quad k + 0.725h > -1$$

The condition (a) and (b) guarantee the existence of the mean and the therefore of the L-moments while (c) and (d) ensure that the parameters are unique given the first four L-moments.

2.6 Profile likelihood estimation

2.6.1 Basic concept of profile likelihood function

Base on the maximum likelihood estimation method, θ is parameter vector with k unknown parameters and Θ is the range of values that Θ, take $\theta = (\theta_i, \theta_{i-1})$ where θ_i represents the interesting component in θ and θ_{i-1} represents remaining component in θ. The log likelihood function of θ can be described as $l(\theta_i, \theta_{i-1})$ and the log likelihood function of θ_i can be described as $l_p(\theta_i) = \max l(\theta_i, \theta_{i-1})$, i.e. using the likelihood function to obtain the maximum value of all components except θ_i.

2.6.2 Profile likelihood estimation of parameters and quantile of extreme value distribution

Assumed that $X_1, X_2, ..., X_n$ are independent random variable which obey the GEV distribution, the basic principle to estimate the confidence interval of key parameters and quantile of GEV distribution using profile likelihood estimation method is as follow:

(i) MLE of parameters and quantiles of extreme distribution, when $k \neq 0$, the log likelihood function of GEV distribution in Equantions 2.12. Maximize the log likelihood function in terms of respect parameter vectors to obtain the likelihood equations. Though there is no analytic solution, based on the given data, the maximum likelihood estimation of each parameter vector of GEV distribution can be calculated by numerical algorithm. With such estimated values of parameters, it is possible to further estimate the quantile, With $0 < p < 1$, the maximum likelihood estimation of p quantile x_p of GEV distribution can be performed according to the following equations:

$$\hat{x}_p = \mu + \frac{\sigma}{k}(1 - (-\log(1-p))^k), \quad k \neq 0,$$

(ii) Profile likelihood estimation of parameters of GEV distribution, suppose $k = k_0; k_o \neq 0$ is constant, calculate the maximum value of Equantions 2.12 about μ and σ, and repeat this step in a certain range of k_0. For each k_0, the maximum value of likelihood function can be obtained, i.e. the profile likelihood function value of k. Then the profile likelihood estimation value $\hat{k}$ of k is obtain by maximizing the profile likelihood function. The approximation of $1-\alpha$ confident interval of k is $C_\alpha = \left\{k : 2(l(\hat{\mu}, \hat{\sigma}, \hat{k})) - \max l(\mu, \sigma, k) \leq c_{1-\alpha}\right\}$, where $c_{1-\alpha}$ is $1-\alpha$ quantile of chi-square distribution with one degree of freedom.

(iii) Profile likelihood estimation of quantile can also be use to estimate the confidence interval of multi parameter function. For in stance, in order to obtain the confident interval of quantile x_p, it is necessary to redefine parameters of the GEV model and introduce x_p into likelihood function.

$$\mu = x_p - \frac{\sigma}{k}(1 - (-\log(1-p))^k), \quad k \neq 0, \tag{2.19}$$

Taking x_p as a constant value and substituting Equation 2.19 into Equation 2.12 we can get the log likelihood function of the GEV model about parameters (x_p, σ, k). According to profile likelihood estimation method of key parameters of GEV distribution, it is possible to obtain the profile likelihood estimation $\hat{x}_p$ and its profile likelihood confidence interval of parameter x_p in GEV distribution.

2.7 Review the generalized extreme value distribution explain to four parameter kappa distribution

Many common probability distributions, including some that have attracted recent interest for flood frequency analysis, may be regarded as spatial cases of four parameter kappa distribution that the generalizes the three parameter kappa distribution, the generalizes extreme value distribution and generalizes logistics distribution. It is the distribution of random variable X that is obtained by applying the transformation

Equation 2.20 to random variable Y distribution as in Equation 2.21. We call this the four parameter kappa distribution. The transformation

$$X = \begin{cases} \mu + \sigma(1 - e^{-kY})/k & \text{if } k \neq 0, \\ \mu + \sigma Y & \text{if } k = 0, \end{cases} \tag{2.20}$$

where X and Y are real valued random variables and μ, σ and k are real value parameter, uderlies several distribution recently used in flood frequency analysis. If Y has cumulative distribution function as

$$F(y) = \begin{cases} (1 - he^{-y})^{1/h} & \text{if } h \neq 0, \\ \exp(-e^{-y}) & \text{if } h = 0, \end{cases} \tag{2.21}$$

The transformation (CDF method).

(i). To find the cumulative distribution when k is not equal to 0

We start with the cdf of X if $k \neq 0$

$$\begin{aligned} F_X(x) &= Pr\left(X \leqslant x\right), \\ &= Pr\left(\mu + \frac{\sigma(1 - e^{-kY})}{k} \leqslant x\right), \\ &= Pr\left(1 - e^{-kY} \leqslant \frac{k(x - \mu)}{\sigma}\right), \\ &= Pr\left(e^{-kY} \geqslant \left[1 - \frac{k(x - \mu)}{\sigma}\right]\right), \\ &= Pr\left(-kY \leqslant \log\left[1 - \frac{k(x - \mu)}{\sigma}\right]\right), \\ &= Pr\left(Y \leqslant \frac{-1}{k}\log\left[1 - \frac{k(x - \mu)}{\sigma}\right]\right), \\ &= F_Y\left(\frac{-1}{k}\log\left[1 - \frac{k(x - \mu)}{\sigma}\right]\right), \\ &= F_Y\left(\text{-log}\left[1 - \frac{k(x - \mu)}{\sigma}\right]^{1/k}\right), \\ F_X(x) &= F_Y\left(u^{-1}(x)\right), \end{aligned}$$

replace $y =$ -log $\left[1 - \dfrac{k(x-\mu)}{\sigma}\right]^{1/k}$ to 2.21 in case $h \neq 0$,

$$
\begin{aligned}
F_X(x) &= F_Y\left(u^{-1}(x)\right), \\
F_X(x) &= \left(1 - h\left[1 - \frac{k(x-\mu)}{\sigma}\right]^{1/k}\right)^{1/h}.
\end{aligned}
$$

Note that X is four parameter kappa distribution if $k \neq 0, h \neq 0$. The properties of the four parameter kappa distribution, and gives an example in which it is applied to modeling the distribution of annual maximum precipitation data in Hosking (1994). If replace $y =$ -log $\left[1 - \dfrac{k(x-\mu)}{\sigma}\right]^{1/k}$ to Equation 2.21 in case $h = 0$,

$$
\begin{aligned}
F_X(x) &= F_Y\left(u^{-1}(x)\right), \\
F_X(x) &= \exp\left(-\left[1 - \frac{k(x-\mu)}{\sigma}\right]^{1/k}\right).
\end{aligned}
$$

Thus, X is generalized extreme value distribution if $k \neq 0, h = 0$.

(ii). To find the cumulative distribution when k is equal to 0

We start with the cdf of X if $k = 0$

$$\begin{aligned}
F_X(x) &= Pr\left(X \leqslant x\right), \\
&= Pr\left(\mu + \sigma Y \leqslant x\right), \\
&= Pr\left(Y \leqslant \frac{(x-\mu)}{\sigma}\right), \\
&= F_Y\left(\frac{(x-\mu)}{\sigma}\right), \\
F_X(x) &= F_Y\left(u^{-1}(x)\right),
\end{aligned}$$

replace $y = \dfrac{(x-\mu)}{\sigma}$ to Equation 2.21 in case $h \neq 0$,

$$\begin{aligned}
F_X(x) &= F_Y\left(u^{-1}(x)\right), \\
F_X(x) &= \left(1 - h \exp\left[\frac{(x-\mu)}{\sigma}\right]\right)^{1/h}.
\end{aligned}$$

Thus, X is generalized three parameter kappa distribution if $k = 0, h \neq 0$.
If replace $y = \dfrac{(x-\mu)}{\sigma}$ to Equation 2.21 in case $h = 0$,

$$\begin{aligned}
F_X(x) &= F_Y\left(u^{-1}(x)\right), \\
F_X(x) &= \exp\left(-\exp\left[-\frac{(x-\mu)}{\sigma}\right]\right).
\end{aligned}$$

Thus, X is gumbel distribution if $k = 0, h = 0$.

2.8 Criterion

To performance the comparison of MLE, MPLE.CD, MPLE.MS1, MPLE.MS2, MPLE.MS3, MPLE.MSP1, MPLE.MSP2, MPLE.MSP3 and L-moments of K4D, simulations have been performed by relative bias (RBIAS) and relative root mean square error (RRMSE) of quantile estimators.

2.8.1 Relative bias (RBIAS) and relative root mean square error (RRMSE)

The RBIAS defined as Castillo et al. (2005),

$$\text{RBIAS} = \frac{1}{M}\sum_{j=1}^{M}\left(\frac{q_j^e - q_j^t}{q_j^t}\right),$$

The RRMSE defined as

$$\text{RRMSE} = \sqrt{\frac{1}{M}\sum_{j=1}^{M}\left(\frac{q_j^e - q_j^t}{q_j^t}\right)^2},$$

where M is number of successful convergence among 1,000 trail, and q_j^e and q_j^t are the estimated and true quantiles, respectively.

2.8.2 Modified prediction absolute error

Modified prediction absolute error (MPAE) is used to assess goodness-of-fit for the above $50-th$ percentile of the distribution for different methods. MPAE is modified from the expected prediction squared error (PSE) of Efron and Tibshirani (1994) as

$$MPAE = \frac{2}{n}\sum_{i=\frac{n}{2}}^{n}\left|x_{(i)} - \hat{x}_{(i)}\right|,$$

where n is the number of observations, $x_{(i)}$ are the ascending order observations, $\hat{x}_{(i)}$ are estimated quantiles obtained from Equation 2.2 using four estimates method and plugged into the plotting position equation $p_i = i/(n+1)$.

2.8.3 Goodness of fit Test

Let X be a continuous random variable with distribution function $F(x)$, and $X_1, X_2, ..., X_n$ be a random sample from X with order statistics $X_{(1)}, X_{(2)}, ..., X_{(n)}$. We wish to test the null hypothesis

$$H_0 : F(x) \equiv F_0(x) \quad \text{for all} \quad x \in (-\infty, \infty)$$

where $F_0(x)$ is a hypothesis distribution function to be testedWhere $F(x)$ is completely know. For other case. The Anderson-Darling test for goodness of fit is based on the test statistics (not denoting the dependence on the parameter)

$$A^2 = n\int_{-\infty}^{\infty}\frac{(F_n(x)-F(x))^2}{F(x)(1-F(x))}dF(x)$$

where F_n is cumulative distribution function of the sample and F is the cumulative distribution function which is to be fitted. A^2 can be computed based on the probability integral transform as The Anderson-Darling statistics

$$A^2 = \frac{-2}{n}\sum_{i=1}^{n}\left[\left(i-\frac{1}{2}\right)\log\left\{F_0(X_{(i)})\right\}+\left(n-i+\frac{1}{2}\right)\log\left\{1-F_0(X_{(i)})\right\}\right]-n$$

where $F_0(X_{(i)})$ is the distribution function evaluated at the i^{th} element of the ordered sample.

2.9 Related research of four kappa distribution

Parida (1999) the random behaviour of the Indian summer monsoon, the possibility of using a generalized four parameter Kappa distribution representing a family of distributions has been explored. An L-moments procedure developed by Hosking (1990), has been used for estimating the reliable rainfall quantiles. Using these estimates, isopluvial maps have been developed for some commonly used recurrence intervals (viz. 20, 50, 100, 200, 500 and 1000 years) which can be used by operational hydro-meteorologists.

Dupuis and Winchester (2001) study the K4D based on simulation the extent of each problem. Maximum likelihood is investigated as an alternative method of estimation. The simulation study compares the performance of L-moments and MLE method of estimation. Maximum likelihood is shown when wind speeds form th Tropical Pacific are examined and the weekly maximum for 10 buoys in the area are analysed.

Park et al. (2001) use the maximum likelihood estimates (MLE) on the summer extreme daily rainfall data at 61 gauging stations over South Korea. This has been made to obtain reliable quantile estimates for several return periods. The distribution of return values for annual maxima of 2-day precipitation (AMP2) is more similar to the climatological features of annual total precipitation of Korea than that of annual maxima of daily precipitation (AMP1). Our results of return values delineate well to the horizontal patterns of the heavy precipitation over the Korean peninsula.

Ahmad et al. (2013) study monsoon rainfall in Pakistan is of great importance because of its needs in agriculture and power generation. The study was conducted to analyse the random behaviour of monsoon rainfall in Pakistan through K4D at 27 meteorological stations for the period 1960-2006. The parameters of this distribution have been estimated using method of L-Moments. Due to the employment of these estimating procedures, it could calculate the quantiles for different return periods from 2 to 500 years. The comparison of estimated quantiles with observed values of rainfall after five years is found to be in good agreement.

Murshed et al. (2014) investigate the effect and feasibility of the high-order L-moments (LH-moments) method for estimating heavy-tail conditions by fitting a K4D. Details of parameter estimation, using LH-moments for the K4D, are described and formulated. Monte-Carlo simulation is performed to illustrate the performance of the LH-moments method in terms of heavy-tail quantiles over all quantiles using K4D and non K4D samples, respectively. The result suggests that the method is either useful (when the method of L-moments estimation fails to give a feasible solution) or as effective as the L-moments approach in handling data following K4D. Applications to the annual maximum flood and sea level data are presented.

Kjeldsen et al. (2017) employ a K4D with the widely used regional goodness-of-fit methods as part of an index flood regional frequency analysis, based on the method of L-moments. The framework was successfully applied to 564 pooling groups and found to significantly improve the probabilistic description of British flood flow compared to the existing procedures. Based on results from an extensive data analysis, it is argued that the successful application of the kappa distribution renders the use of the traditional three-parameter distributions such as the generalized extreme value (GEV) and generalized logistic (GLO) distributions obsolete, except for large and relatively dry catchments. The importance of these findings is discussed in terms of the sensitivity of design floods to distribution choice.

2.10 Related research of the r largest order statistics model

Smith (1986) present a family of statistical distributions and estimators for extreme values based on a fixed number $r \geqslant 1$ of the largest annual events. The distributions are based on the asymptotic joint distribution of the r largest values in a single sample, and the method of estimation is numerical maximum likelihood. The method is illustrated by an application to the sea levels in Venice, with particular attention to questions concerning trend and periodicity. Theoretical calculations are given for the asymptotic efficiency of the method.

Dupuis (1997) consider the robust estimation of parameters in such families of distributions. The estimation technique, which is based on optimal B-robust estimates, will assign weights to each observation and give estimates of the parameters based on the data which are well modeled by the distribution. Thus, observations which are not consistent with the proposed distribution can be identified and the validity of the model can be assessed. The method is illustrated on Venice sea level data

An and Pandey (2007) paper presents the statistical estimation of extreme wind speed using annually r largest order statistics (r-LOS) extracted from the time series of wind data. The method is based on a joint generalized extreme value distribution of r-LOS derived from the theory of Poisson process. The parameter estimation is based on the method of maximum likelihood. The hourly wind speed data collected at 30 stations in Ontario, Canada. The results of r-LOS method are compared with those obtained from the method of independent storms (MIS) and specifications of the Canadian National Building Code (CNBC-1995). The CNBC estimates are apparently conservative upper bound due to large sampling error associated with annual maxima analysis. Using the r-LOS method, the paper shows that the wind pressure data can be suitably modelled by the Gumbel distribution.

CHAPTER 3

METHODOLOGY

In this chapter, the research methodology will be presented in two parts following the research objectives.

3.1 Research methodology of propose and compare the efficiencies maximum penalized likelihood estimator

3.1.1 Maximum penalized likelihood estimation using penalty of Cole and Dixon

Penalized likelihood is a straightforward method of incorporating into an inference information that is supplementary to that provided by the data. A standard application is to non-parametric smoothing, in which the appropriate likelihood is balanced by a function that penalizes roughness. Coles and Dixon (1999) use a penalty function to provide the likelihood with the information that the value of k is smaller than 1, and that values close to 1 are less likely than smaller values. This application is closer in spirit to a Bayesian formulation in which structural information on k and h is formulated as a prior distribution. After some experimentation we restricted attention to penalty functions of the form and we call it penalty function Coles-Dixon (CD) follow as

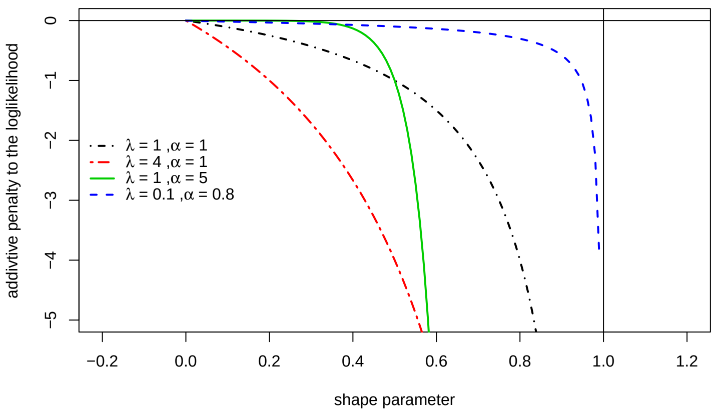


Figure 3.1: Logarithm of the penalty function for various of α and λ plotted against k

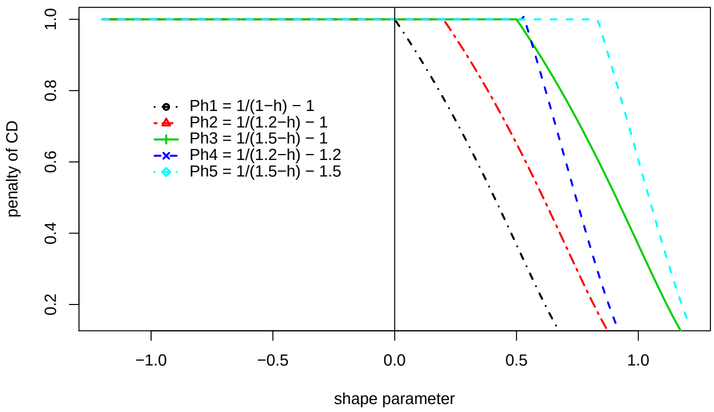


Figure 3.2: Penalty function for various of plotted against h

$$P(k) = \begin{cases} 1 & \text{if } k \leq 0, \\ \exp\left\{-\lambda\left(\dfrac{1}{1-k} - 1\right)^{\alpha}\right\} & \text{if } 0 < k < 1, \\ 0 & \text{if } k \geq 0, \end{cases}$$

$$P_1(h) = \begin{cases} 1 & \text{if } h \leq 0, \\ \exp\left\{-\lambda\left(\dfrac{1}{1-h} - 1\right)^{\alpha}\right\} & \text{if } 0 < h < 1, \\ 0 & \text{if } h \geq 0, \end{cases}$$

and we propose penalty function as $P_2(h), P_3(h), P_3(h), P_4(h)$ and $P_5(h)$

$$P_2(h) = \begin{cases} 1 & \text{if } h \leq 0.2, \\ \exp\left\{-\lambda\left(\dfrac{1}{1.2-h} - 1\right)^{\alpha}\right\} & \text{if } 0.2 < h < 1.2, \\ 0 & \text{if } h \geq 1.2, \end{cases}$$

$$P_3(h) = \begin{cases} 1 & \text{if } h \leq 0.5, \\ \exp\left\{-\lambda\left(\dfrac{1}{1.5-h} - 1\right)^{\alpha}\right\} & \text{if } 0.5 < h < 1.5, \\ 0 & \text{if } h \geq 1.5, \end{cases}$$

$$P_4(h) = \begin{cases} 1 & \text{if } h \leq 0.52, \\ \exp\left\{-\lambda\left(\dfrac{1}{1.2-h} - 1.2\right)^{\alpha}\right\} & \text{if } 0.52 < h < 1.2, \\ 0 & \text{if } h \geq 1.2, \end{cases}$$

$$P_5(h) = \begin{cases} 1 & \text{if } h \leq 0.84, \\ \exp\left\{-\lambda\left(\dfrac{1}{1.2-h} - 1.2\right)^{\alpha}\right\} & \text{if } 0.84 < h < 1.5, \\ 0 & \text{if } h \geq 1.5, \end{cases}$$

for a rang of non-negative value of α and λ. The corresponding penalized likelihood function is

$$L_{pen(j)}(\mu, \sigma, k, h) \;=\; L(\mu, \sigma, k, h) \times P(k) \times P_j(h), \quad j = 1, 2, 3, 4, 5 \qquad (3.1)$$

where L is the likelihood function define by Equation 2.14. Large values of α in the penalty function correspond to a more severe relative penalty for value of k which are large, but less than 1, while λ determines the overall weighting attached to the penalty. The penalty function (on a logarithm scale) is show in Figure 3.1 for various value of α and λ. After further experimentation, it was found that the combination $\alpha = \lambda = 1$ leads to a reasonable performance. In Figure 3.2 show Penalty function for various of plotted against h. The value of which maximizes Equation 3.1 is the maximum penalized likelihood estimator is call MPLE.CD1, MPLE.CD2, MPLE.CD3, MPLE.CD4 and MPLE.CD5 respectively. The penalized negative log likelihood function is

$$\begin{aligned} \tau_{pen(j)} &= \log L_{pen(j)}(\mu, \sigma, k, h) \\ &= m \log \sigma - (1-h) \sum_{i=1}^{m} \log F(x_i) - \frac{(1-k)}{k} \sum_{i=1}^{m} \log G_i \\ &\quad + C(k) + C_j(h), \end{aligned}$$

where $C(k) = \lambda\left(\dfrac{1}{1-k} - 1\right)^{\alpha}$, $C_1(h) = \lambda\left(\dfrac{1}{1-h} - 1\right)^{\alpha}$, $C_2(h) = \lambda\left(\dfrac{1}{1.2-h} - 1\right)^{\alpha}$,

$C_3(h) = \lambda\left(\frac{1}{1.5-h} - 1\right)^{\alpha}$,$C_4(h) = \lambda\left(\frac{1}{1.2-h} - 1.2\right)^{\alpha}$ and $C_5(h) = \lambda\left(\frac{1}{1.5-h} - 1.5\right)^{\alpha}$.

The first derivatives of $\tau_{pen(j)}$ with respect to μ, σ, k, h are show, respectively as

$$\frac{\partial \tau_{pen(j)}}{\partial \mu} = \sigma^{-1}\sum_{i=1}^{m} G_i^{-1}(k-1-H_i)$$

$$\frac{\partial \tau_{pen(j)}}{\partial \sigma} = \frac{m}{\sigma} + \sigma^{-2}\sum_{i=1}^{m} W_i(k-1-H_i)$$

$$\frac{\partial \tau_{pen(j)}}{\partial h} = -(h-1)h^{-1}\sum_{i=1}^{m} G_i^{(1/k)}F(x_i)^{-h} + h^{-1}\sum_{i=1}^{m}\ln F(x_i) - D_j(h)$$

$$\frac{\partial \tau_{pen(j)}}{\partial k} = k^{-1}\left[\sum_{i=1}^{m} H_i(k^{-1}\log\, G_i + \sigma^{-1}W_i) + \sum_{i=1}^{m} k^{-1}(\log\, G_i - (k-1)\sigma^{-1}W_i) - D(k)\right],$$

where $W_i = G_i^{-1}(x-\mu)$, $H_i = (h-1)G_i^{1/k}F(x_i)^{-h}$, $D(k) = \alpha\,[k(1-k)]^{-1}\,C(k)$ and $D_j(h) = \alpha\,[h(1-h)]^{-1}\,C_j(h)$.

3.1.2 Maximum penalized likelihood estimation using penalty of Martins and Stedinger

In year 2000 Martins and Stedinger (2000) propose penalty function as a reasonable penalty function for the shape parameter as show in Figure 3.3, the penalty function employed here is the beta distribution.

$$P(k) = \frac{(0.5+k)^{p-1}(0.5-k)^{q-1}}{B(p,q)}, \tag{3.2}$$

with $p = 6, q = 9$, where $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ we call MS penalty function. To propose MS penalty function show in Equation 3.4 to 3.5

$$P_1(h) = \frac{(0.5+h)^{p-1}(0.5-h)^{q-1}}{B(p,q)}, \tag{3.3}$$

$$P_2(h) = \frac{(1.2+h)^{p-1}(1.2-h)^{q-1}}{B_E(p,q)}, \tag{3.4}$$

$$P_3(h) = \frac{(1.2+h)^{p-1.2}(1.2-h)^{q-1.2}}{B_E(p,q)}. \tag{3.5}$$

Where $B_E(p,q) = \int\limits_{-1.2}^{1.2}(1.2+h)^{p-1}(1.2-h)^{q-1}dh$. Park (2005) recommened to use

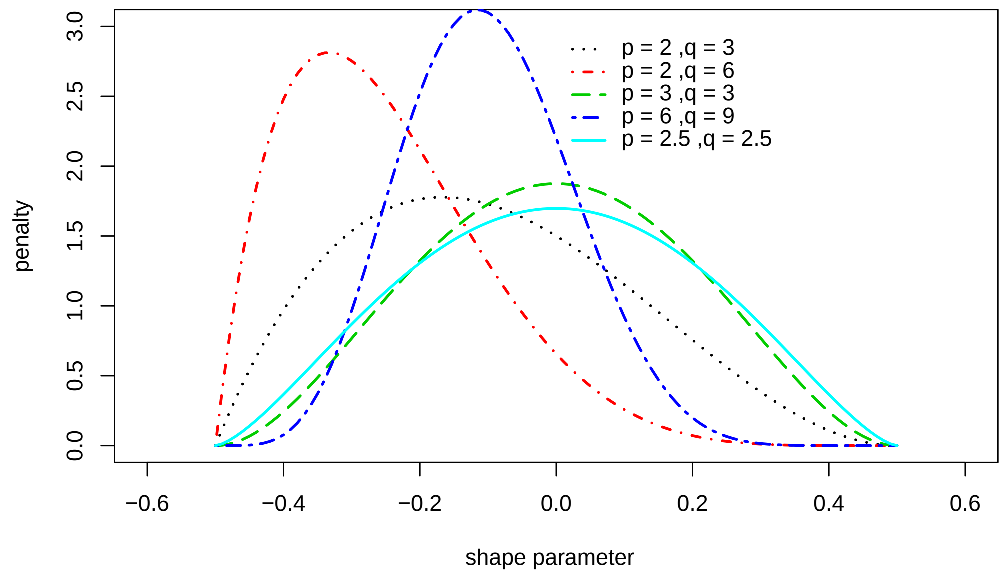


Figure 3.3: The penalty function for various of p and q plotted against k

$p = 2.5, q = 2.5$, We call MSP penalty function. The MSP penalty function show in Equation 3.6 and develop penalty function of MSP show in Equation 3.7 to Equation 3.8.

$$P_{p1}(h) = \frac{(0.5+h)^{p-1}(0.5-h)^{q-1}}{B(p,q)}, \tag{3.6}$$

$$P_{p2}(h) = \frac{(1.2+h)^{p-1}(1.2-h)^{q-1}}{B_E(p,q)}, \tag{3.7}$$

$$P_{p3}(h) = \frac{(1.2+h)^{p-1.2}(1.2-h)^{q-1.2}}{B_E(p,q)}. \tag{3.8}$$

The penalized likelihood function of MS penalty and MSP penalty function is

$$L_{pen(t)}(\mu,\sigma,k,h) = L(\mu,\sigma,k,h) \times P(k) \times P_t(h), \quad t = 1,2,3 \tag{3.9}$$

$$L_{pen(pt)}(\mu,\sigma,k,h) = L(\mu,\sigma,k,h) \times P(k) \times P_{pt}(h), \quad t = 1,2,3 \tag{3.10}$$

where L is the likelihood function defined by Equation 2.14. The penalized negative log likelihood function by use MS penalty function is

$$\tau_{pen(t)} = \log L_{pen(t)}(\mu,\sigma,k,h),$$

$$= m\log\sigma - (1-h)\sum_{i=1}^{m}\log F(x_i) - \frac{(1-k)}{k}\sum_{i=1}^{m}\log G_i$$
$$- E(k) - E_t(h) + 2\log B(p,q),$$

and penalized negative log likelihood function by use MSP penalty function is

$$\tau_{pen(pt)} = \log L_{pen(pt)}(\mu,\sigma,k,h),$$
$$= m\log\sigma - (1-h)\sum_{i=1}^{m}\log F(x_i) - \frac{(1-k)}{k}\sum_{i=1}^{m}\log G_i$$
$$- E(k) - E_t(h) + 2\log B(p,q),$$

with $E(k) = (p-1)\log(0.5+k) - (q-1)(0.5-k)$, $E_1(h) = (p-1)\log(0.5+h) - (q-1)(0.5-h)$, $E_2(h) = (p-1)\log(1.2+h) - (q-1)(1.2-h)$ and $E_3(h) = (p-1.2)\log(1.2+h) - (q-1.2)(1.2-h)$.

The first derivatives of $\tau_{pen(t)}$ and $\tau_{pen(pt)}$ with respect to μ,σ,k,h are show, respectively as

$$\frac{\partial\tau_{pen(t)}}{\partial\mu} = \sigma^{-1}\sum_{i=1}^{m}G_i^{-1}(k-1-H_i)$$
$$\frac{\partial\tau_{pen(t)}}{\partial\sigma} = \frac{m}{\sigma} + \sigma^{-2}\sum_{i=1}^{m}W_i(k-1-H_i)$$
$$\frac{\partial\tau_{pen(t)}}{\partial h} = -(h-1)h^{-1}\sum_{i=1}^{m}G_i^{(1/k)}F(x_i)^{-h} + h^{-1}\sum_{i=1}^{m}\ln F(x_i) + M(h)$$
$$\frac{\partial\tau_{pen(t)}}{\partial k} = k^{-1}\sum_{i=1}^{n}\left[H_i\left(k^{-1}\ln G_i + \sigma^{-1}W_i\right) + \left(k^{-1}\ln G_i - (k-1)\sigma^{-1}W_i\right)\right] + M(k),$$

and

$$\frac{\partial\tau_{pen(pt)}}{\partial\mu} = \sigma^{-1}\sum_{i=1}^{m}G_i^{-1}(k-1-H_i)$$
$$\frac{\partial\tau_{pen(pt)}}{\partial\sigma} = \frac{m}{\sigma} + \sigma^{-2}\sum_{i=1}^{m}W_i(k-1-H_i)$$
$$\frac{\partial\tau_{pen(pt)}}{\partial h} = -(h-1)h^{-1}\sum_{i=1}^{m}G_i^{(1/k)}F(x_i)^{-h} + h^{-1}\sum_{i=1}^{m}\ln F(x_i) + M(h)$$
$$\frac{\partial\tau_{pen(pt)}}{\partial k} = k^{-1}\sum_{i=1}^{n}\left[H_i\left(k^{-1}\ln G_i + \sigma^{-1}W_i\right) + \left(k^{-1}\ln G_i - (k-1)\sigma^{-1}W_i\right)\right] + M(k),$$

where $W_i = G_i^{-1}(x-\mu)$, $H_i = (h-1)G_i^{1/k}F(x_i)^{-h}$, $M(k) = \frac{(p-1)}{(0.5+k)} - \frac{(q-1)}{(0.5-k)}$, $M_1(h) = \frac{(p-1)}{(0.5+h)} - \frac{(q-1)}{(0.5-h)}$, $M_2(h) = \frac{(p-1)}{(1.2+h)} - \frac{(q-1)}{(1.2-h)}$ and $M_3(h) = \frac{(p-1.2)}{(1.2+h)} - \frac{(q-1.2)}{(1.2-h)}$.

The maximum penalized likelihood estimator (MPLE.MS1, MPLE.MS2, MPLE.MS3, MPLE.MSP1, MPLE.MSP2 and MPLE.MSP3) of (μ, σ, k, h) can be identified by maximizing Equation 3.9 and 3.10 by replacement of $P(k)$, $P_j(h)$ and $P_{pj}(h)$ is the beta distribution in Equation 3.2 to Equation 3.8 .

3.1.3 Step of Simulation study

The objective of this part is to construct maximum penalized likelihood estimator of K4D. The investigation of the accuracy of estimations method by the Monte-Carlo method following

Step 1. fix parameter $\mu = 0$, $\sigma = 1$, value of $-0.4 \leq k \leq 0.4$ and $-1.2 \leq h \leq 1.2$.

Step 2. The probability of quantile function, $x(F) = 0.90, 0.95, 0.99, 0.995,\ 0.999$.

Step 3. The sample size (n) are $30, 50$ and 100.

Step 4. Use these fixed parameter μ, σ and combination shape parameters k and h for sample size in step 1 and 2 to generate a random variable from K4D with parameter μ, σ, k and h.

Step 5. Estimated parameter $\hat{\mu}$, $\hat{\sigma}, \hat{k}$ and $\hat{h}$ by MLE, MPLE.CD1, MPLE.CD2, MPLE.CD3, MPLE.CD4, MPLE.CD5, MPLE.MS1, MPLE.MS2,MPLE.MS3, MPLE.MSP1, MPLE.MSP2, MPLE.MSP3 and L-moments method.

Step 6. Repeat step 3 to 4, with in this research we repeat for 1,000 times and calculate RBIAS and RRMSE of the quantiles estimator of fix probability follow step 3. After that consider the best estimator by smallest RBIAS and RRMSE value.

3.1.4 Step of applications with hydrology data

The objective of this part is to construct maximum penalized likelihood estimator of K4D. The investigation of the accuracy of estimations method by applications with two hydrology data sets. The annual maximum rainfall data (in mm) was measured at Pattaya of Thailand, recorded over the period of 1984 to 2014 for 31 year and The annual maximum temperature data was measured at Surin of Thailand, recorded over the period of 1990 to 2018 for 29 year.

Step 1. Prepare the annual maximum rainfall and annual maximum temperature data.

Step 2. Estimated parameter $\hat{\mu}, \hat{\sigma}, \hat{k}$ and $\hat{h}$ by MLE, MPLE.CD1, MPLE.CD2, MPLE.CD3, MPLE.CD4, MPLE.CD5, MPLE.MS1, MPLE.MS2,MPLE.MS3, MPLE.MSP1, MPLE.MSP2, MPLE.MSP3 and L-moments method.

Step 3. Calculate MPAE and Anderson-Darling test for goodness of fit is based on the test statistics for select the best model by consider the p-value > 0.05 and smallest MPAE value.

Step 4. Fitted model base on histogram with fitted density and Q-Q plot for rainfall data and temperature data.

Step 5. Find a 95% confidence interval for 20-year return level used profile likelihood method for rainfall data and temperature data.

3.2 To propose the r-largest order statistics for four parameter kappa distribution (r-K4D)

The objective of this part is to construct develop the r-largest order statistics for four parameter kappa distribution on the basic concept of the r-largest order statistics in GEV.

Step 1. To study the r-largest order statistics model.

Step 2. Develop the r-largest order Studying of the r-largest order statistics model for four parameter kappa distribution.

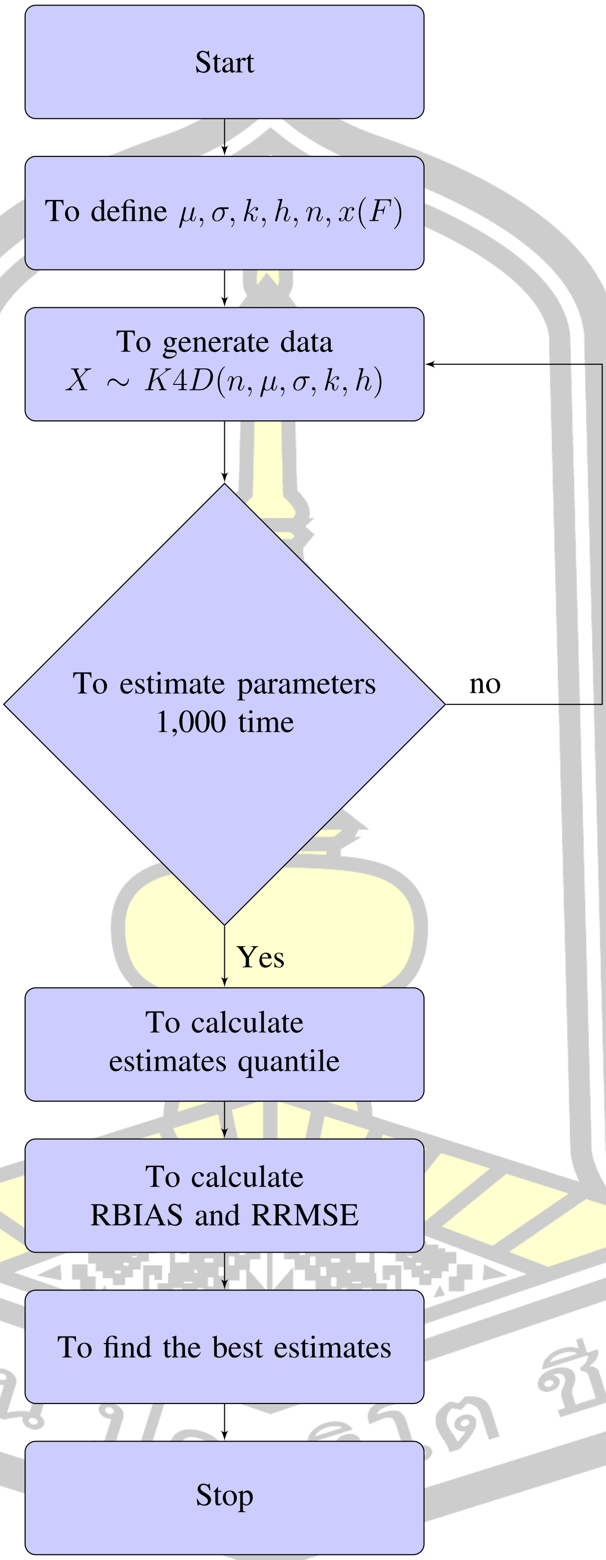


Figure 3.4: Flow work for Simulation study

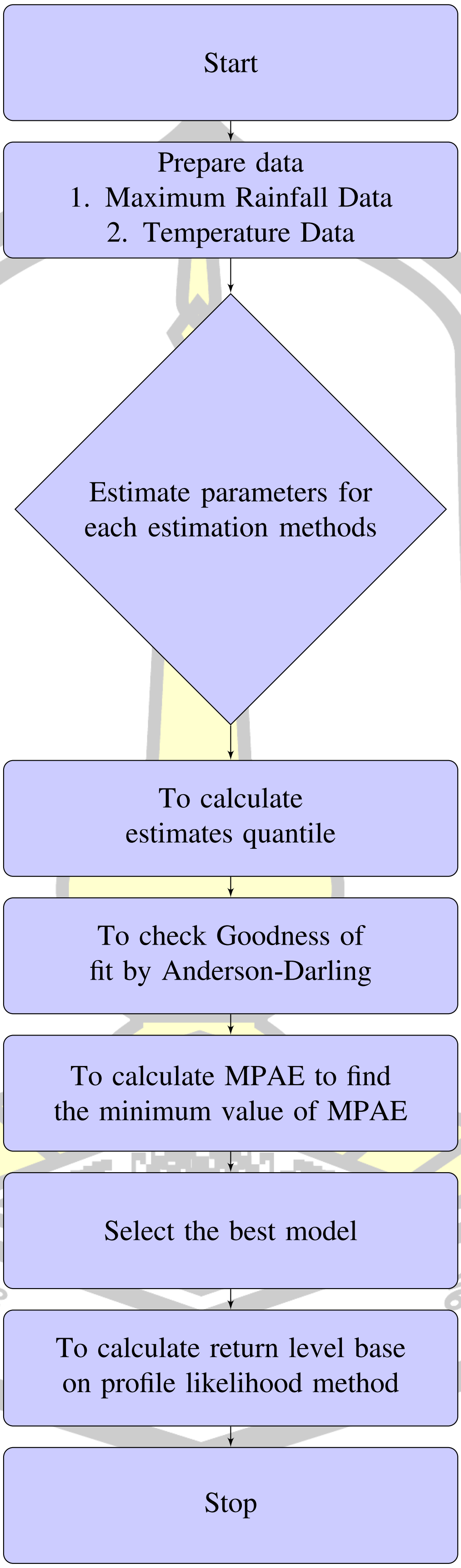


Figure 3.5: Flow work for applications with hydrology data

CHAPTER 4

RESULTS AND DISCUSSION

In this chapter, we present the analytical and numerical results in two parts following the research objectives.

4.1 Result of compare the performance of maximum penalized likelihood estimator (MPLE) with maximum likelihood estimator and L-moment estimator for four parameter kappa distribution

4.1.1 Simulation Study

The Monte-Carlo simulations have been performed to investigate the effect of using MLE, MPLE.CD1, MPLE.CD2, MPLE.CD3, MPLE.CD4, MPLE.CD5, MPLE.MS1, MPLE.MS2, MPLE.MS3, MPLE.MSP1, MPLE.MSP2, MPLE.MSP3 and LM on heavy-quantile estimation $x(F = 0.90)$, $x(F = 0.95)$, $x(F = 0.99)$, $x(F = 0.995)$ and $x(F = 0.999)$ for sample sizes 30, 50 and 100. We replicate the data for each sample size for $1,000$ simulation runs. A simulation study for samples are used to obtain RBIAS and RRMSE of quantile estimators. The results are summarize in the following.

Section (i) Comparisons the performance of maximum penalized likelihood estimator (MPLE) with maximum likelihood estimator and L-moment estimator for four parameter kappa distribution for n=30

Section (ii) Comparisons the performance of maximum penalized likelihood estimator (MPLE) with maximum likelihood estimator and L-moment estimator for four parameter kappa distribution for n=50

Section (iii) Comparisons the performance of maximum penalized likelihood estimator (MPLE) with maximum likelihood estimator and L-moment estimator for four parameter kappa distribution for n=100

Section (iv) Comparisons the performance of maximum penalized likelihood estimator (MPLE) with maximum likelihood estimator and L-moment estimator for four parameter kappa distribution

(i) Comparisons the performance of maximum penalized likelihood estimator (MPLE) with maximum likelihood estimator and L-moment estimator for four parameter kappa distribution for n=30

In Table 4.1 to Table 4.4 illustrate the RBIAS and RRMSE of the MLE parameter estimation method. The MPLE.CD5 was found the most efficient method among MPLE.CD1 to MPLE.CD5. The MPLE.MS3 was found the most efficient method among MPLE.MS1 to MPLE.MS3 and MPLE.MSP1 to MPLE.MSP3 when the k value was negative and close to zero. The MPLE.MS3 method was found the most efficient method among MPLE.MS1 to MPLE.MS3 and among MPLE.MSP1 to MPLE.MSP3 when the k value was positive (see Appendix A) and the LM method in all quantiles. In case that the two shape parameters were $-0.4 \leqslant k \leqslant 0.4$ and h was between -1.2 to 1.2, it was found that the quantiles and parameters were different, resulting in the difference in the RBIAS and RRMSE of parameter estimation methods

Figure 4.1 to 4.10 indicate the RBIAS and RRMSE of all parameters and quantiles studied. Figures 4.1 to 4.4, Figures 4.5 to 4.6, and Figures 4.7 to 4.10 illustrate the RBIAS and RRMSE when the k value was negative, close to 0 and positive, respectively. Concerning the RBIAS when the h value was between -1.2 to 1.2;

- In the MLE and MPLE.CD5 methods, there was a decrease in the RBIAS when the h value was negative [-1.2, -0.2], but when the h value was positive [0.2,1.2], the RBIAS would immediately increase;
- The RBIAS decreased in the LM method;
- The MPLE.MSP3 method had an increase in the RBIAS when the quantile was low ($x(F = 0.90, 0.95)$). When the quantile was higher than 0.95, the RBIAS would decrease, except when the k value was between 0.2 to 0.4, the RBIAS value would increase; and
- The MPLE.MS3 method had an increase in the RBIAS when the k value was negative, but when the k value was close to 0, the RBIAS would decrease when

the h value was negative in a case that the quantile was 0.90 and 0.95. The RBIAS would continuously increase when the h value was between -1.2 to 1.2 in case that the quantile was 0.99 and 0.999. However, considering the RRMSE, it was found that all parameter estimation methods experienced an increase in RRMSE when the h value was between -1.2 to 1.2.

Figures 4.1 to 4.10, in case that the k and h values were negative and the quantile was between 0.99 to 0.999, illustrate that the MPLE.CD5 method had the least RBIAS, compared to that of the MLE, MPLE.MS3, MPLE.MSP3 and LM methods, except when k was -0.1, the MPLE.MS3 method had the least RBIAS. In case that the k value was positive and the h value was negative, the MPLE.CD5 and MPLE.MSP3 methods had the least RBIAS, respectively in case that the quantile is 0.99 to 0.999. However, considering the RRMSE, there was slightly difference in the RRMSE in the MPLE.MS3 and MPLE.MSP3 methods. For the k value that was negative and close to 0, the MPLE.MS3 method had the least RRMSE, compared to that of the LM and MLE methods. For the k value that was positive, the MPLE.MSP3 method had the least RRMSE, compared to that of all methods in the quantiles studied.

Table 4.1: The RBIAS and RRMSE value of the estimates of K4D obtained by four estimation methods for $k = -0.4$, $-1.2 \leq h \leq 1.2$ and $n = 30$.

k	h	$x(F)$	RBIAS(RRMSE)			
			MLE	MPLE.CD5	MPLE.MS3	LM
		0.90	**0.0160**(0.1374)	0.0272(0.1491)	-0.1331(**0.0867**)	0.0212(0.1268)
		0.95	-0.0290(0.1714)	0.0375(0.2153)	-0.1861(**0.0847**)	**-0.0276**(0.1383)
	-1.2	0.99	-0.1567(0.5200)	**-0.0168**(0.6218)	-0.3446(**0.1572**)	-0.2319(0.2360)
		0.995	-0.1969(0.5147)	**0.0115**(0.9183)	-0.4000(**0.1975**)	-0.2802(0.3747)
		0.999	-0.2055(2.1522)	**0.1334**(3.6886)	-0.5278(**0.3106**)	-0.3973(0.5667)
		0.90	**0.0181**(0.1415)	0.0333(0.1747)	-0.1363(**0.0854**)	0.0227(0.1280)
		0.95	-0.0352(0.1817)	**0.0197**(0.2795)	-0.1855(**0.0896**)	-0.0352(0.1423)
	-1	0.99	-0.1378(0.4973)	**-0.0042**(0.5862)	-0.3418(**0.1557**)	-0.2001(0.2520)
		0.995	-0.1763(0.5524)	**0.0368**(1.0165)	-0.3970(**0.1941**)	-0.2755(0.2918)
		0.999	**-0.1443**(1.4343)	0.1469(3.3738)	-0.5264(**0.3074**)	0.3866(0.5395)
		0.90	**0.0314**(0.1391)	0.0390(0.1537)	-0.1268(**0.0785**)	0.0408(0.1340)
		0.95	-0.0297(0.1884)	**0.0222**(0.2597)	-0.1918(**0.0925**)	-0.0305(0.1576)
	-0.8	0.99	-0.1259(0.4004)	**-0.0182**(0.6252)	-0.3439(**0.1587**)	-0.1961(0.2449)
		0.995	-0.1668(0.6355)	**0.0017**(0.8767)	-0.4113(**0.2047**)	-0.2846(0.3018)
		0.999	-0.1898(1.3305)	**0.0775**(2.1393)	-0.5334(**0.3122**)	-0.3917(0.5793)
		0.90	**0.0121**(0.1321)	0.0219(0.1482)	-0.1390(**0.0849**)	0.0207(0.1248)
		0.95	-0.0406(0.1726)	**-0.0016**(0.1940)	-0.2081(**0.0904**)	-0.0451(0.1340)
-0.4	-0.6	0.99	-0.1530(0.3476)	**-0.0133**(0.4752)	-0.3511(**0.1607**)	-0.2033(0.2459)
		0.995	-0.1584(0.6452)	**0.0326**(1.0422)	-0.4115(**0.2043**)	-0.2561(0.3873)
		0.999	**-0.1130**(1.6120)	0.1951(3.4144)	-0.5429(**0.3227**)	-0.3945(0.4848)
		0.9	**0.0290**(0.1398)	0.0371(0.1542)	-0.1372(**0.0800**)	0.0348(0.1286)
		0.95	-0.0264(0.1949)	0.0304(0.2458)	-0.2041(**0.0895**)	**-0.0246**(0.1403)
	-0.4	0.99	-0.1621(0.4520)	**-0.0213**(0.5711)	-0.3739(**0.1726**)	-0.2315(0.2305)
		0.995	-0.1632(0.5144)	**-0.0082**(0.7851)	-0.4292(**0.2140**)	-0.2787(0.2758)
		0.999	**-0.0381**(2.3713)	0.2427(3.1172)	-0.5418(**0.3193**)	-0.3441(0.4926)
		0.9	**0.0138**(0.1194)	0.0251(0.1295)	-0.1551(**0.0780**)	0.0215(0.1097)
		0.95	-0.0371(0.1746)	**0.0116**(0.2517)	-0.2279(**0.0990**)	-0.0437(0.1366)
	-0.2	0.99	-0.0888(0.4743)	**-0.0542**(0.6204)	-0.3726(**0.1753**)	-0.1984(0.2397)
		0.995	-0.1354(0.6943)	**0.0357**(0.8431)	-0.4360(**0.2195**)	-0.2754(0.3441)
		0.999	**-0.0351**(1.7359)	0.3247(10.1177)	-0.5463(**0.3257**)	-0.3375(0.5501)
		0.9	**-0.0060**(0.1119)	0.0120(0.1239)	-0.1775(**0.0812**)	0.0089(0.1049)
		0.95	**-0.0247**(0.1638)	0.0361(0.2302)	-0.2391(**0.0957**)	-0.0331(0.1325)
	-0.01	0.99	**-0.0463**(0.8155)	0.1810(2.5584)	-0.3737(**0.1723**)	-0.1749(0.2311)
		0.995	**-0.0489**(0.9775)	0.1895(1.5338)	-0.4415(**0.2250**)	-0.2249(0.3967)
		0.999	**-0.0324**(3.7587)	0.4884(6.7076)	-0.5570(**0.3335**)	-0.3128(0.8815)

Table 4.2: The RBIAS and RRMSE value of the estimates of K4D obtained by four estimation methods for $k = -0.4$, $-1.2 \leq h \leq 1.2$ and $n = 30$. (Cont.)

k	h	$x(F)$	RBIAS(RRMSE)			
			MLE	MPLE.CD5	MPLE.MS3	LM
		0.90	**0.0034**(0.1165)	0.0226(0.1609)	-0.1760(**0.0784**)	0.0157(0.1086)
		0.95	-0.0352(0.1722)	**-0.0231**(0.2675)	-0.2469(**0.1049**)	-0.0411(0.1362)
	0.01	0.99	**-0.1004**(0.4832)	**0.1116**(1.5378)	-0.3871(**0.1799**)	-0.1882(0.2689)
		0.995	**-0.0678**(0.7549)	0.1552(2.9436)	-0.4390(**0.2247**)	-0.2323(0.3447)
		0.999	**0.0693**(1.9988)	0.5630(2272.3560)	-0.5480(**0.3264**)	-0.2768(0.7026)
		0.90	**0.0252**(0.1490)	0.0543(0.1968)	-0.1827(**0.0828**)	0.0320(0.1258)
		0.95	**0.0000**(0.1887)	0.0740(0.2677)	-0.2558(**0.1051**)	-0.0262(0.1357)
	0.2	0.99	**-0.0378**(0.6034)	0.2060(1.7128)	-0.3955(**0.1859**)	-0.1843(0.2463)
		0.995	**0.0147**(1.0045)	0.3058(2.3474)	-0.4473(**0.2284**)	-0.2085(0.3216)
		0.999	**0.2226**(8.9399)	0.7685(19.1099)	-0.5613(**0.3390**)	-0.2845(0.7733)
		0.90	**0.0095**(0.1234)	0.0455(0.1614)	-0.2126(**0.0879**)	0.0111(0.1027)
		0.95	**-0.0097**(0.1743)	0.0874(0.2710)	-0.2720(**0.1146**)	-0.0468(0.1221)
	0.4	0.99	**0.0214**(0.6784)	0.3112(3.8617)	-0.4194(**0.2036**)	-0.1725(0.2303)
		0.995	0.1960(1.7973)	0.6700(5.5061)	-0.4511(**0.2309**)	**-0.1682**(0.3648)
		0.999	0.3894(10.3758)	1.1748(18.4287)	-0.5619(**0.3358**)	**-0.2724**(0.6794)
		0.90	**0.0173**(0.1467)	0.0738(0.2042)	-0.2158(**0.0920**)	-0.0190(0.1118)
		0.95	**0.0045**(0.1820)	0.1295(0.3648)	-0.2968(0.1240)	-0.0458(**0.1215**)
	0.6	0.99	**0.2021**(1.3033)	0.6562(6.2293)	-0.4200(**0.2037**)	-0.1347(0.2425)
		0.995	0.3724(6.2066)	0.9469(10.6333)	-0.4743(**0.2514**)	**-0.1840**(0.3899)
-0.4		0.999	0.6068(12.5262)	2.0637(48.5011)	-0.5761(**0.3523**)	**-0.2486**(0.7694)
		0.90	0.0536(0.3057)	0.1167(0.3362)	-0.2362(**0.0937**)	**0.0254**(0.1128)
		0.95	0.0865(0.4475)	0.2497(0.7321)	-0.3033(**0.1291**)	**-0.0213**(0.1331)
	0.8	0.99	0.2160(1.2177)	0.7548(5.4607)	-0.4469(**0.2243**)	**-0.1572**(0.2456)
		0.995	0.4152(2.9224)	1.4207(76.4584)	-0.4897(**0.2614**)	**-0.1672**(0.3540)
		0.999	1.3415(34.9599)	3.1923(108.8934)	-0.5911(**0.3671**)	**-0.2404**(0.6737)
		0.9	0.0619(0.2338)	0.1589(0.8592)	-0.2463(**0.1001**)	**0.0272**(0.1121)
		0.95	0.1217(0.4935)	0.2642(0.7413)	-0.3367(0.1457)	**-0.0444(0.1253)**
	1.0	0.99	0.6077(4.8476)	1.2953(12.0419)	-0.4660(**0.2398**)	**-0.1318**(0.2552)
		0.995	1.3454(39.5469)	2.5971(90.2281)	-0.5063(**0.2773**)	**-0.1702**(0.3622)
		0.999	4.6422(1000.4180)	8.1629(1490.5780)	-0.5928(**0.3686**)	**-0.1944**(0.8010)
		0.9	0.0786(0.2098)	0.2053(0.4323)	-0.2656(0.1062)	**0.0238(0.1028)**
		0.95	0.2371(3.5271)	0.5294(6.1596)	-0.3508(0.1527)	**-0.0329(0.1318)**
	1.2	0.99	0.7343(8.0170)	1.9958(38.1632)	-0.4771(0.2499)	**-0.1312(0.2471)**
		0.995	1.6581(304.3207)	3.8068(412.1524)	-0.5245(**0.2939**)	**-0.1652**(0.3611)
		0.999	8.8344(7205.491)	15.8362(8786.9190)	-0.6021(**0.3793**)	**-0.1519**(0.9439)

Table 4.3: The RBIAS and RRMSE value of the estimates of K4D obtained by four estimation methods for $k = 0.4$, $-1.2 \leq h \leq 1.2$ and $n = 30$.

k	h	$x(F)$	RBIAS(RRMSE)			
			MLE	MPLE.CD5	MPLE.MSP3	LM
0.4	-1.2	0.90	0.0362(0.0277)	-0.0542(0.0260)	-0.0345(**0.0226**)	**0.0223**(0.0239)
		0.95	-0.0286(0.0191)	-0.0669(0.0217)	**-0.0030**(**0.0168**)	-0.0333(0.0212)
		0.99	-0.1073(0.0250)	-0.0518(**0.0190**)	**0.0487**(0.0197)	-0.1036(0.0331)
		0.995	-0.1264(0.0317)	**-0.0395**(**0.0221**)	0.0623(0.0241)	-0.1261(0.0425)
		0.999	-0.1537(0.0422)	**0.0126**(0.0347)	0.0817(**0.0289**)	-0.1523(0.0519)
	-1	0.90	0.0330(0.0259)	-0.0549(0.0250)	-0.0367(**0.0222**)	**0.0216**(0.0230)
		0.95	-0.0261(0.0168)	-0.0619(0.0187)	**-0.0026**(**0.0152**)	**-0.0248**(0.0180)
		0.99	-0.1045(0.0250)	**-0.0448**(0.0201)	0.0504(**0.0200**)	-0.0783(0.0288)
		0.995	-0.1323(0.0319)	**-0.0367**(0.0238)	0.0561(**0.0216**)	-0.1041(0.0353)
		0.999	0.0292(0.0258)	**-0.0488**(0.0243)	-0.0354(**0.0216**)	0.0192(0.0228)
	-0.8	0.90	0.0292(0.0258)	-0.0488(0.0243)	-0.0354(**0.0216**)	**0.0192**(0.0228)
		0.95	-0.0187(0.0159)	-0.0508(0.0177)	**0.0003**(**0.0148**)	-0.0106(0.0171)
		0.99	-0.1129(0.0277)	-0.0444(0.0230)	**0.0371**(**0.0211**)	-0.0684(0.0305)
		0.995	-0.1301(0.0315)	**-0.0303**(0.0234)	0.0555(**0.0226**)	-0.0858(0.0342)
		0.999	-0.1492(0.0480)	**0.0330**(0.0493)	0.0815(**0.0317**)	-0.0917(0.0416)
	-0.6	0.90	0.0308(0.0212)	-0.0385(0.0192)	-0.0313(**0.0167**)	**0.0219**(0.0179)
		0.95	-0.0205(0.0154)	-0.0448(0.0170)	**0.0014**(**0.0146**)	**-0.0024**(0.0163)
		0.99	-0.1097(0.0265)	**-0.0387**(0.0228)	0.0401(**0.0201**)	-0.0560(0.0283)
		0.995	-0.1253(0.0302)	**-0.0120**(0.0276)	0.0620(**0.0234**)	-0.0544(0.0316)
		0.999	-0.1484(0.0473)	**0.0494**(0.0552)	0.0880(**0.0312**)	-0.0864(0.0432)
	0.4	0.9	0.0281(0.0223)	-0.0332(0.0203)	-0.0350(**0.0182**)	**0.0163**(0.0192)
		0.95	-0.0296(0.0166)	-0.0481(0.0184)	-0.0092(**0.0154**)	-0.0071(0.0168)
		0.99	-0.1061(0.0267)	**-0.0189**(0.0279)	0.0450(**0.0206**)	-0.0372(0.0271)
		0.995	-0.1223(0.0311)	**0.0015**(0.0274)	0.0598(**0.0212**)	-0.0458(0.0289)
		0.999	-0.1262(0.0480)	0.0970(0.0727)	0.1019(**0.0342**)	**-0.0445**(0.0442)
	-0.2	0.9	0.0242(0.0209)	-0.0278(0.0180)	-0.0332(**0.0158**)	**0.0156**(0.0170)
		0.95	-0.0289(0.0151)	-0.0388(0.0173)	-0.0071(**0.0140**)	**-0.0019**(0.0156)
		0.99	-0.1035(0.0253)	**-0.0080**(0.0253)	0.0437(**0.0201**)	-0.0267(0.0261)
		0.995	-0.1120(0.0286)	0.0278(0.0363)	0.0709(**0.0246**)	**-0.0228**(0.0315)
		0.999	-0.1257(0.0454)	0.1376(0.1011)	0.1132(**0.0376**)	**-0.0315**(0.0454)
	-0.01	0.9	0.0160(0.0195)	-0.0282(0.0182)	-0.0367(**0.0160**)	**0.0108**(0.0171)
		0.95	-0.0357(0.0162)	-0.0424(0.0175)	-0.0153(**0.0143**)	**-0.0077**(0.0155)
		0.99	-0.0894(0.0225)	0.0210(0.0296)	0.0570(**0.0199**)	**-0.0084**(0.0244)
		0.995	-0.1096(0.0310)	0.0602(0.0471)	0.0759(**0.0252**)	**-0.0080**(0.0322)
		0.999	-0.1074(0.0480)	0.2156(0.1598)	0.1308(**0.0425**)	**0.0046**(0.0514)

Table 4.4: The RBIAS and RRMSE value of the estimates of K4D obtained by four estimation methods for $k = 0.4$, $-1.2 \leq h \leq 1.2$ and $n = 30$. (Cont.)

k	h	$x(F)$	RBIAS(RRMSE)			
			MLE	MPLE.CD5	MPLE.MS3	LM
		0.90	0.0174(0.0202)	-0.0277(0.0183)	-0.0354(**0.0160**)	**0.0128**(0.0177)
		0.95	-0.0414(0.0160)	-0.0475(0.0174)	-0.0209(**0.0141**)	**-0.0138**(0.0153)
		0.99	-0.0910(0.0238)	0.0221(0.0325)	0.0536(**0.0201**)	**-0.0101**(0.0237)
		0.995	-0.1095(0.0313)	0.0626(0.0468)	0.0742(**0.0260**)	**-0.0132**(0.0310)
		0.999	-0.1095(0.0475)	0.2127(0.1514)	0.1301(**0.0420**)	**0.0090**(0.0546)
		0.90	0.0086(0.0175)	-0.0310(0.0170)	-0.0403(**0.0152**)	**0.0079**(0.0162)
		0.95	-0.0381(0.0158)	-0.0346(0.0180)	-0.0176(**0.0140**)	**-0.0093**(0.0149)
	0.2	0.99	-0.0859(0.0242)	0.0540(0.0423)	0.0553(**0.0209**)	**-0.0080**(0.0255)
		0.995	-0.1051(0.0303)	0.1064(0.0692)	0.0820(**0.0261**)	**-0.0065**(0.0337)
		0.999	-0.0975(0.0574)	0.3101(0.3042)	0.1430(**0.0463**)	**0.0136**(0.0607)
		0.90	0.0097(0.0178)	-0.0457(0.0183)	-0.0526(**0.0164**)	**-0.0072**(0.0164)
		0.95	-0.0353(0.0132)	**-0.0161**(0.0170)	-0.0125(**0.0124**)	**-0.0033**(0.0129)
	0.4	0.99	-0.0825(0.0226)	0.1068(0.0689)	0.0654(**0.0216**)	**0.0006**(0.0237)
		0.995	-0.0931(0.0326)	0.1773(0.1284)	0.0920(**0.0287**)	**0.0076**(0.0344)
		0.999	-0.0765(0.725)	0.4470(0.5275)	0.1531(**0.0507**)	**0.0262**(0.0634)
		0.90	-0.0234(0.0156)	-0.0399(0.0179)	-0.04997(**0.0145**)	**-0.0014**(0.0149)
		0.95	-0.0400(0.0142)	0.0019(0.0213)	-0.0105(**0.0127**)	**-0.0019**(0.0137)
	0.6	0.99	0.0744(0.0221)	0.1567(0.0918)	0.0684(**0.0213**)	**0.0045**(0.0228)
		0.995	-0.0789(0.0345)	0.2524(0.2021)	0.0996(**0.0304**)	**0.0081**(0.0356)
0.4		0.999	-0.0603(0.0871)	0.6042(2.6955)	0.1679(**0.0630**)	**0.0322**(0.0716)
		0.90	-0.0342(0.0146)	-0.0294(0.0176)	-0.0464(**0.0142**)	**0.0042**(0.0155)
		0.95	-0.0451(0.0161)	0.0101(0.0276)	-0.0158(**0.0126**)	**-0.0036**(0.0136)
	0.8	0.99	-0.0436(0.0420)	0.2188(0.1912)	0.0701(**0.0220**)	**0.0057**(0.0243)
		0.995	-0.0523(0.0408)	0.3631(0.4967)	0.1058(**0.0316**)	**0.0153**(0.0353)
		0.999	**0.0357**(0.4373)	0.8096(2.6470)	0.1801(**0.0753**)	0.0494(0.0839)
		0.9	-0.0466(0.0182)	-0.0223(0.0232)	-0.0556(**0.0153**)	**0.0003**(0.0157)
		0.95	-0.0390(0.0225)	0.0317(0.0451)	-0.0243(**0.0123**)	**-0.0080**(0.0131)
	1.0	0.99	0.0306(0.2079)	0.4195(2.6772)	0.0715(**0.0244**)	**0.0025**(0.0260)
		0.995	0.1110(1.5891)	0.5348(2.3242)	0.1140(**0.0377**)	**0.0171**(0.0379)
		0.999	0.2252(1.9934)	1.3621(22.5272)	0.2030(0.0906)	**0.0469**(**0.0846**)
		0.9	-0.0199(**0.0103**)	-0.0063(0.0244)	-0.0604(0.0148)	**-0.0016**(0.0143)
		0.95	-0.0161(0.0155)	0.0795(0.0765)	-0.0270(**0.0123**)	**-0.0044**(0.0127)
	1.2	0.99	0.0548(0.4030)	0.4973(1.3643)	0.0789(0.0256)	**0.0033**(**0.0236**)
		0.995	0.0621(0.2660)	0.7159(3.3050)	0.1279(0.0443)	**0.0152**(**0.0349**)
		0.999	0.1986(2.1758)	2.4818(94.5909)	0.2554(0.1326)	**0.0627**(**0.0886**)

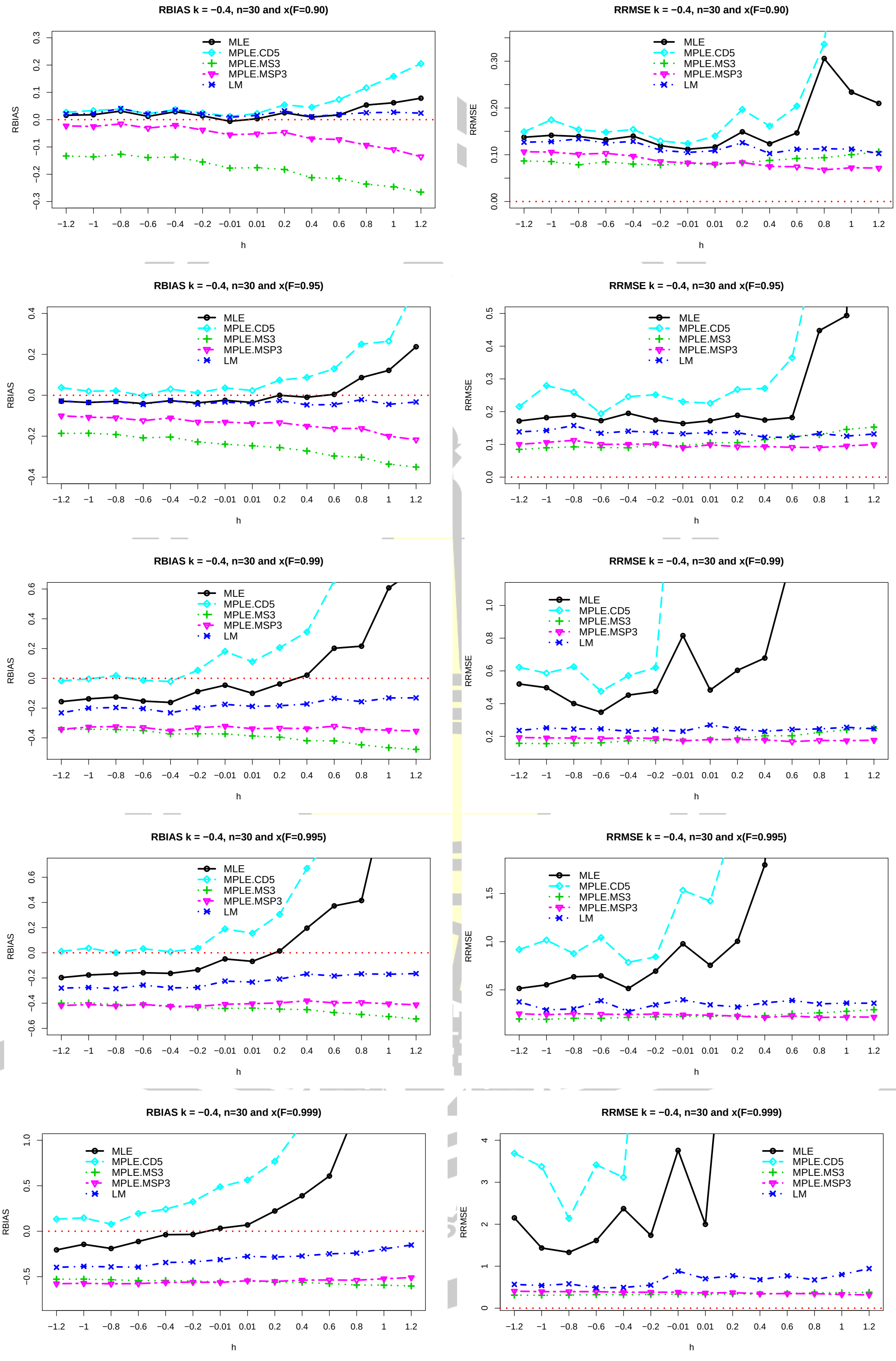


Figure 4.1: RBIAS and RRMSE of the all quantile estimators of MLE, MPLE.CD, MPLE.MS3, MPLE.MSP3 and LM for value of $k = -0.4$ and sample size $n = 30$.

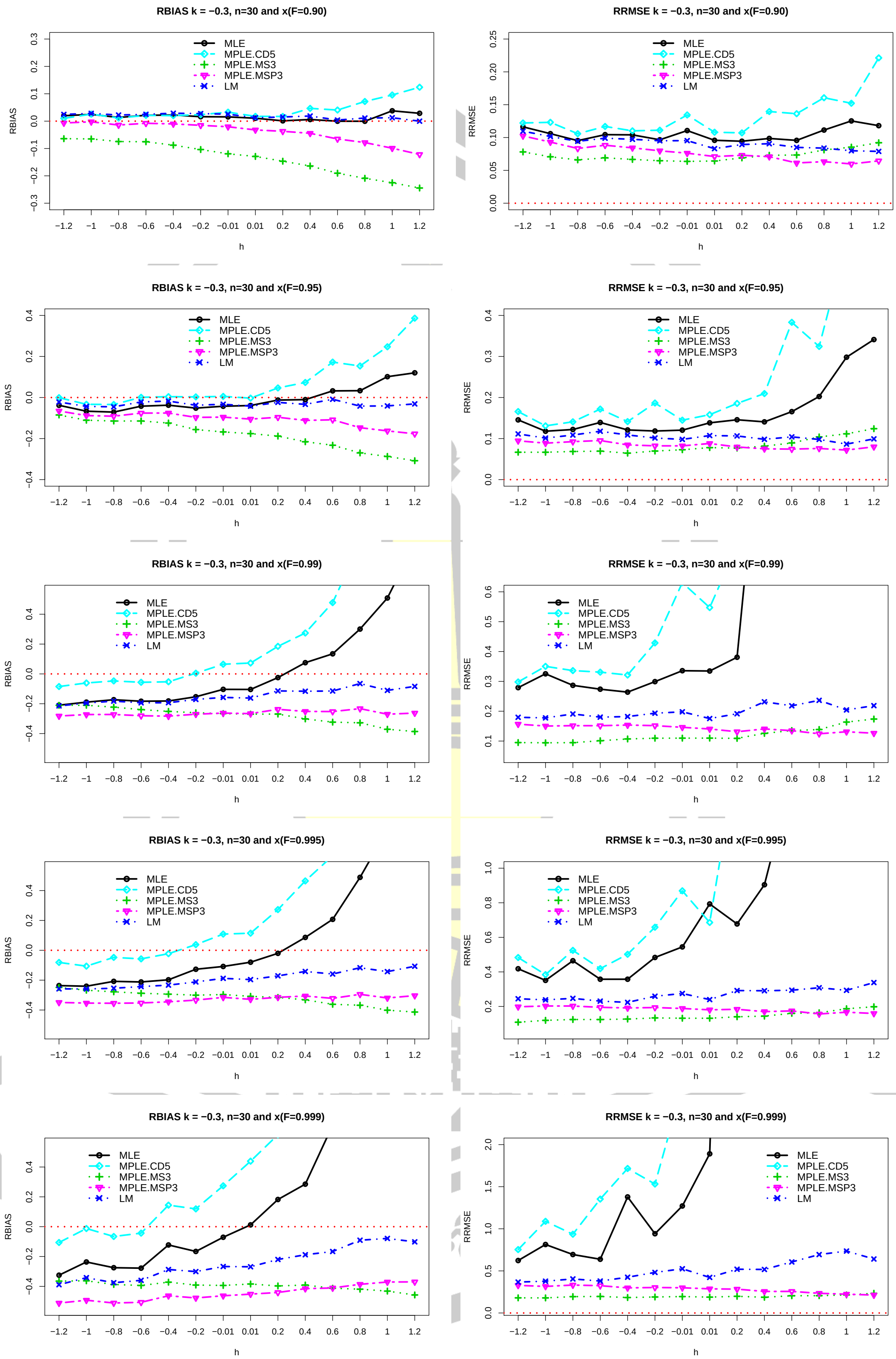


Figure 4.2: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MLE, MPLE.CD, MPLE.MSP3 and LM for value of $k = -0.3$ and sample size $n = 30$.

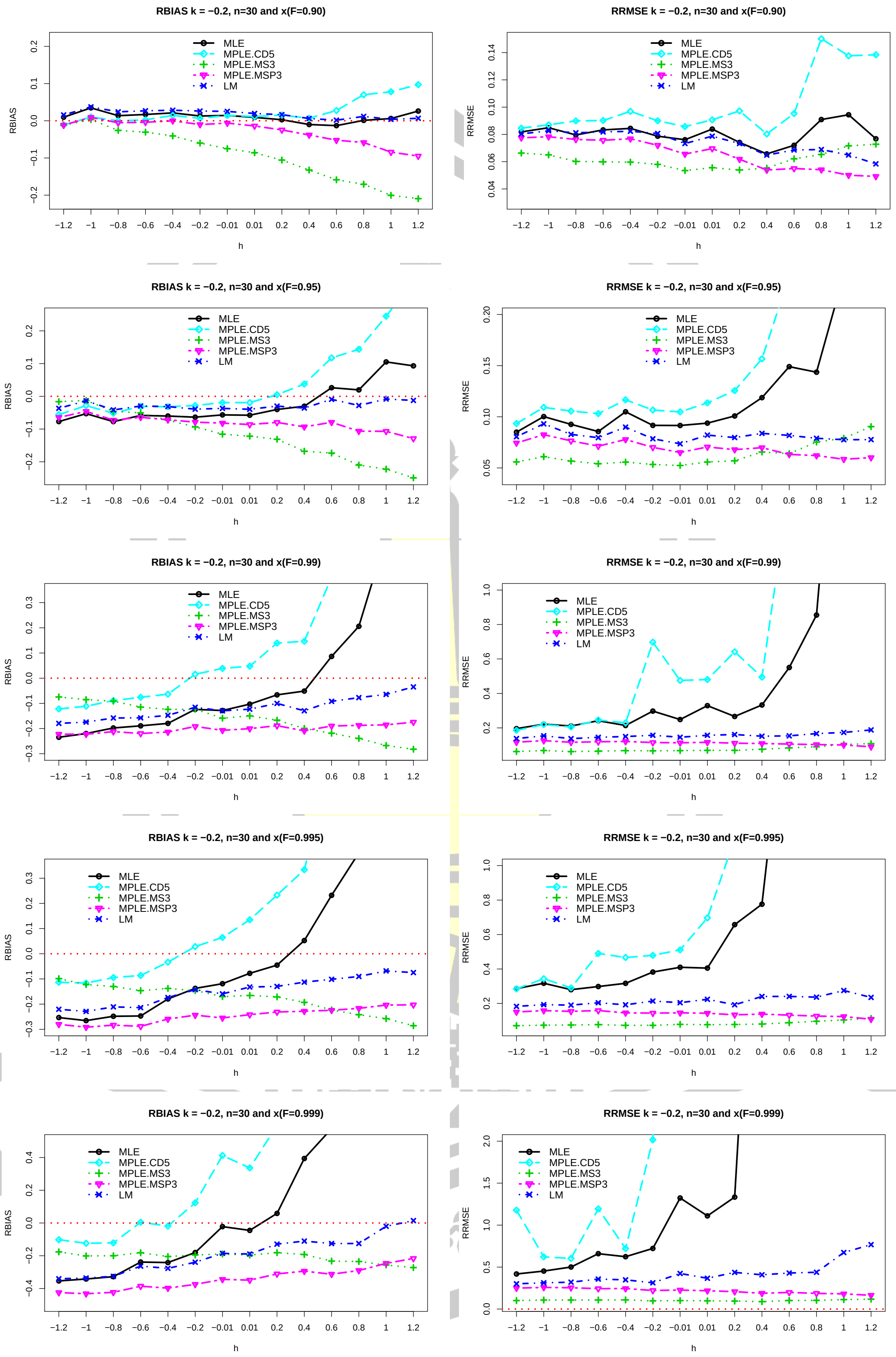


Figure 4.3: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MLE, MPLE.CD, MPLE.MSP3 and LM for value of $k = -0.2$ and sample size $n = 30$.

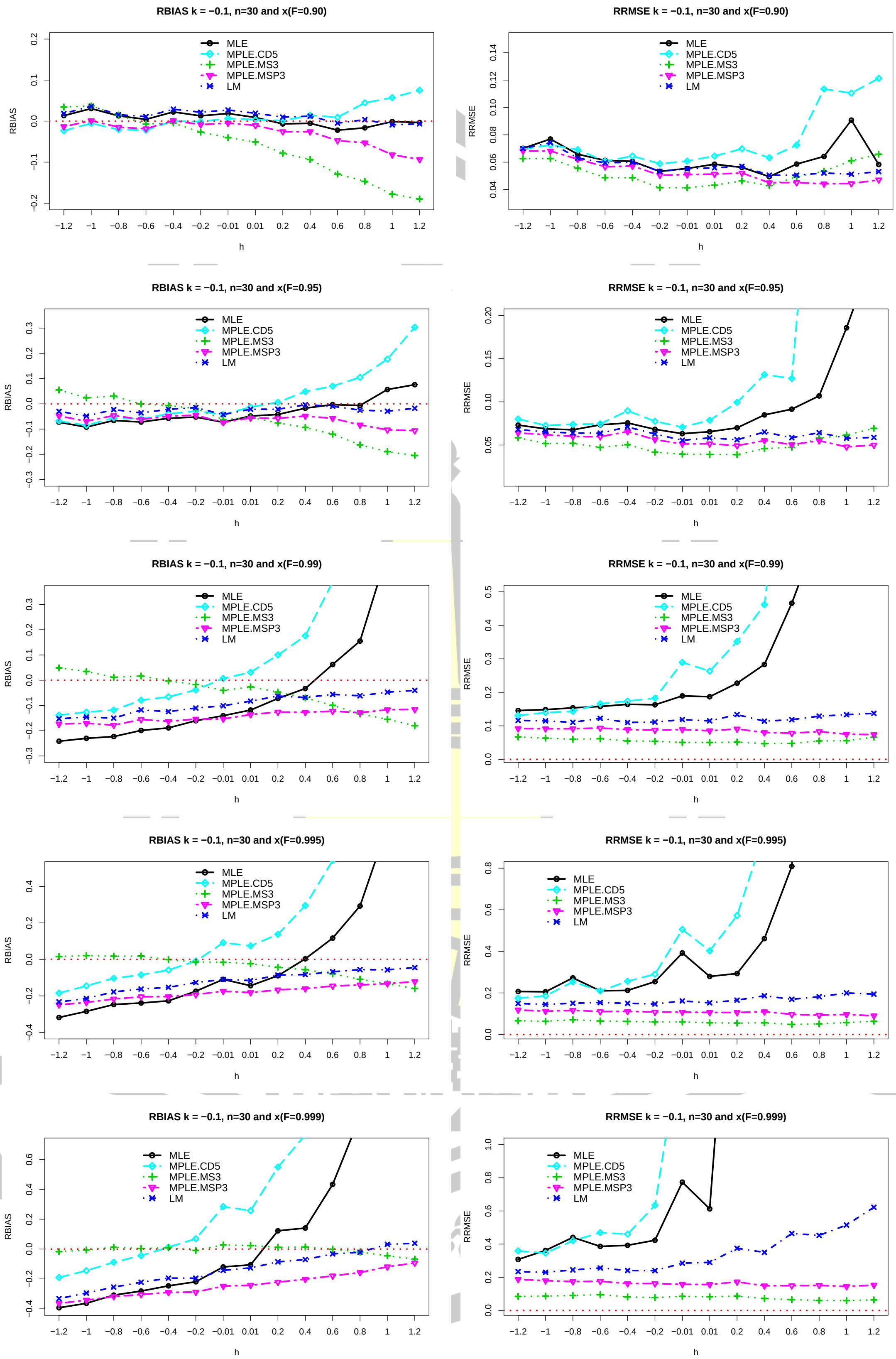


Figure 4.4: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MLE, MPLE.CD, MPLE.MSP3 and LM for value of $k = -0.1$ and sample size $n = 30$.

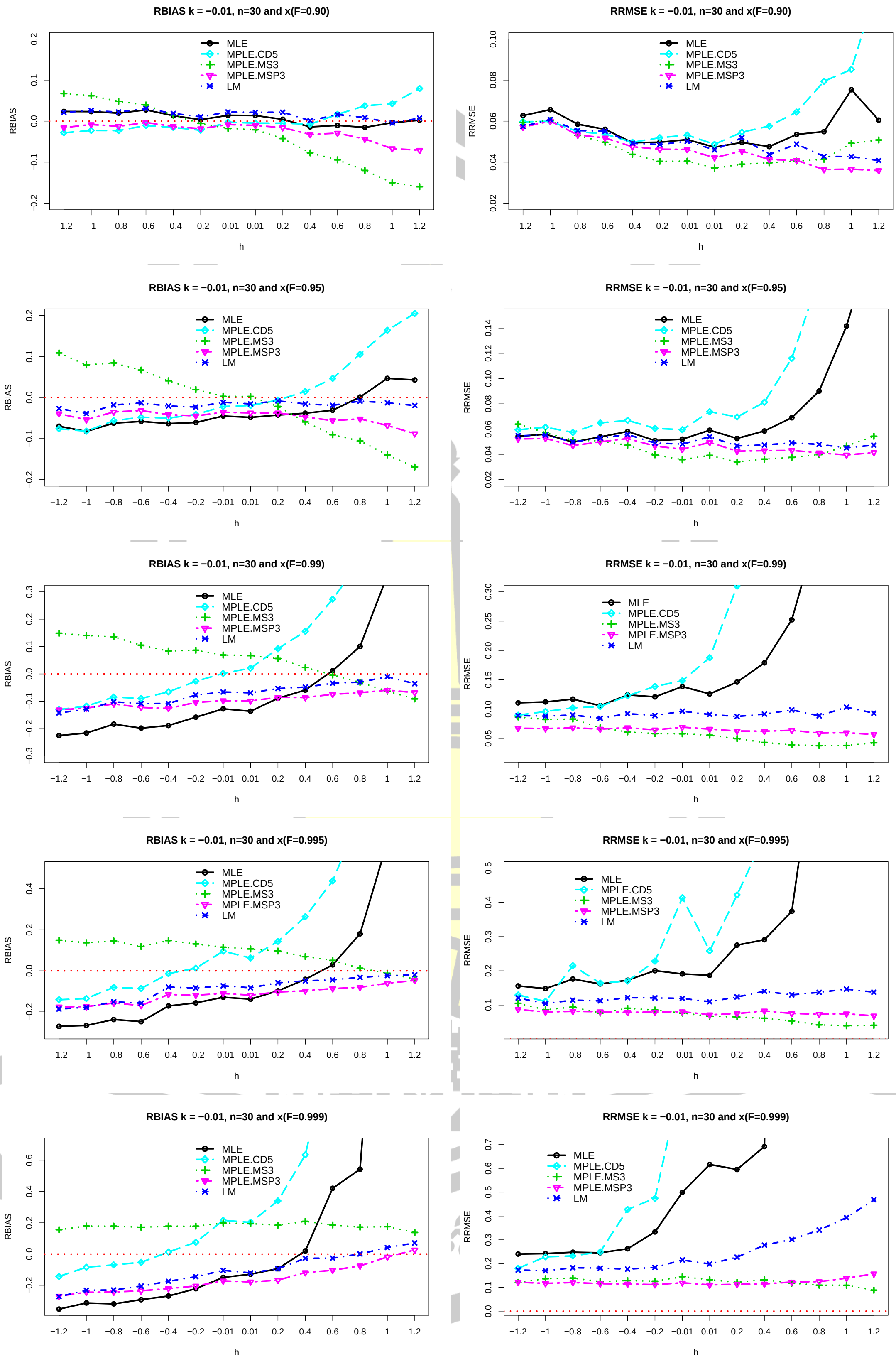


Figure 4.5: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MLE, MPLE.CD, MPLE.MSP3 and LM for value of $k = -0.01$ and sample size $n = 30$.

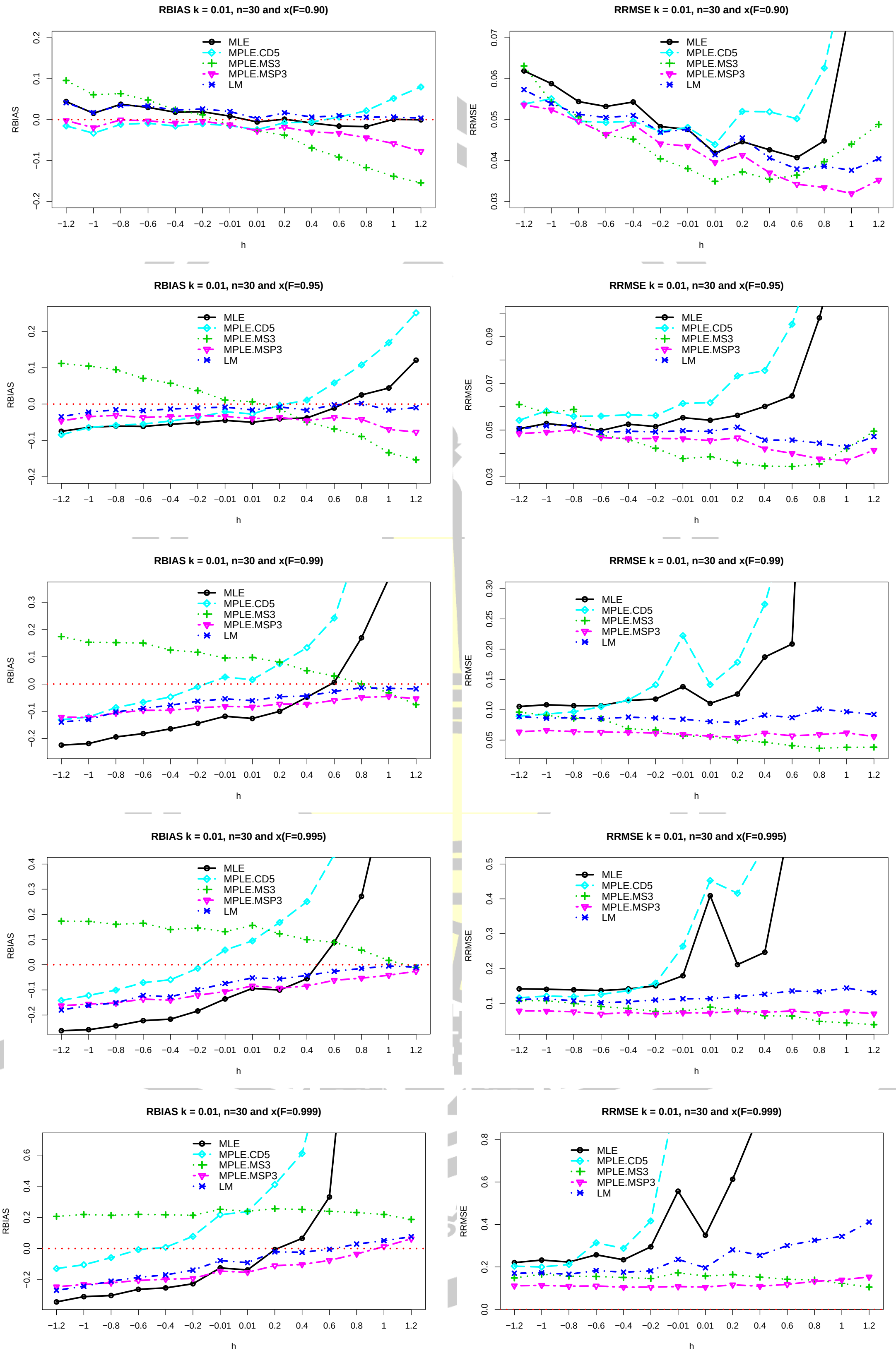


Figure 4.6: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MLE, MPLE.CD, MPLE.MSP3 and LM for value of $k = 0.01$ and sample size $n = 30$.

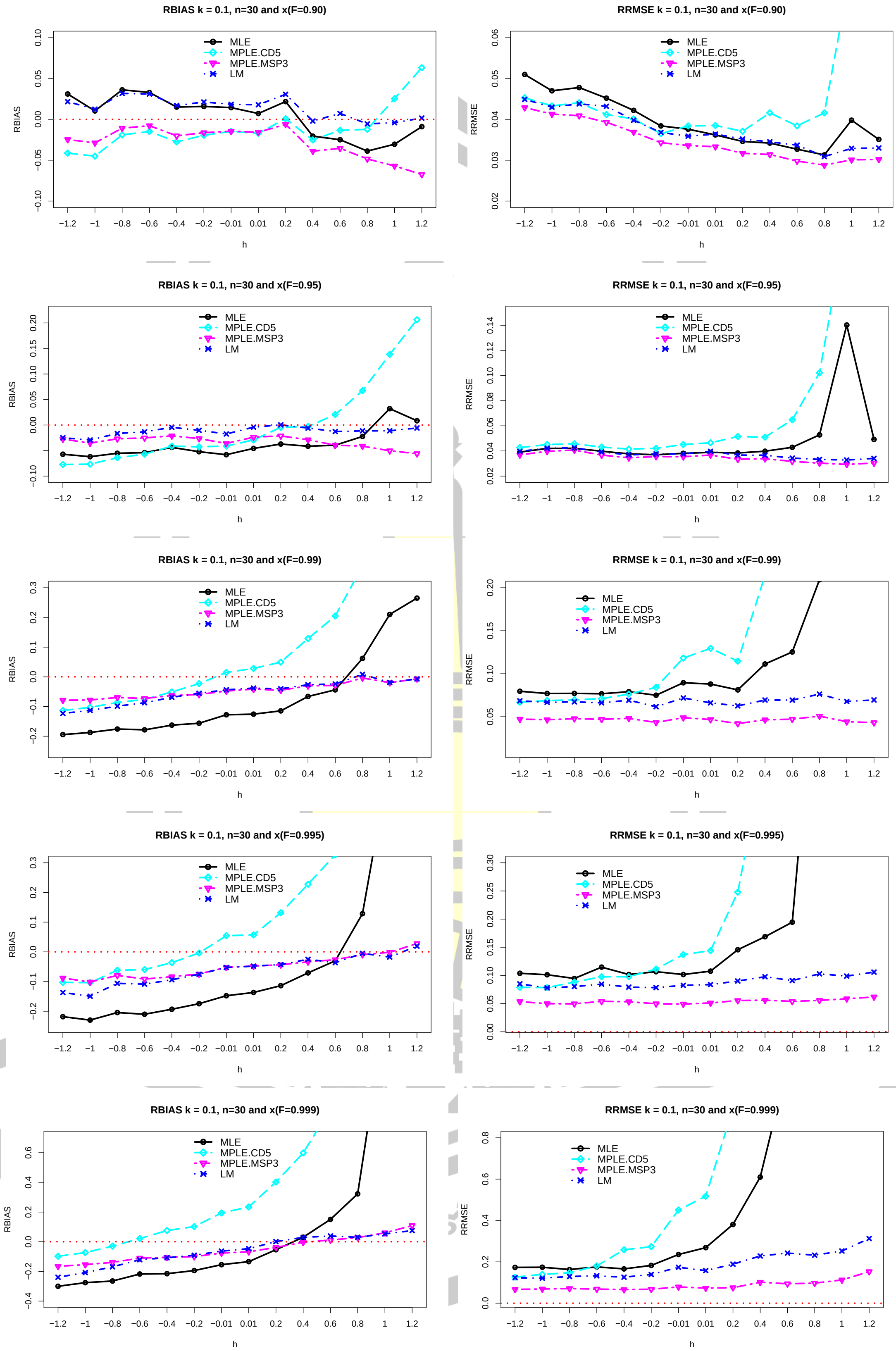


Figure 4.7: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MLE, MPLE.CD, MPLE.MSP3 and LM for value of $k = 0.1$ and sample size $n = 30$.

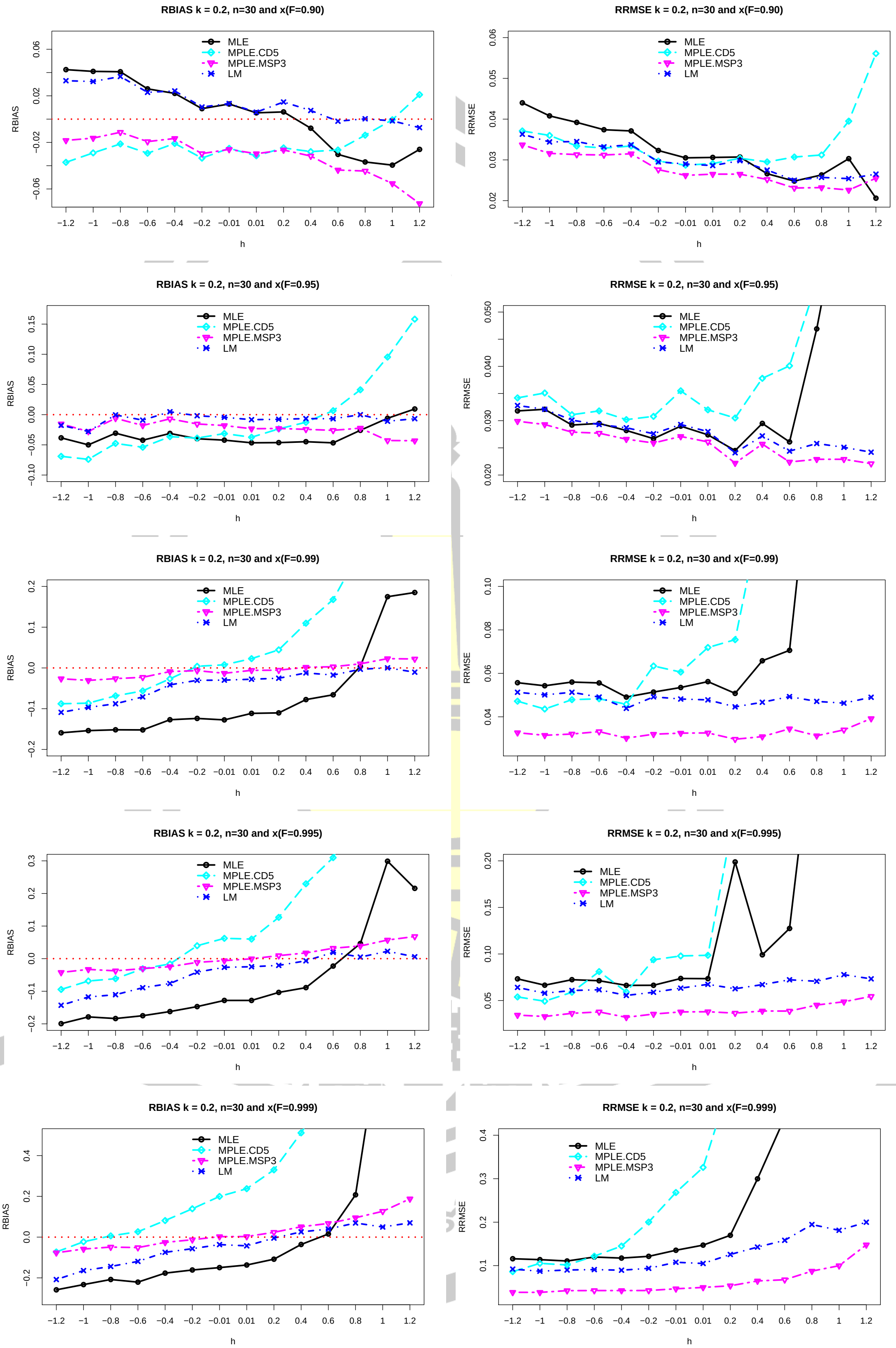


Figure 4.8: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MLE, MPLE.CD, MPLE.MSP3 and LM for value of $k = 0.2$ and sample size $n = 30$.

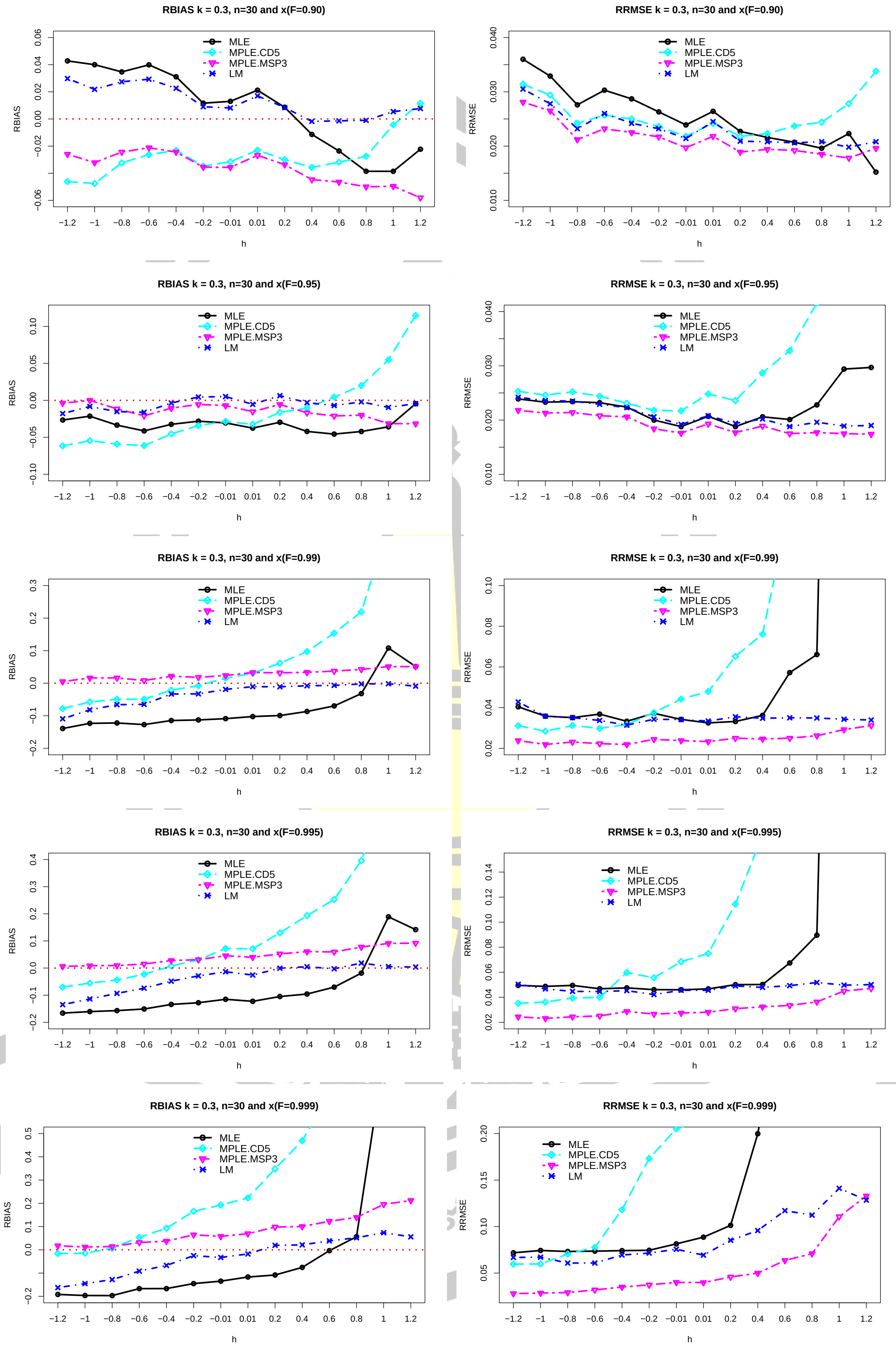


Figure 4.9: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MLE, MPLE.CD, MPLE.MSP3 and LM for value of $k = 0.3$ and sample size $n = 30$.

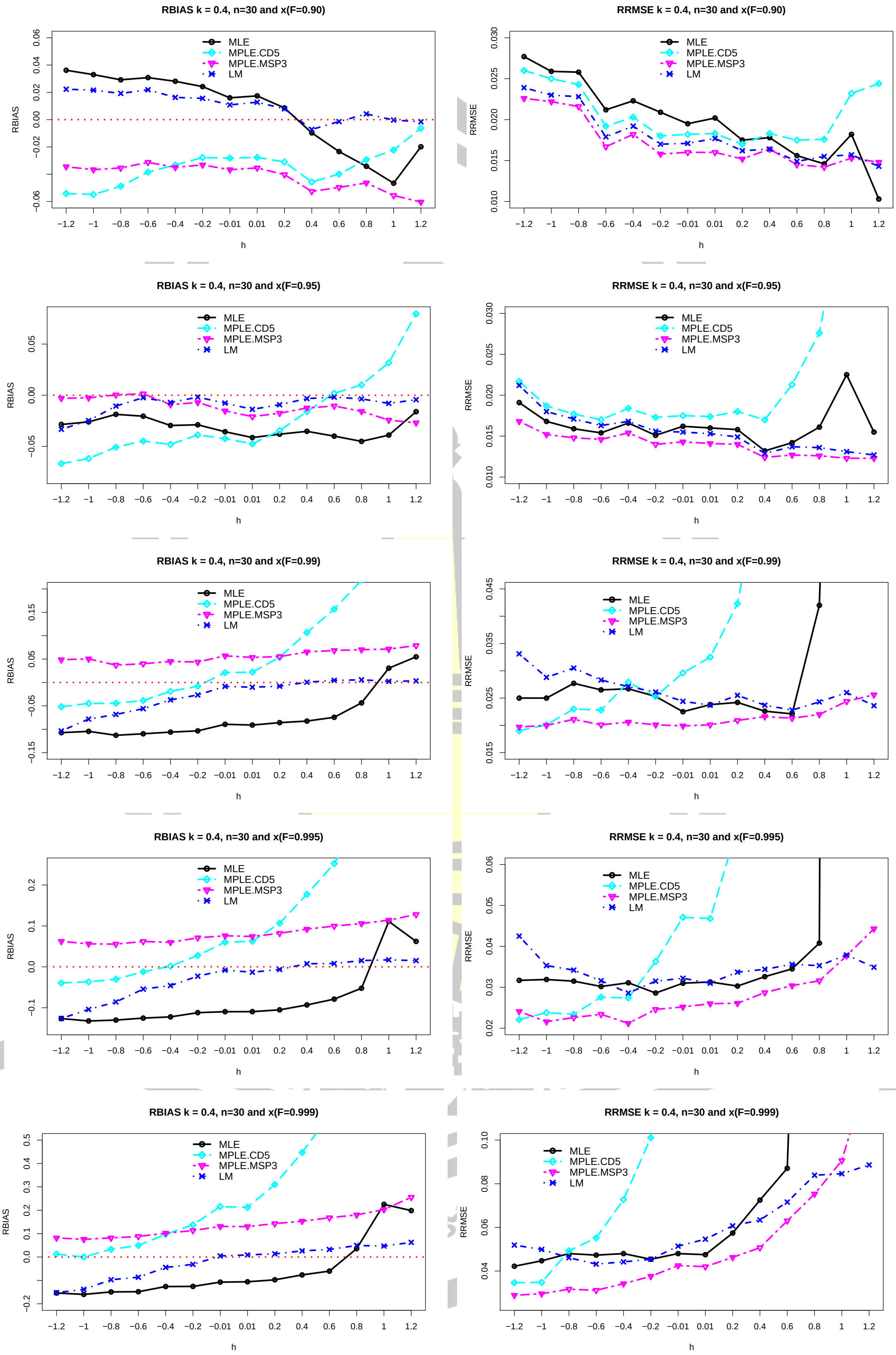


Figure 4.10: RBIAS and RRMSE of the all quantile estimators of MLE, MPLE.CD, MPLE.MSP3 and LM for value of $k = 0.4$ and sample size $n = 30$.

(ii) Comparisons the performance of maximum penalized likelihood estimator (MPLE) with maximum likelihood estimator and L-moment estimator for four parameter kappa distribution for n=50

In Table 4.5 to Table 4.8 illustrate the RBIAS and RRMSE values of the MLE MPLE.CD5, MPLE.MS3, MPLE.MSP3 and LM methods in all quantiles. In case that both two shape parameters which had the k value at -0.4 and 0.4 and the h value was between -1.2 to 1.2, the quantiles and parameters were different, resulting in the difference in the RBIAS and RRMSE values of parameter estimation.

Figure 4.11 to 4.20 show the RBIAS and RRMSE of all parameters and quantiles being researched. The Figures 4.11 to 4.14, Figures 4.15 to 4.16 and 4.17 to 4.20 indicate the RBIAS and RRMSE in case that the k value was negative, close to 0, and positive respectively. Concerning the RBIAS, when the h value ranged from -1.2 to 1.2;

- The MLE and MPLE.CD5 methods had a decrease in the RBIAS when the h value was negative [-1.2, -0.2];
- The LM method had a gradual drop in the RBIAS;
- The MPLE.MSP3 method experienced an increase in the RBIAS when the quantile was low $(x(F = 0.90, 0.95))$. If the quantile was over 0.95, the RBIAS would drop, except when the k value ranged between 0.2 to 0.4, the RBIAS would rise; and
- The MPLE.MS3 method, the RBIAS rose when the k value was negative. If the k value was close to 0, the RBIAS would decline. When the h value was negative in case the quantile was 0.90 and 0.95. Given the quantile was 0.99 to 0.999, the RBIAS rose continuously. Nevertheless, considering the RRMSE, it was found that all estimation methods had a rise in RRMSE when the h value reached -1.2 to 1.2.

Figure 4.11 to 4.20 concerned the h value was negative and the quantile was 0.99 to 0.999. It was found that MPLE.MSP3 method was the method that had the least RBIAS when the k value was 0.4, 0.3, and 0.2. The MPLE.CD5 had the least

RBIAS when the k value was 0.01, -0.01, -0.1, and -0.2. The MPLE.MS3 had the least RBIAS when the k value was 0.1, -0.3, and -0.4. Regarding the RRMSE, there are no significant difference in the MPLE.MS3 and MPLE.MSP3 methods. Concerning the k value, when the k value was negative and close to 0, it was found that the MPLE.MS3 had the least RRMSE and the MPLE.MSP3 had the lesser RRMSE than the LM and MLE methods. Concerning the case that the k value was positive, the MPLE.MSP3 had the least RRMSE compared to that of other methods in the quantiles studied.

Table 4.5: The RBIAS and RRMSE value of the estimates of K4D obtained by four estimation methods for $k = -0.4$, $-1.2 \leq h \leq 1.2$ and $n = 50$.

k	h	$x(F)$	RBIAS(RRMSE)			
			MLE	MPLE.CD5	MPLE.MS3	LM
-0.4	-1.2	0.90	0.1950(0.0795)	**0.0141**(0.0814)	-0.0870(**0.0549**)	0.0334(0.0815)
		0.95	-0.0606(0.0909)	**-0.0535**(0.0953)	-0.1608(**0.0656**)	-0.0543(0.0822)
		0.99	-0.1692(0.2082)	**-0.1354**(0.2039)	-0.2943(**0.1208**)	-0.1968(0.1682)
		0.995	-0.1831(0.3269)	**-0.1361**(0.3489)	-0.3513(**0.1592**)	-0.2485(0.2436)
		0.999	-0.2514(0.4579)	**-0.2098**(0.4337)	-0.4854(**0.2610**)	-0.3847(0.3477)
	-1	0.90	**0.0008**(0.0809)	-0.0041(0.0810)	-0.0943(**0.0577**)	0.0165(0.0827)
		0.95	-0.0463(0.0898)	-0.0402(0.0944)	-0.1506(**0.0603**)	**-0.0363**(0.0816)
		0.99	-0.1590(0.2160)	**-0.1318**(0.2151)	-0.2987(**0.1217**)	-0.1957(0.1666)
		0.995	-0.1967(0.2968)	**-0.1638**(0.2934)	-0.3634(**0.1644**)	-0.2600(0.2247)
		0.999	-0.2465(0.4656)	**-0.1919**(0.4950)	-0.4905(**0.2662**)	0.3755(0.3489)
	-0.8	0.90	0.0079(0.0689)	**0.0048**(0.0704)	-0.0932(**0.0509**)	0.0246(0.0713)
		0.95	-0.0255(0.1012)	**-0.0178**(0.1089)	-0.1445(**0.0604**)	-0.0200(0.0903)
		0.99	-0.1540(0.1989)	**-0.1278**(0.1928)	-0.3006(**0.1212**)	-0.1902(0.1601)
		0.995	-0.1851(0.2677)	**-0.1536**(0.2553)	-0.3650(**0.1634**)	-0.2553(0.1984)
		0.999	-0.1657(0.6838)	**-0.1179**(0.7120)	-0.4839(**0.2600**)	-0.3318(0.4116)
	-0.6	0.90	**-0.0030**(0.0747)	-0.0055(0.0771)	-0.1036(**0.0554**)	0.0136(0.0774)
		0.95	-0.0551(0.0908)	-0.0491(0.0944)	-0.1685(**0.0636**)	**-0.0450**(0.0824)
		0.99	-0.1539(0.1964)	**-0.1305**(0.2017)	-0.3143(**0.1297**)	-0.1955(0.1513)
		0.995	-0.1674(0.2760)	**-0.1330**(0.2850)	-0.3676(**0.1639**)	-0.2414(0.2027)
		0.999	-0.1794(0.5554)	**-0.1253**(0.5787)	-04864(**0.2631**)	-0.3206(0.3774)
	-0.4	0.9	**0.0048**(0.0738)	0.0061(0.0777)	-0.1051(**0.0512**)	0.0214(0.0731)
		0.95	-0.0514(0.0865)	**-0.0413**(0.0899)	-0.1760(**0.0644**)	-0.0439(0.0784)
		0.99	-0.1234(0.1990)	**-0.1005**(0.2108)	-0.3188(**0.1309**)	-0.1774(0.1544)
		0.995	-0.1431(0.2857)	**-0.1094**(0.3052)	-0.3708(**0.1658**)	-0.2248(0.2100)
		0.999	-0.1853(0.4778)	**-0.1288**(0.5666)	-0.4998(**0.2741**)	-0.3381(0.3447)
	-0.2	0.9	0.0056(0.0710)	**0.0027**(0.0716)	-0.1120(**0.0518**)	0.0178(0.0704)
		0.95	-0.0228(0.0909)	**-0.0147**(0.0983)	-0.1682(**0.0620**)	-0.0198(0.0802)
		0.99	-0.0980(0.2237)	**-0.0708**(0.2288)	-0.3258(**0.1347**)	-0.1609(0.1586)
		0.995	-0.1423(0.2685)	**-0.1008**(0.2914)	-0.3849(**0.1731**)	-0.2220(0.2246)
		0.999	-0.0860(0.6282)	**-0.0243**(0.7343)	-0.4911(**0.2659**)	-0.2860(0.3670)
	-0.01	0.9	-0.0016(0.0615)	**-0.0049**(0.0618)	-0.1303(**0.0499**)	0.0135(0.0619)
		0.95	-0.0595(0.0895)	**-0.0528**(0.0929)	-0.2039(**0.0725**)	-0.0563(0.0788)
		0.99	-0.0780(0.2018)	**-0.0392**(0.2242)	-0.3232(**0.1295**)	-0.1373(0.1508)
		0.995	-0.1069(0.3345)	**-0.0596**(0.3494)	-0.3902(**0.1781**)	-0.2123(0.2217)
		0.999	-0.0312(0.8882)	**0.0540**(0.9882)	-0.4891(**0.2620**)	-0.2500(0.4256)

Table 4.6: The RBIAS and RRMSE value of the estimates of K4D obtained by four estimation methods for $k = -0.4$, $-1.2 \leq h \leq 1.2$ and $n = 50$. (Cont.)

k	h	$x(F)$	RBIAS(RRMSE)			
			MLE	MPLE.CD5	MPLE.MS3	LM
	0.01	0.90	-0.0072(0.0602)	**-0.0068**(0.0633)	-0.1355(**0.0502**)	0.0076(0.0604)
		0.95	-0.0352(0.0887)	**-0.0259**(0.0933)	-0.1927(**0.0679**)	-0.0344(0.0792)
		0.99	0.0826(0.2498)	**-0.0342**(0.3111)	-0.3296(**0.1352**)	-0.1537(0.1561)
		0.995	-0.0688(0.3470)	**0.0028**(0.4389)	-0.3827(**0.1719**)	-0.1858(0.2209)
		0.999	**-0.0204**(1.0812)	0.0613(1.1937)	-0.4965(**0.2707**)	-0.2697(0.4234)
	0.2	0.90	**0.0063**(0.0683)	0.0069(0.0706)	-0.1384(**0.0524**)	0.0232(0.0697)
		0.95	-0.0281(0.0875)	**-0.0148**(0.0961)	-0.2121(**0.0712**)	-0.0370(0.0712)
		0.99	-0.0481(0.2366)	**0.0152**(0.2905)	-0.3398(**0.1404**)	-0.1330(0.1539)
		0.995	-0.0516(0.3853)	**0.0088**(0.4126)	-0.3925(**0.1788**)	-0.1910(0.2309)
		0.999	**-0.0101**(0.6384)	0.0983(0.7893)	-0.4977(**0.2685**)	-0.2598(0.3323)
	0.4	0.90	0.0096(0.0618)	**-0.0023**(0.0691)	-0.1671(**0.0587**)	0.0047(0.0624)
		0.95	-0.0263(0.0906)	**-0.0055**(0.1090)	-0.2319(0.0795)	-0.0356(**0.0755**)
		0.99	**0.0015**(0.2786)	0.0750(0.3125)	-0.3516(**0.1474**)	-0.1266(0.1523)
		0.995	**0.0134**(0.4251)	0.1344(0.6337)	-0.3993(**0.1804**)	-0.1605(0.2179)
		0.999	**0.1798**(1.7160)	0.3751(2.1605)	-0.4976(**0.2683**)	-0.2068(0.4244)
	0.6	0.90	-0.0122(0.0631)	-0.0058(0.0648)	-0.1873(0.0631)	**-0.0040**(**0.0575**)
		0.95	**-0.0151**(0.0945)	0.0163(0.1291)	-0.2491(0.087)	-0.0374(**0.0747**)
		0.99	**0.0341**(0.3031)	0.1647(0.4683)	-0.3681(**0.1568**)	-0.1155(0.1622)
		0.995	**0.0472**(0.5073)	0.2740(1.5029)	-0.4155(**0.1925**)	-0.1591(0.2435)
-0.4		0.999	0.2392(1.9837)	0.6909(4.9391)	-0.5072(**0.2766**)	**-0.2064**(0.4325)
	0.8	0.90	**0.0057**(0.0644)	0.0247(0.0711)	-0.1902(0.0629)	0.0167(**0.0594**)
		0.95	**0.0197**(0.1238)	0.0775(1561)	-0.2614(0.0906)	-0.0223(**0.0785**)
		0.99	0.1158(0.3979)	0.3050(0.7023)	-0.3781(0.1620)	**-0.1079**(**0.1571**)
		0.995	0.2793(5.700)	0.5691(3.5636)	-0.4190(**0.1958**)	**-0.0978**(0.2853)
		0.999	0.6083(9.9792)	1.3731(17.3816)	-0.5146(**0.2828**)	**-0.1446**(0.5415)
	1.0	0.9	-0.0049(0.0674)	0.0243(0.0767)	-0.2176(0.0709)	**0.0026**(**0.0567**)
		0.95	**0.0256**(0.1468)	0.1037(0.2385)	-0.2901(0.1066)	-0.0454(**0.0737**)
		0.99	0.3069(1.5287)	0.5923(2.5530)	-0.4017(0.1800)	**-0.1067**(**0.1596**)
		0.995	0.3748(2.0018)	0.8230(4.4421)	-0.4410(**0.2124**)	**-0.1163**(0.2312)
		0.999	1.0866(8.1173)	2.3321(34.8162)	-0.5201(**0.2861**)	**-0.1287**(0.4989)
	1.2	0.9	0.0317(0.0653)	0.0785(0.1100)	-0.2162(0.0730)	**0.0234**(**0.0605**)
		0.95	0.0635(0.1587)	0.2062(0.4644)	-0.3082(0.1166)	**-0.0410**(**0.0794**)
		0.99	0.3921(2.9921)	1.1745(9.8150)	-0.4144(0.1886)	**-0.0883**(**0.1710**)
		0.995	0.6533(19.4787)	1.7060(27.9580)	-0.4594(**0.2291**)	**-0.1292**(0.2605)
		0.999	1.9365(269.4655)	4.5864(337.0829)	-0.5333(**0.3005**)	**-0.1347**(0.6218)

Table 4.7: The RBIAS and RRMSE value of the estimates of K4D obtained by four estimation methods for $k = 0.4$, $-1.2 \leq h \leq 1.2$ and $n = 50$.

k	h	$x(F)$	RBIAS(RRMSE)			
			MLE	MPLE.CD5	MPLE.MSP3	LM
0.4	-1.2	0.90	0.0272(0.0168)	-0.0380(0.0147)	-0.0291(**0.0132**)	**0.0231**(0.0140)
		0.95	-0.0107(0.0105)	-0.0454(0.0116)	**-0.0066(0.0091)**	0.0182(0.0113)
		0.99	-0.0719(0.0136)	-0.0484(0.0116)	**0.0114(0.0086)**	-0.0923(0.0231)
		0.995	-0.0875(0.0171)	-0.0411(0.0134)	**0.0217(0.0090)**	-0.1054(0.0264)
		0.999	-0.1019(0.0253)	**-0.0145**(0.0229)	0.0257(**0.0109**)	-0.1494(0.0407)
	-1.0	0.90	0.0230(0.0155)	-0.0376(0.0138)	-0.0317(**0.0121**)	**0.0184**(0.0129)
		0.95	-0.0108(0.0091)	-0.0429(0.0104)	**-0.0103(0.0083)**	-0.0141(0.0100)
		0.99	-0.0713(0.0130)	-0.0458(0.0114)	**0.0081(0.0080)**	-0.0713(0.0174)
		0.995	-0.0843(0.0165)	-0.0388(0.0131)	**0.0132(0.0093)**	-0.0961(0.0246)
		0.999	-0.1030(0.0259)	**-0.0114**(0.0248)	0.0280(**0.0113**)	-0.1134(0.0337)
	-0.8	0.90	0.0181(0.0150)	-0.0362(0.0139)	-0.03451(**0.0123**)	**0.014**(0.0129)
		0.95	-0.0153(0.0091)	-0.0440(0.0105)	-0.0155(**0.0085**)	**-0.0108**(0.0099)
		0.99	-0.0704(0.0127)	-0.0420(0.0112)	**-0.0107(0.0081)**	-0.0531(0.0153)
		0.995	-0.0853(0.0165)	-0.0370(0.0136)	**0.0167(0.0093)**	-0.0679(0.0203)
		0.999	-0.0987(0.0255)	**-0.0016**(0.0284)	0.0345(**0.0117**)	-0.0872(0.0271)
	-0.6	0.90	0.0124(0.0138)	-0.0350(0.0129)	-0.0367(**0.0115**)	**0.0089(0.0115)**
		0.95	-0.0145(0.0098)	-0.0385(0.0106)	-0.0125(**0.0089**)	**-0.0041**(0.0103)
		0.99	-0.0654(0.0126)	-0.0300(0.0113)	**0.0186(0.0087)**	-0.0322(0.0147)
		0.995	-0.0757(0.0158)	**-0.0183**(0.0146)	0.0209(**0.0095**)	-0.0456(0.0184)
		0.999	-0.0932(0.0257)	**0.0286**(0.0329)	0.0439(**0.0125**)	-0.0553(0.0252)
	-0.4	0.90	0.0118(0.0129)	-0.0285(0.0124)	-0.0327(**0.0107**)	**0.0095**(0.0113)
		0.95	-0.0185(0.0087)	-0.0387(0.0093)	-0.0158(**0.0077**)	**-0.0018**(0.0084)
		0.99	-0.0635(0.0126)	-0.0211(0.0113)	**0.0155(0.0089)**	-0.0253(0.0147)
		0.995	-0.0733(0.0154)	**-0.0022**(0.0136)	0.0280(**0.0103**)	-0.0286(0.0194)
		0.999	-0.0743(0.0278)	0.0617(0.0388)	0.0508(**0.0142**)	**-0.0340**(0.0281)
	-0.2	0.90	**0.0059**(0.0122)	-0.0282(0.0116)	-0.0340(**0.0106**)	0.0066(0.0110)
		0.95	-0.0198(0.0092)	-0.0377(0.0097)	-0.0155(**0.0081**)	**0.0005**(0.0089)
		0.99	-0.0631(0.0124)	-0.0159(0.0113)	0.0167(**0.0089**)	**-0.0138**(0.0148)
		0.995	-0.0664(0.0158)	0.0111(0.0162)	0.0363(**0.0115**)	**-0.0086**(0.0195)
		0.999	-0.0620(0.0298)	0.0899(0.0443)	-0.0592(**0.0154**)	**-0.0095**(0.0307)
	-0.01	0.90	0.0051(0.0122)	-0.0280(0.0113)	-0.0360(**0.0100**)	**0.0045**(0.0105)
		0.95	-0.0243(0.0085)	-0.0367(0.0091)	-0.0165(**0.0077**)	**-0.0012**(0.0083)
		0.99	-0.0611(0.0119)	**-0.0043**(0.0113)	0.0219(**0.0086**)	-0.0068(0.0137)
		0.995	-0.0533(0.0145)	0.0347(0.0170)	0.0469(**0.0121**)	**0.0071**(0.0209)
		0.999	-0.0411(0.0323)	0.1310(0.0576)	0.0747(**0.0188**)	**0.0083**(0.0329)

Table 4.8: The RBIAS and RRMSE value of the estimates of K4D obtained by four estimation methods for $k = 0.4$, $-1.2 \leq h \leq 1.2$ and $n = 50$. (Cont.)

k	h	$x(F)$	RBIAS(RRMSE)			
			MLE	MPLE.CD5	MPLE.MSP3	LM
0.4	0.01	0.90	-0.0017(0.0121)	-0.0321(0.0112)	-0.0382(**0.0100**)	**0.0007**(0.0103)
		0.95	-0.0262(0.0091)	-0.0421(0.0096)	-0.0199(**0.0081**)	**-0.0039**(0.0087)
		0.99	-0.0548(0.0115)	0.0039(0.0116)	0.0264(**0.0097**)	**-0.0024**(0.0147)
		0.995	-0.0540(0.0147)	0.0355(0.0171)	0.0427(**0.0113**)	**-0.0003**(0.0191)
		0.999	-0.0486(0.0317)	0.1262(0.0591)	0.0727(**0.0179**)	**0.0066**(0.0352)
	0.2	0.90	0.0007(0.0110)	-0.0354(0.0102)	-0.0410(**0.0094**)	**-0.0006**(**0.0094**)
		0.95	-0.0199(0.0095)	-0.0328(0.0099)	-0.0145(**0.0081**)	**0.0012**(0.0092)
		0.99	-0.0496(0.0117)	0.0148(0.0136)	0.0306(**0.0099**)	**0.0000**(0.0155)
		0.995	-0.0565(0.0163)	0.0512(0.0241)	0.0424(**0.0116**)	**-0.0014**(0.0184)
		0.999	-0.0417(0.0382)	0.1767(0.0887)	0.0803(**0.0200**)	**0.0130**(0.0371)
	0.4	0.90	-0.0017(0.0126)	-0.0401(0.0117)	-0.0425(**0.0104**)	**-0.0014**(0.0108)
		0.95	-0.0212(0.0082)	-0.0313(0.0089)	-0.0161(**0.0072**)	**-0.0018**(0.0080)
		0.99	-0.0490(0.0127)	0.0310(0.0194)	0.0306(**0.0102**)	**0.0030**(0.0151)
		0.995	-0.0505(0.0181)	0.0756(0.0343)	0.0494(**0.0127**)	**0.0057**(0.0192)
		0.999	**-0.0101**(0.0488)	0.2469(0.1475)	0.0945(**0.0226**)	0.0247(0.0390)
	0.6	0.90	-0.0089(0.0107)	-0.0445(0.0109)	-0.0434(**0.0093**)	**-0.0036**(0.0095)
		0.95	-0.0244(0.0086)	-0.0292(0.0101)	-0.0206(**0.0078**)	**-0.0050**(0.0083)
		0.99	-0.0428(0.0108)	0.0494(0.0223)	0.0392(**0.0102**)	**0.0098**(0.0145)
		0.995	-0.0397(0.0196)	0.0995(0.0480)	0.0592(**0.0140**)	**0.0123**(0.0205)
		0.999	-0.0352(0.0529)	0.2636(0.2178)	0.0882(**0.0243**)	**0.0160**(0.0407)
	0.8	0.90	-0.0177(0.0088)	-0.0403(0.0093)	-0.0416(**0.0084**)	**-0.0015**(0.0088)
		0.95	-0.0308(0.0081)	-0.0261(0.0105)	-0.0220(**0.0074**)	**-0.0060**(0.0079)
		0.99	-0.0432(0.0136)	0.0531(0.0325)	0.0320(**0.0095**)	**-0.0017**(0.0126)
		0.995	-0.0267(0.0215)	0.1131(0.0784)	0.0607(**0.0142**)	**0.0123**(0.0209)
		0.999	**-0.0141**(0.0635)	0.2685(0.2893)	0.1040(**0.0303**)	0.0292(0.0410)
	1.0	0.90	-0.0297(0.0087)	-0.0376(0.0095)	-0.0431(**0.0083**)	**-0.0001**(0.0089)
		0.95	-0.0347(0.0097)	-0.0209(0.0112)	-0.0254(**0.0080**)	**-0.0067**(0.0085)
		0.99	-0.0234(0.0215)	0.0650(0.0534)	0.0285(**0.0111**)	**0.0096**(0.0139)
		0.995	-0.0057(0.0293)	0.1152(0.1004)	0.0621(**0.0158**)	**0.0023**(0.0190)
		0.999	0.0558(0.1238)	0.3657(0.8081)	0.1286(**0.0399**)	**0.0363**(0.0438)
	1.2	0.90	-0.0128(**0.0064**)	-0.0321(0.0097)	-0.0509(0.0093)	**-0.0011**(0.0089)
		0.95	-0.0134(**0.0055**)	-0.0135(0.0130)	-0.0286(0.0073)	**-0.0054**(0.0073)
		0.99	**0.0005**(0.0157)	0.0971(0.1341)	0.0453(**0.0144**)	-0.0011(0.0145)
		0.995	0.0216(0.0692)	0.1824(0.3719)	0.0915(0.0280)	**0.0158**(**0.0206**)
		0.999	0.0527(0.1684)	0.4235(2.1169)	0.1725(0.0742)	**0.0273**(**0.0442**)

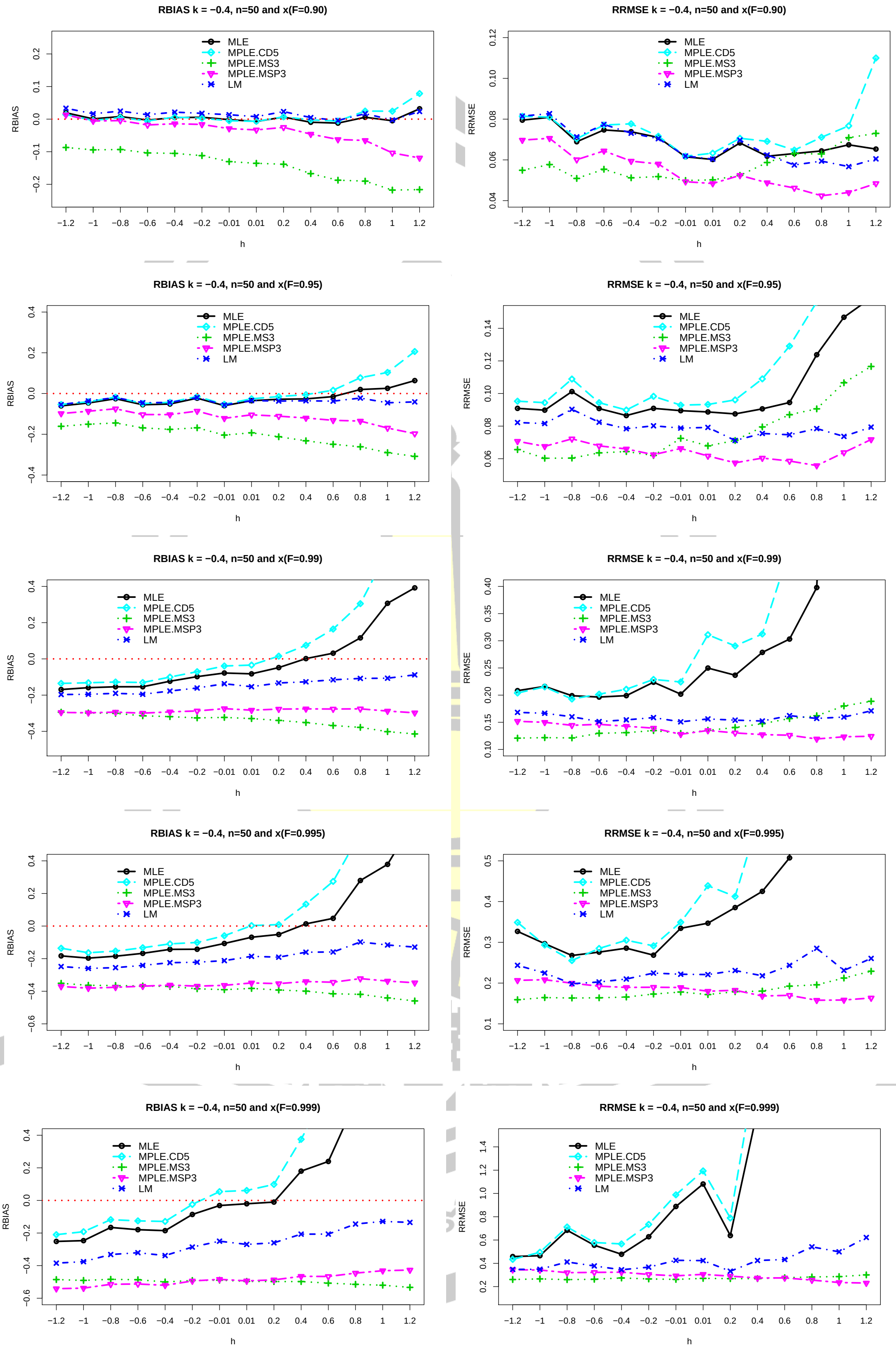


Figure 4.11: RBIAS and RRMSE of the all quantile estimators of MLE, MPLE.CD5, MPLE.MS3, MPLE.MSP3 and LM for value of $k = -0.4$ and sample size $n = 50$.

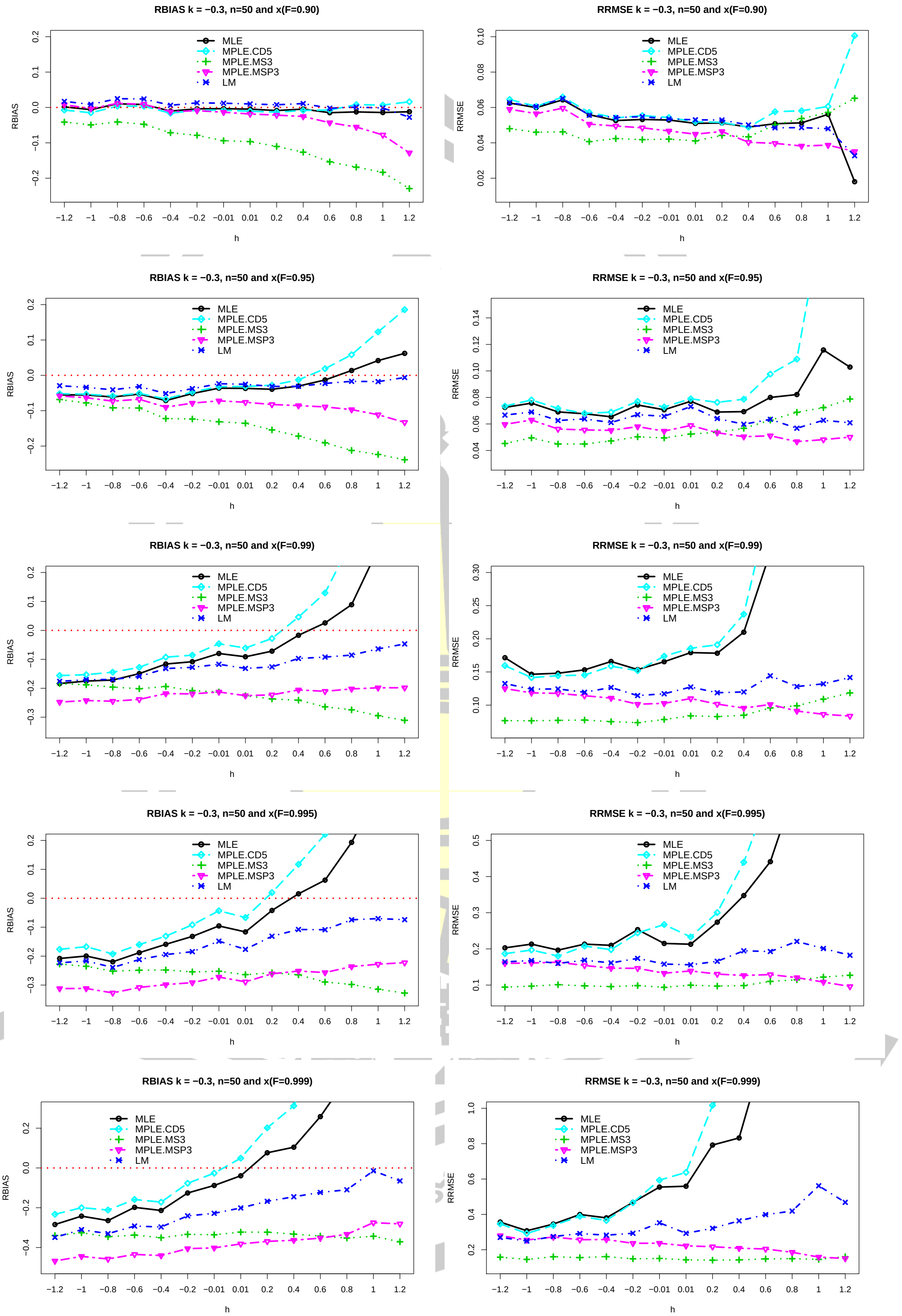


Figure 4.12: RBIAS and RRMSE of the all quantile estimators of MLE, MPLE.CD5, MPLE.MS3, MPLE.MSP3 and LM for value of $k = -0.3$ and sample size $n = 50$.

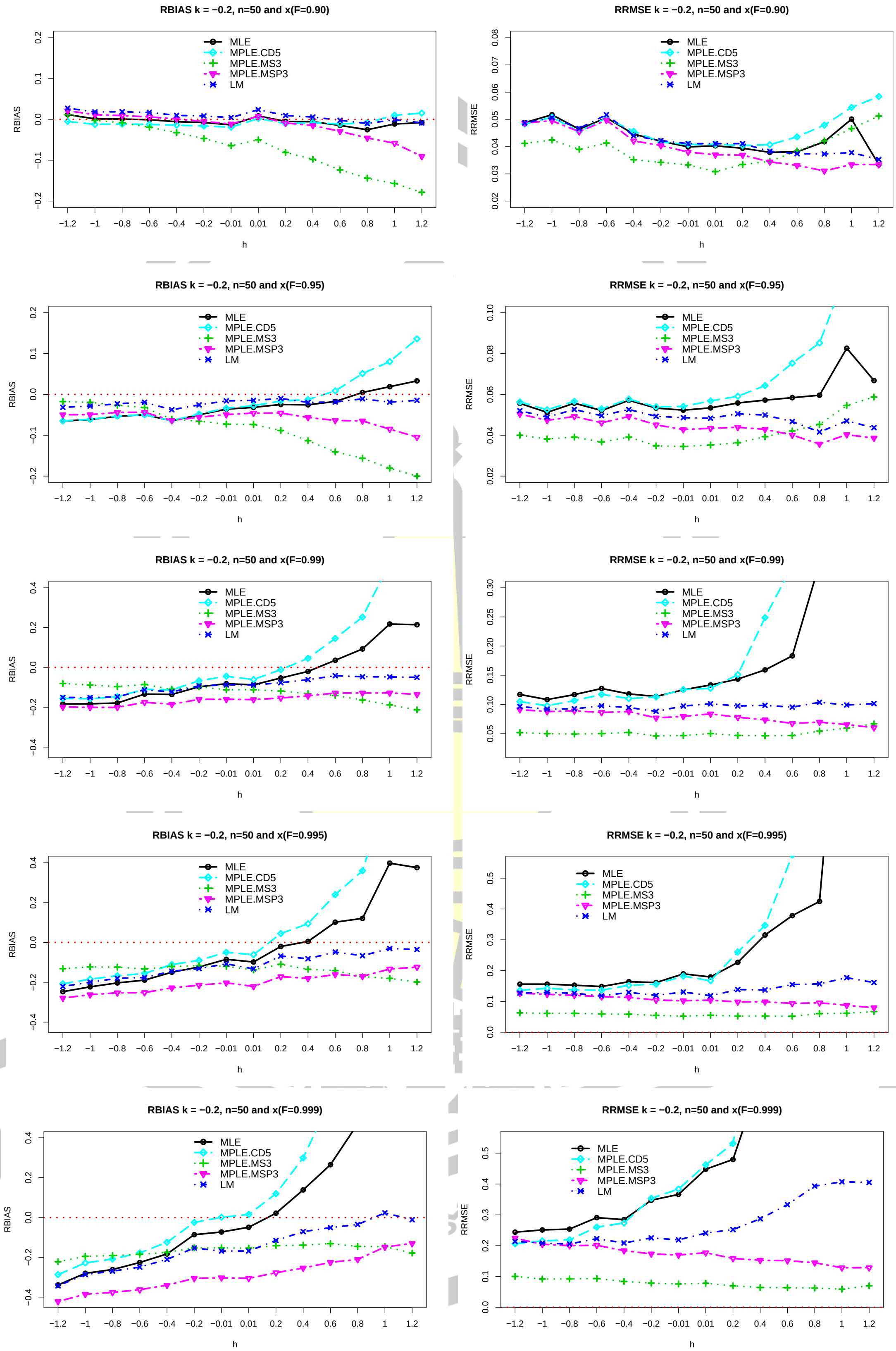


Figure 4.13: RBIAS and RRMSE of the all quantile estimators of MLE, MPLE.CD5, MPLE.MS3, MPLE.MSP3 and LM for value of $k = -0.2$ and sample size $n = 50$.

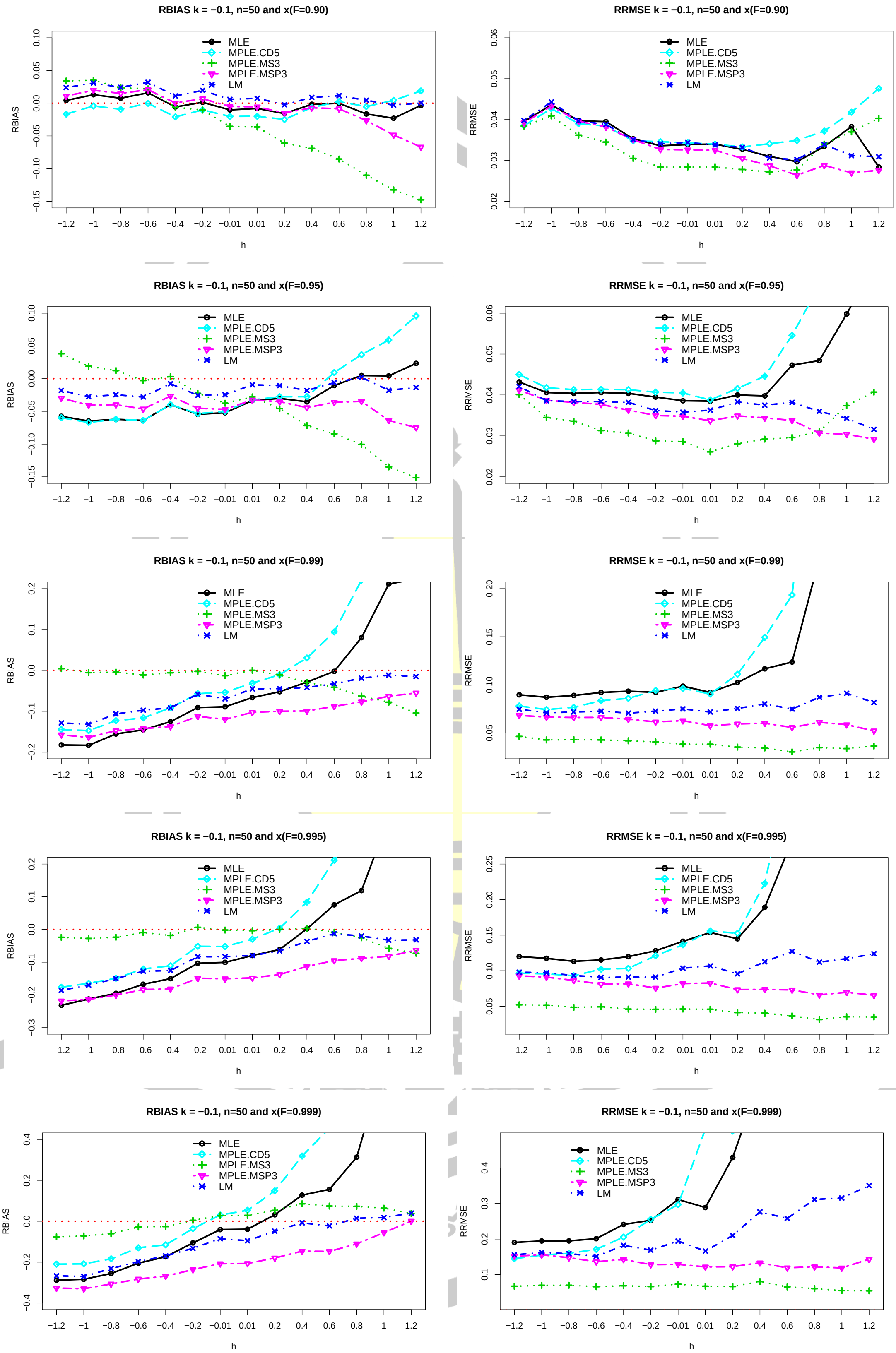


Figure 4.14: RBIAS and RRMSE of the all quantile estimators of MLE, MPLE.CD5, MPLE.MS3, MPLE.MSP3 and LM for value of $k = -0.1$ and sample size $n = 50$.

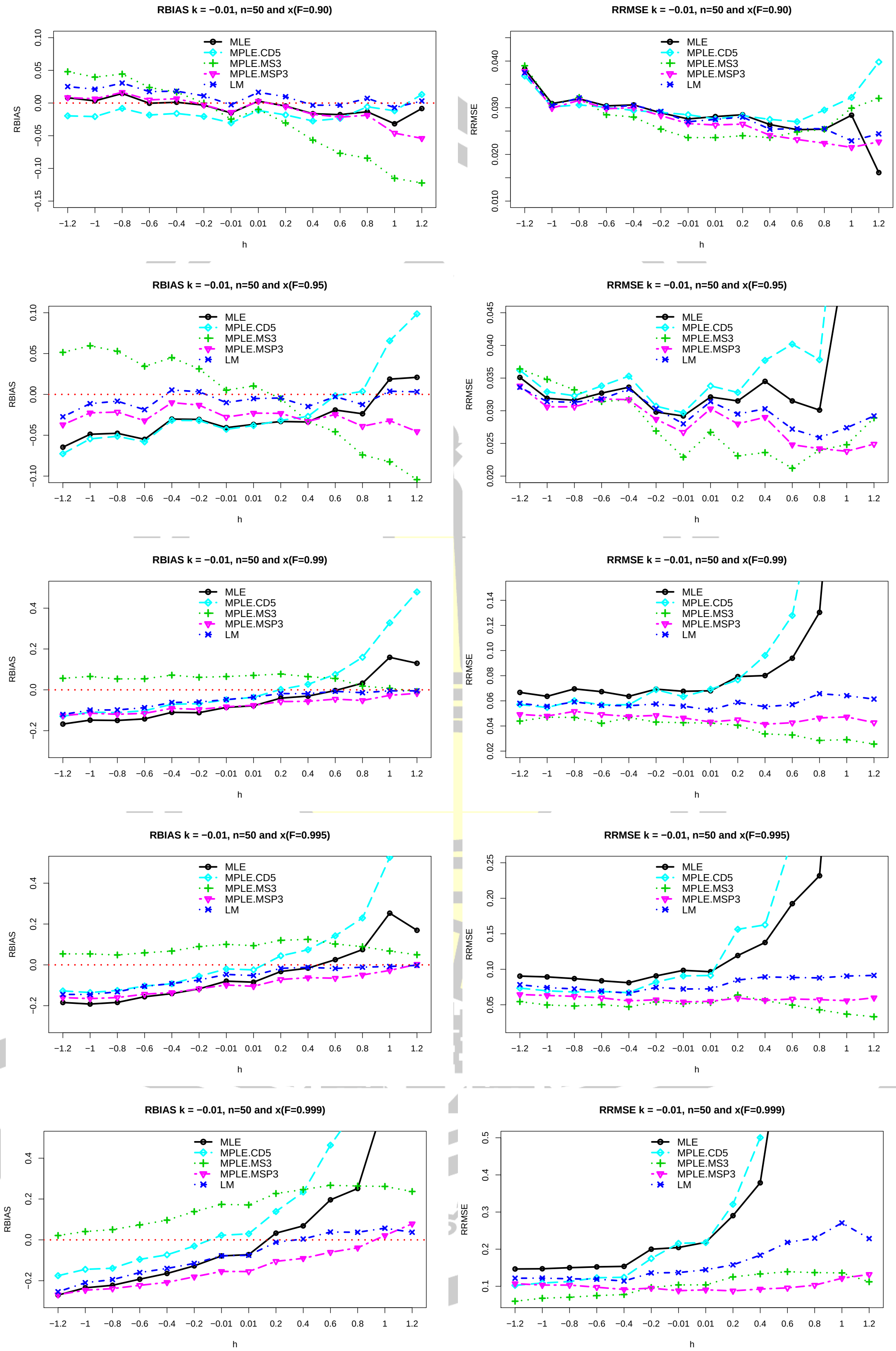


Figure 4.15: RBIAS and RRMSE of the all quantile estimators of MLE, MPLE.CD5, MPLE.MS3, MPLE.MSP3 and LM for value of $k = -0.01$ and sample size $n = 50$.

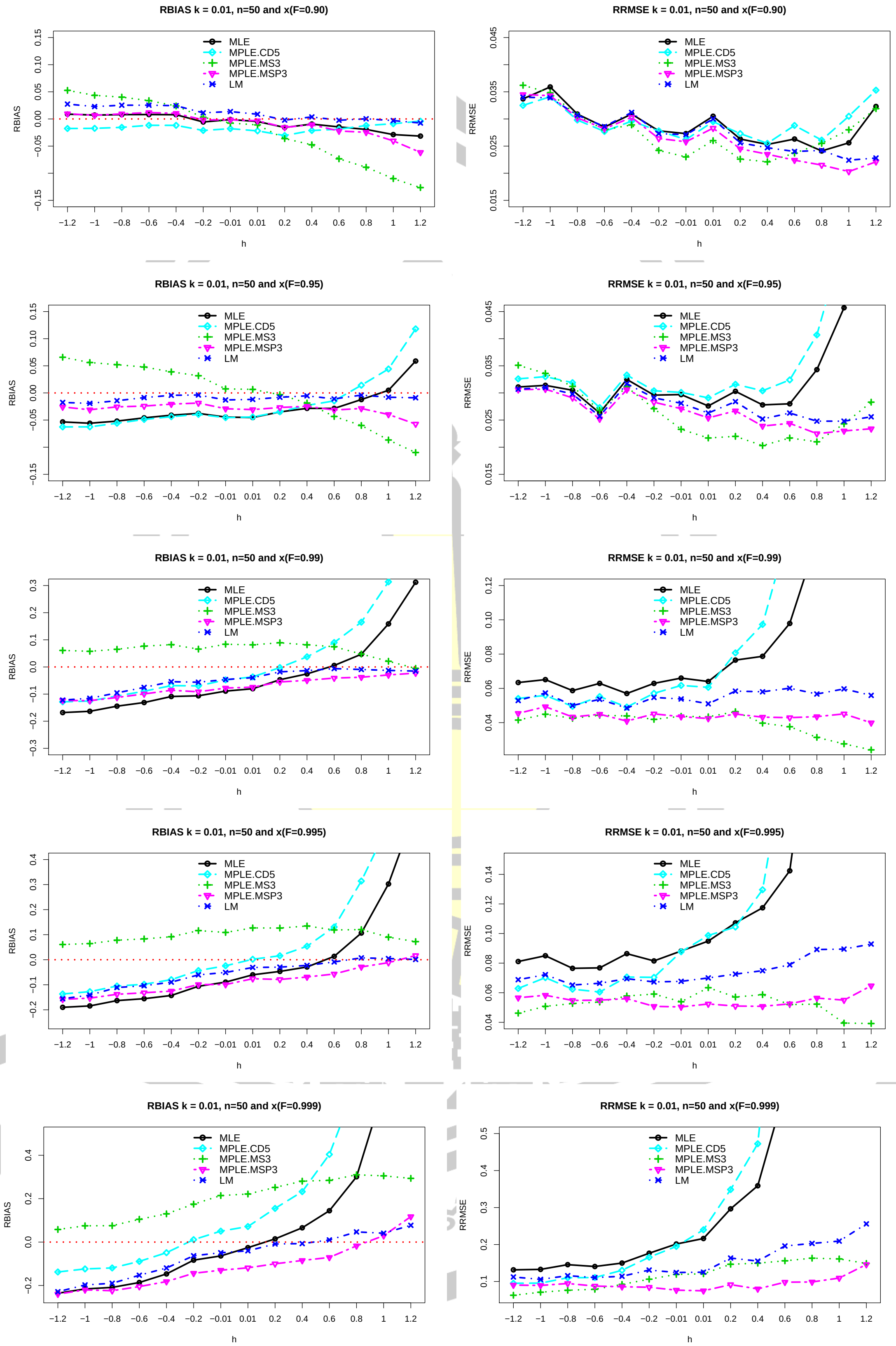


Figure 4.16: RBIAS and RRMSE of the all quantile estimators of MLE, MPLE.CD5, MPLE.MSP3 and LM for value of $k = 0.01$ and sample size $n = 50$.

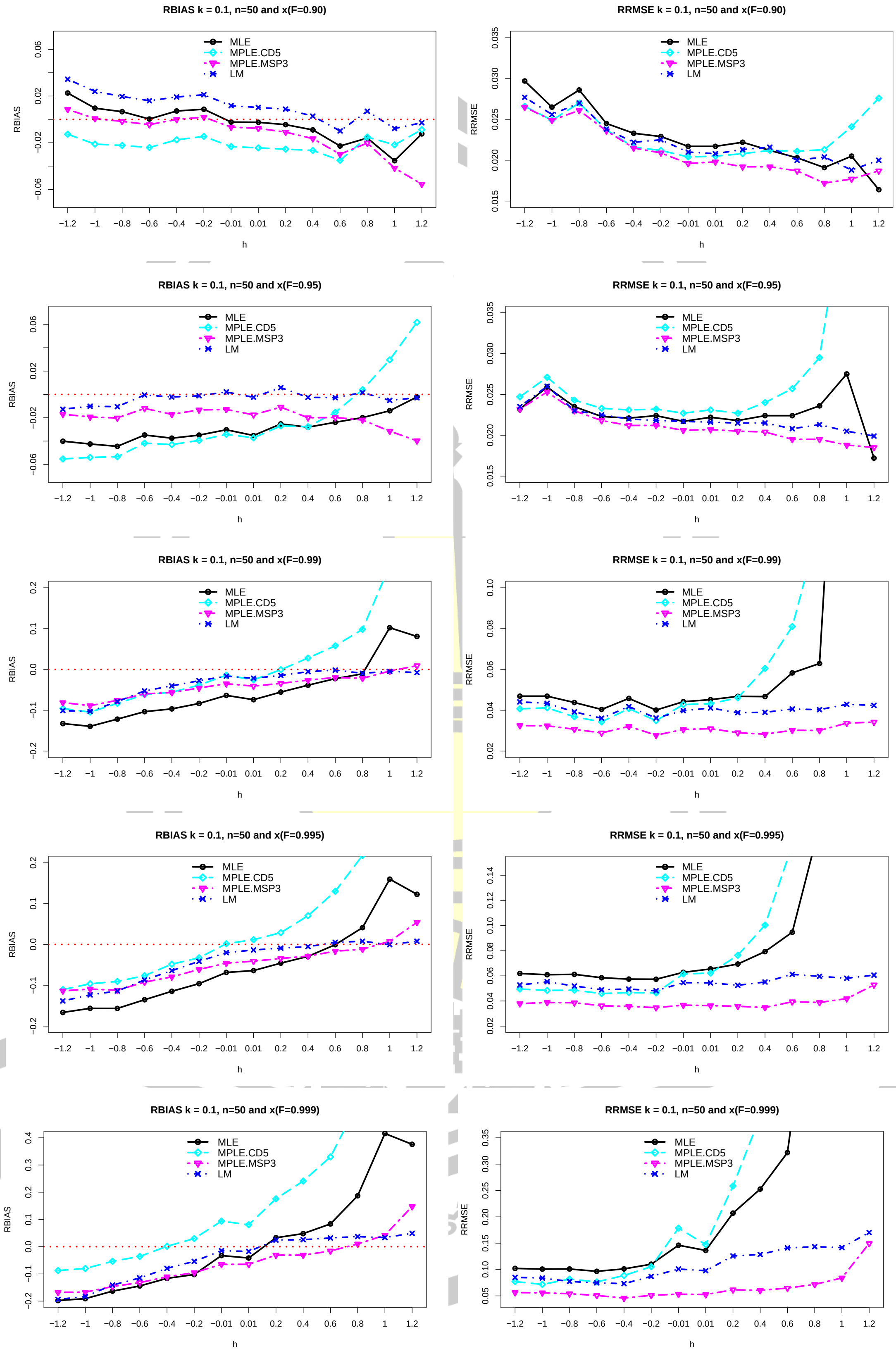


Figure 4.17: RBIAS and RRMSE of the all quantile estimators of MLE, MPLE.CD5, MPLE.MSP3 and LM for value of $k = 0.1$ and sample size $n = 50$.

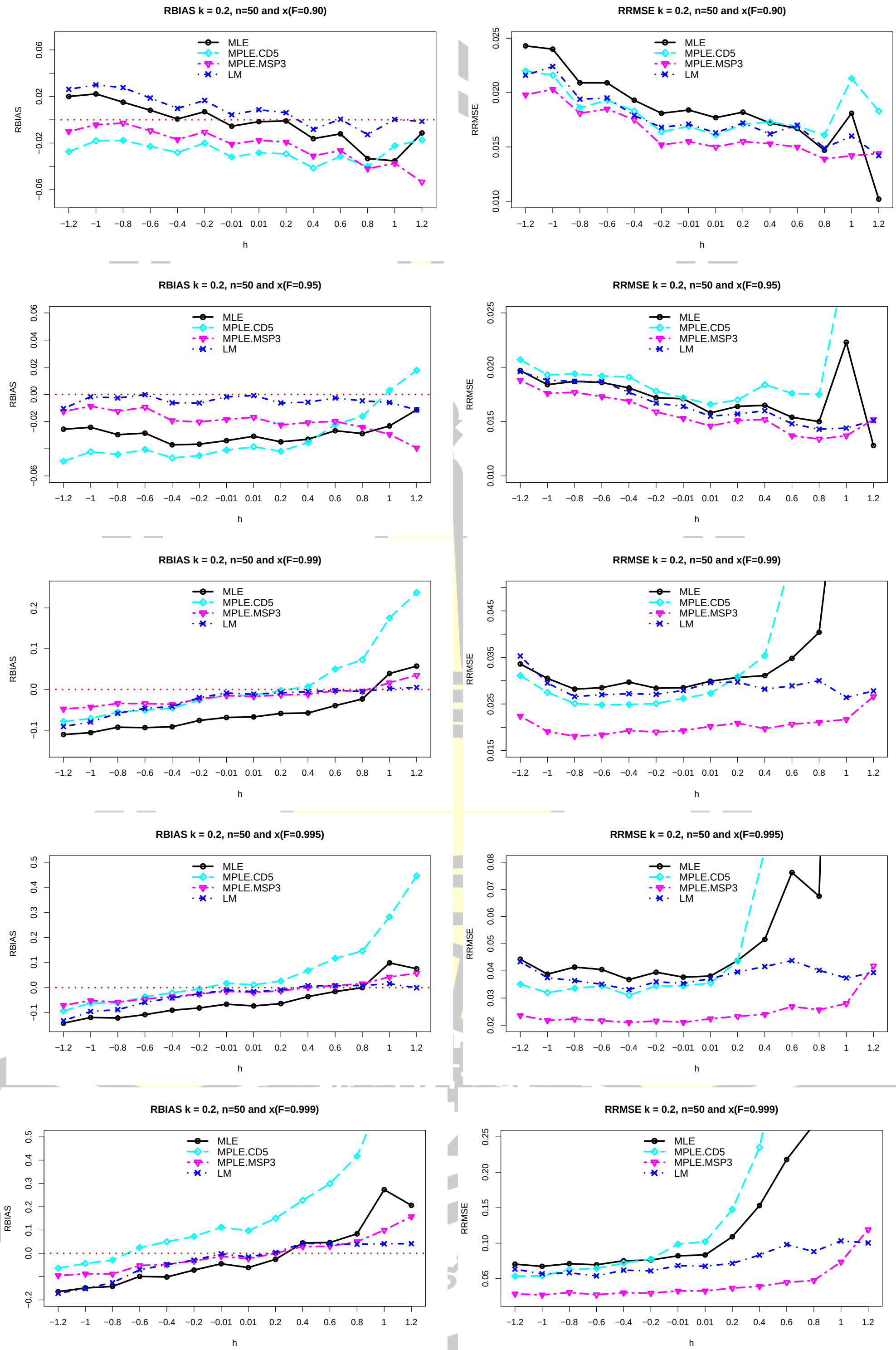


Figure 4.18: RBIAS and RRMSE of the all quantile estimators of MLE, MPLE.CD, MPLE.MSP3 and LM for value of $k = 0.2$ and sample size $n = 50$.

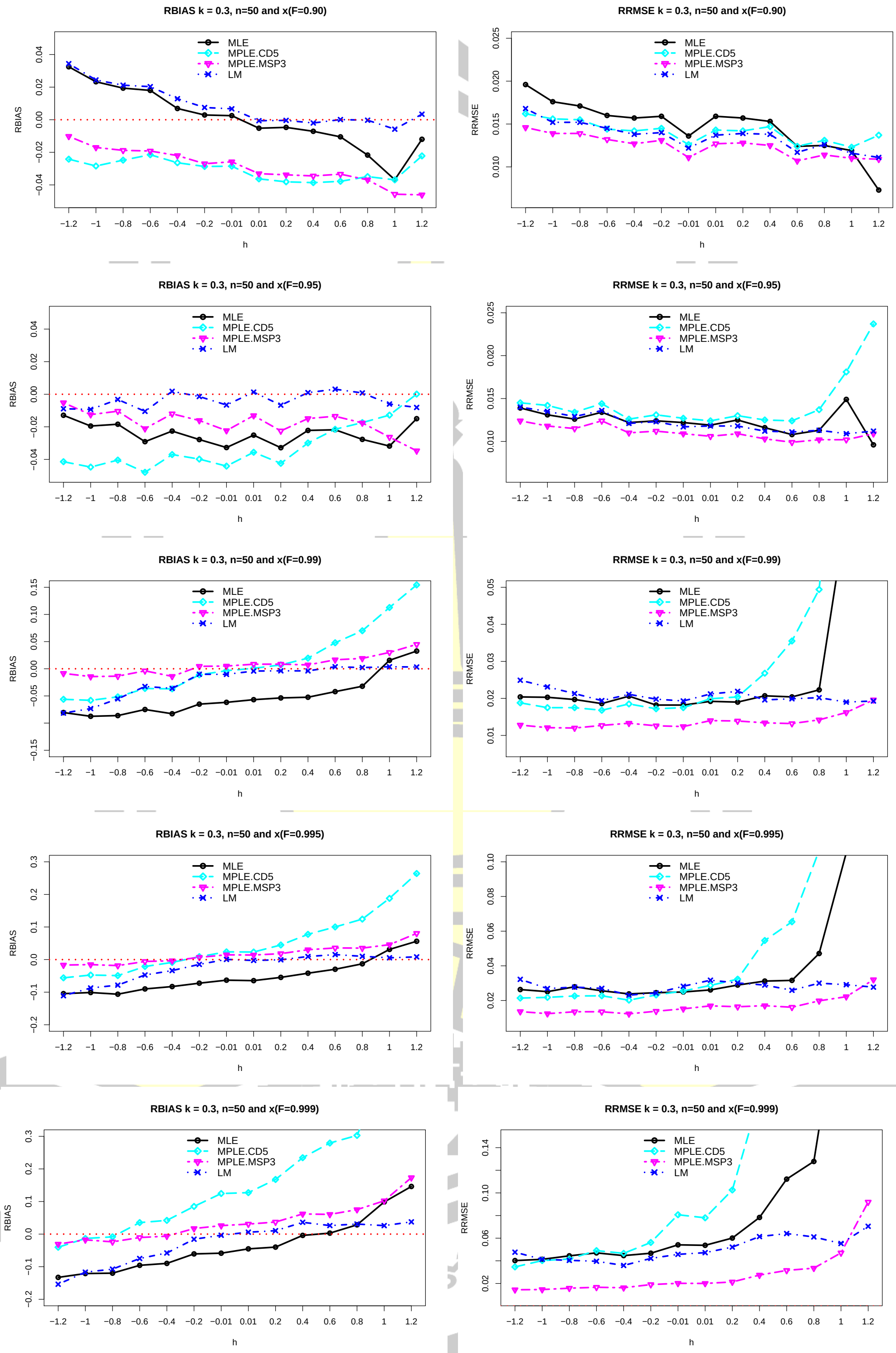


Figure 4.19: RBIAS and RRMSE of the all quantile estimators of MLE, MPLE.CD, MPLE.MSP3 and LM for value of $k = 0.3$ and sample size $n = 50$.

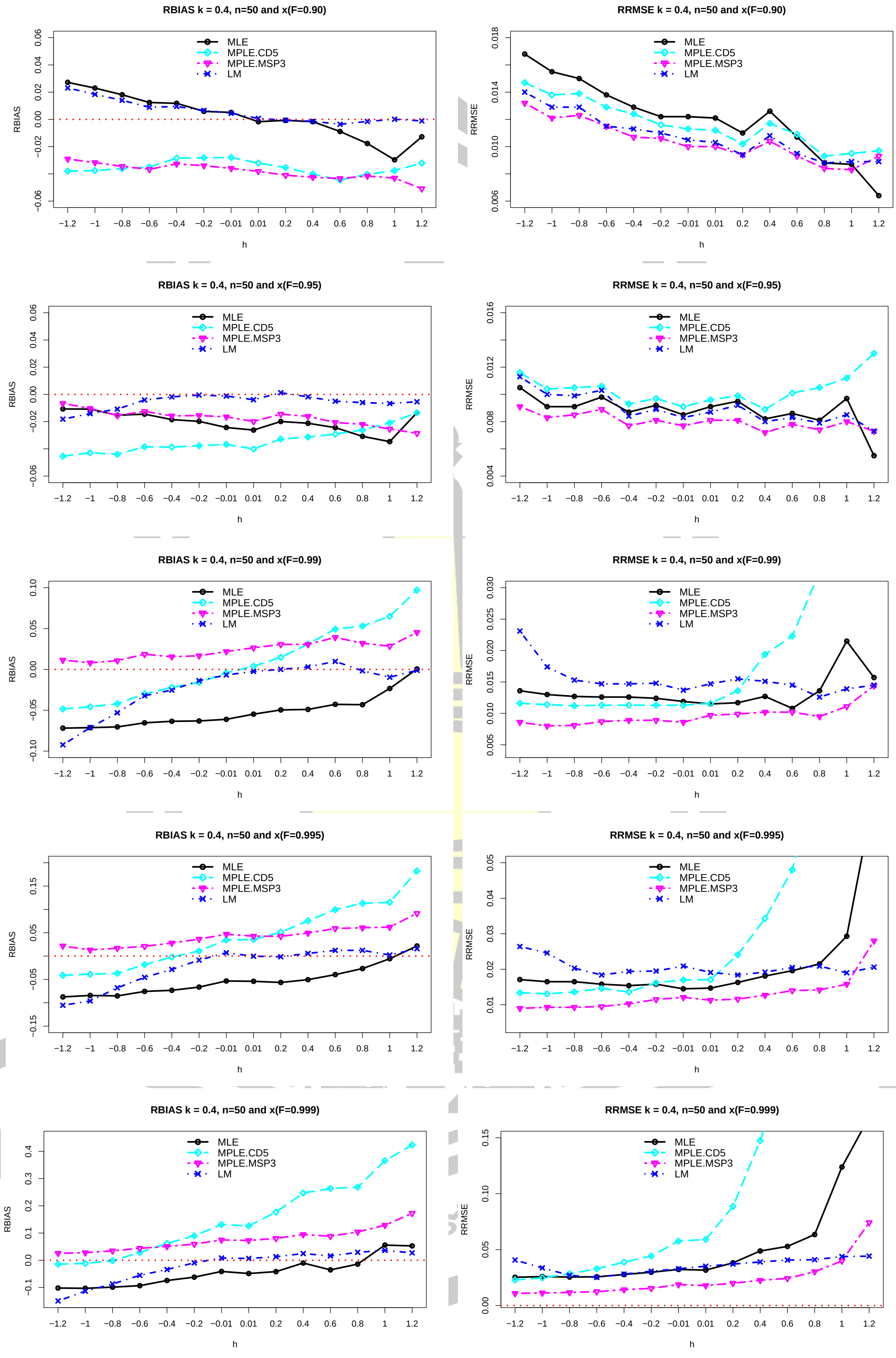


Figure 4.20: RBIAS and RRMSE of the all quantile estimators of MLE, MPLE.CD5, MPLE.MS3, MPLE.MSP3 and LM for value of $k = 0.4$ and sample size $n = 50$.

(iii) the performance of maximum penalized likelihood estimator (MPLE) with maximum likelihood estimator and L-moment estimator for four parameter kappa distribution for n=100

In Table 4.9 to Table 4.12 show the RBIAS and RRMSE of the MLE MPLE.CD5, MPLE.MS3, MPLE.MSP3, and LM parameter estimation methods in all quantiles. The two shape parameters were: the k value was -0.4 and 0.4, and the h value was -1.2 to 1.2. It was found that the differences in the quantiles and parameters results the differences in the RBIAS and RRMSE of the parameter estimation methods.

Considering Figure 4.21 to 4.30,Figure 4.21 to 4.30 illustrate the RBIAS and RRMSE of all parameters and quantiles studied. Figures 4.21 to 4.24, Figures 4.25 to 4.26, and Figures 4.27 to 4.30 show the RBIAS and RRMSE when the k value was negative, close to zero and positive respectively. Considering the RBIAS, when the h value ranged between -1.2 to 1.2;

- The MLE and MPLE.CD5 methods had a decrease in the RBIAS when the h value was negative [-1.2, -0.2];
- For the LM method, there was a gradual drop in the RBIAS;
- For the MPLE.MSP3 method, the RBIAS increased when the quantile was low $(x(F = 0.90, 0.95))$. If the quantile was over 0.95, the RBIAS would drop, except when the k value ranged between 0.2 to 0.4, the RBIAS would rise;
- For the MPLE.MS3 method, the RBIAS rose when the k value was negative. If the k value was close to 0, the RBIAS would decline. When the h value was negative in case the quantile was 0.90 and 0.95. Given the quantile was 0.99 to 0.999, the RBIAS rose continuously. Nevertheless, considering the RRMSE, it was found that all estimation methods had a rise in RRMSE when the h value reached -1.2 to 1.2.

Figure 4.21 to 4.30 concern when the k value was 0.4 and 0.3 and the quantiles were 0.99 to 0.999. It was found that the MPLE.MSP3 had the least RBIAS when the h value was negative. The MPLE.CD5 had the least RBIAS when the k value was 0.01, -0.01, and -0.1. The MPLE.MS3 had the least RRMSE when the k value was 0.2, 0.1, -0.2, -0.3, and -0.4. Concerning the RRMSE, there was no significance difference in the MPLE.MS3 and MPLE.MSP3. Concerning the k value when it was negative and close to 0, the MPLE.MS3 had the least RRMSE and MPLE.MSP3 had the lesser RRMSE than that of the LM and MLE methods. Concerning the k value when it was positive, the MPLE.MSP3 had the least RRMSE, compared to that of others in the quantiles researched.

Table 4.9: The RBIAS and RRMSE value of the estimates of K4D obtained by four estimation methods for $k = -0.4$, $-1.2 \leq h \leq 1.2$ and $n = 100$.

k	h	$x(F)$	RBIAS(RRMSE)			
			MLE	MPLE.CD5	MPLE.MS3	LM
		0.90	**-0.0078**(0.0365)	-0.0084(0.0366)	-0.0505(**0.0299**)	0.0084(0.0373)
		0.95	-0.0456(0.0478)	-0.0461(0.0480)	-0.1016(**0.0368**)	**-0.0372**(0.0441)
	-1.2	0.99	-0.1234(0.1010)	**-0.1215**(0.1001)	-0.2392(**0.0814**)	-0.1601(0.0901)
		0.995	-0.1333(0.1285)	**-0.1306**(0.1286)	-0.2865(**0.1067**)	-0.1888(0.1216)
		0.999	-0.2125(0.2292)	**-0.2098**(0.2262)	-0.4284(**0.2071**)	-0.3221(0.2128)
		0.90	**-0.0078**(0.0361)	-0.0091(0.0363)	-0.0545(**0.0290**)	0.0081(0.0365)
		0.95	-0.0467(0.0424)	-0.0468(0.0425)	-0.1089(**0.0349**)	**-0.0382**(0.0402)
	-1	0.99	-0.1201(0.0958)	**-0.1182**(0.0946)	-0.2414(**0.0816**)	-0.1548(0.0854)
		0.995	-0.1413(0.1259)	**-0.1402**(0.1250)	-0.2972(**0.1115**)	-0.1992(0.1187)
		0.999	**-0.1932**(0.2274)	-0.1913(0.2252)	-0.4303(**0.2093**)	-0.3006(0.2218)
		0.90	**-0.0075**(0.0316)	-0.0078(0.0317)	-0.0546(**0.0260**)	0.0105(0.0330)
		0.95	-0.0353(0.0419)	**-0.0354**(0.0420)	-0.1025(**0.0330**)	**-0.0269**(0.0394)
	-0.8	0.99	-0.1292(0.0950)	**-0.1274**(0.0941)	-0.2540(**0.0867**)	-0.1626(0.0883)
		0.995	-0.1486(0.1241)	**-0.1478**(0.1235)	-0.3078(**0.1172**)	-0.2067(0.1196)
		0.999	-0.1943(0.2177)	**-0.1914**(0.2187)	-0.4340(0.2105)	-0.3111(**0.2101**)
		0.90	0.0020(0.0339)	**0.0010**(0.0339)	-0.0507(**0.0268**)	0.0193(0.0351)
		0.95	-0.0408(0.0436)	-0.0410(0.0436)	-0.1162(**0.0360**)	**-0.0321**(0.0417)
-0.4	-0.6	0.99	-0.1255(0.0966)	**-0.1249**(0.0961)	-0.2599(**0.0896**)	-0.1590(0.0923)
		0.995	-0.1188(0.1305)	**-0.1172**(0.1313)	-0.30116(**0.1147**)	-0.1799(0.1205)
		0.999	-0.1887(0.2089)	**-0.1877**(**0.2079**)	-0.4404(0.2158)	-0.3052(0.2110)
		0.9	**-0.0103**(0.0333)	-0.0108(0.0333)	-0.0696(**0.0279**)	**0.0007**(0.0334)
		0.95	-0.0408(0.0421)	**-0.0407**(0.0421)	-0.1245(**0.0361**)	**-0.0344**(0.0403)
	-0.4	0.99	-0.1044(0.0931)	**-0.1042**(0.0930)	-0.2616(**0.0897**)	-0.1448(0.0908)
		0.995	-0.1049(0.1307)	**-0.1031**(0.1298)	-0.3064(**0.1156**)	-0.1730(0.1199)
		0.999	-0.1508(0.2367)	**-0.1492**(0.2367)	-0.4340(**0.2100**)	-0.2775(0.2191)
		0.9	-0.0113(0.0331)	-0.0119(0.0332)	-0.0774(**0.0287**)	**0.0064**(0.0343)
		0.95	-0.0409(0.0430)	**-0.0410**(0.0430)	-0.1376(**0.0392**)	**-0.0344**(0.0408)
	-0.2	0.99	-0.0850(0.0979)	**-0.0839**(0.0978)	-0.2602(**0.0879**)	-0.1344(0.0879)
		0.995	-0.0998(0.1428)	**-0.0990**(0.1422)	-0.3158(**0.1218**)	-0.1799(0.1243)
		0.999	-0.1395(0.2608)	**-0.1368**(0.2581)	-0.4424(**0.2168**)	-0.2749(0.2334)
		0.9	-0.0119(0.0278)	-0.0128(0.0279)	-0.0841(**0.0260**)	**0.0048**(0.0289)
		0.95	-0.0354(0.0427)	-0.0355(0.0428)	-0.1422(**0.0395**)	**-0.0308**(0.0408)
	-0.01	0.99	-0.0567(0.1123)	**-0.0558**(0.1119)	-0.2637(**0.0896**)	-0.1164(0.0928)
		0.995	-0.0806(0.1395)	**-0.0777**(0.1433)	-0.3213(0.1228)	-0.1669(**0.1213**)
		0.999	-0.0762(0.2510)	**0.0726**(0.2503)	-0.4311(0.2048)	-0.2539(**0.1983**)

Table 4.10: The RBIAS and RRMSE value of the estimates of K4D obtained by four estimation methods for $k = -0.4$, $-1.2 \leq h \leq 1.2$ and $n = 100$. (Cont.)

k	h	$x(F)$	RBIAS(RRMSE)			
			MLE	MPLE.CD5	MPLE.MS3	LM
	0.01	0.90	**-0.0054**(0.0304)	-0.0057(0.0307)	-0.0822(**0.0267**)	0.0118(0.0319)
		0.95	-0.0269(0.0416)	-0.0262(0.0420)	-0.1377(**0.0385**)	**-0.0248**(0.0385)
		0.99	0.0486(0.1120)	**-0.0466**(0.1110)	-0.2630(**0.0888**)	-0.1108(0.0948)
		0.995	-0.0476(0.1548)	**-0.0460**(0.1542)	-0.3125(**0.1173**)	-0.1426(0.1275)
		0.999	-0.0789(0.2720)	**-0.0753**(0.2708)	-0.4371(0.2102)	-0.2523(**0.2046**)
-0.4	0.2	0.90	-0.0177(0.0312)	-0.0186(0.0310)	-0.1008(**0.0303**)	**-0.0015**(0.0319)
		0.95	-0.0291(0.0408)	**-0.0285**(0.0417)	-0.1503(0.0401)	-0.0300(**0.0369**)
		0.99	-0.0385(0.1116)	**0.0373**(0.1116)	-0.2759(0.0926)	-0.1132(**0.0850**)
		0.995	-0.0175(0.1698)	**-0.0106**(0.1687)	-0.3190(**0.1212**)	-0.1316(0.1289)
		0.999	**0.0515**(0.4076)	0.0609(0.4106)	-0.4248(**0.2014**)	-0.1901(0.2360)
	0.4	0.90	0.0080(0.0302)	-0.0097(0.0301)	-0.1060(**0.0300**)	**0.0071**(0.0310)
		0.95	-0.0183(0.0431)	**-0.0176**(0.0432)	-0.1614(0.0436)	-0.0237(**0.0382**)
		0.99	**0.0042**(0.1358)	0.0159(0.1436)	-0.2727(0.0915)	-0.0890(**0.0874**)
		0.995	**0.0002**(0.1764)	0.0206(0.1871)	-0.3210(0.1199)	-0.1283(**0.1152**)
		0.999	**0.0952**(0.5737)	0.1107(0.5813)	-0.4213(**0.1963**)	-0.1725(0.2592)
	0.6	0.90	-0.0142(**0.0292**)	-0.0155(0.0301)	-0.1216(0.0325)	**-0.0025**(0.0296)
		0.95	-0.0264(0.0438)	**-0.0212**(0.0454)	-0.1815(0.0487)	-0.0350(**0.0386**)
		0.99	**0.0115**(0.1448)	0.0382(0.1592)	-0.2850(0.0984)	-0.0867(**0.0959**)
		0.995	**0.0675**(0.2627)	0.1069(0.2864)	-0.3228(**0.1214**)	-0.0925(0.1372)
		0.999	0.1701(0.6524)	0.2439(0.7716)	-0.4233(**0.1975**)	**0.1314**(0.2760)
	0.8	0.90	**-0.0081**(0.0274)	-0.0074(0.0280)	-0.1345(0.0323)	-0.0034(**0.0259**)
		0.95	**-0.0138**(0.0464)	-0.0031(0.0510)	-0.1921(0.0514)	-0.0304(**0.0366**)
		0.99	0.0429(0.1572)	0.1089(0.2021)	-0.2952(0.1013)	**0.0752(0.0884)**
		0.995	0.0860(0.2600)	0.1897(0.3881)	-0.3371(**0.1292**)	**-0.1012**(0.1250)
		0.999	0.2604(0.9734)	0.4819(1.5096)	-0.4270(**0.1994**)	**-0.0935**(0.3394)
	1.0	0.9	-0.0046(0.0323)	**0.0027**(0.0354)	-0.1473(0.0366)	0.0036(**0.0291**)
		0.95	**0.0139**(0.0563)	0.0431(0.0662)	-0.2000(0.0550)	-0.0151(**0.0385**)
		0.99	0.1390(0.2366)	0.2572(0.4562)	-0.3022(0.1050)	**-0.0626(0.0897)**
		0.995	0.1790(0.5464)	0.3159(0.6498)	-0.3516(0.1373)	**-0.1035(0.1340)**
		0.999	0.5290(2.3277)	0.9353(4.7184)	-0.4293(**0.2014**)	**-0.0949**(0.3636)
	1.2	0.9	0.0042(**0.0218**)	0.0111(0.0368)	-0.1649(0.0404)	**0.0006**(0.0273)
		0.95	0.0344(0.0475)	0.0827(0.0857)	-0.2132(0.0589)	**-0.0101(0.0372)**
		0.99	0.0166(0.4269)	0.4332(1.0801)	-0.3161(0.1138)	**-0.0566(0.0957)**
		0.995	0.1997(0.5009)	0.5262(1.5611)	-0.3591(**0.1424**)	**-0.0779**(0.1540)
		0.999	0.6219(5.5301)	1.6245(13.1193)	-0.4381(**0.2074**)	**-0.0729**(0.3733)

Table 4.11: The RBIAS and RRMSE value of the estimates of K4D obtained by four estimation methods for $k = 0.4$, $-1.2 \leq h \leq 1.2$ and $n = 100$.

k	h	$x(F)$	RBIAS(RRMSE)			
			MLE	MPLE.CD5	MPLE.MS3	LM
0.4	-1.2	0.90	0.0154(0.0082)	-0.0139(0.0073)	**-0.0024(0.0063)**	0.0220(0.0073)
		0.95	**-0.0074(0.0045)**	-0.0257(0.0050)	0.0399(0.0058)	-0.0159(0.0051)
		0.99	-0.0317(0.0057)	**-0.0305(0.0055)**	0.1033(0.0162)	-0.0714(0.0123)
		0.995	-0.0464(0.0064)	**-0.0310(0.0057)**	0.1234(0.0210)	-0.0919(0.0161)
		0.999	-0.0621(0.0116)	**-0.0281(0.0111)**	0.1535(0.0321)	-0.1195(0.0239)
	-1	0.90	0.0126(0.0071)	-0.0145(0.0065)	**-0.0022(0.0056)**	0.0159(0.0064)
		0.95	**-0.0048(0.0047)**	-0.0217(0.0049)	0.0387(0.0060)	-0.0088(0.0051)
		0.99	-0.0361(0.0051)	**-0.0295(0.0050)**	0.1028(0.0156)	-0.0538(0.0095)
		0.995	-0.0429(0.0062)	**-0.0275(0.0059)**	0.1241(0.0212)	-0.0661(0.0128)
		0.999	-0.0590(0.0103)	**-0.0277(0.0100)**	0.1666(0.0364)	-0.0906(0.0189)
	-0.8	0.90	0.0096(0.0069)	-0.0157(0.0069)	**-0.0063(0.0056)**	0.0111(0.0062)
		0.95	-0.0030(**0.0045**)	-0.0186(0.0046)	0.0386(0.0056)	**-0.0018**(0.0046)
		0.99	-0.0376(0.0050)	**-0.0309(0.0048)**	0.1067(0.0167)	-0.0390(0.0086)
		0.995	-0.0430(0.0062)	**-0.0271(0.0058)**	0.1329(0.0248)	0.0430(0.0106)
		0.999	-0.0558(**0.0110**)	**-0.0229**(0.0116)	0.1744(0.0392)	-0.0636(0.0156)
	-0.6	0.90	**0.0070**(0.0061)	-0.0158(0.0056)	-0.0083(**0.0050**)	0.0076(0.0054)
		0.95	-0.0092(**0.0046**)	-0.0230(0.0049)	0.0324(0.0057)	**-0.0041**(0.0048)
		0.99	-0.0335(0.0051)	-0.0242(**0.0050**)	0.1126(0.0184)	**-0.0227**(0.0073)
		0.995	-0.0407(0.0065)	**-0.0219(0.0062)**	0.1410(0.0268)	-0.0264(0.0099)
		0.999	-0.0467(**0.0120**)	**-0.0045**(0.0147)	0.1957(0.0485)	-0.0376(0.0139)
	-0.4	0.9	-0.0019(0.0061)	-0.0217(0.0060)	-0.0135(**0.0051**)	**0.0010**(0.0055)
		0.95	-0.0132(0.0044)	-0.0260(0.0046)	0.0309(0.0052)	**-0.0022(0.0043)**
		0.99	-0.0310(0.0050)	-0.0174(**0.0049**)	0.1207(0.0211)	**-0.0124**(0.0074)
		0.995	-0.0312(**0.0057**)	**-0.0040**(0.0066)	-0.1593(0.0328)	-0.0124(0.0093)
		0.999	-0.0382(**0.0128**)	0.0168(0.0191)	0.2229(0.0602)	**-0.0127**(0.0143)
	-0.2	0.9	-0.0038(0.0058)	-0.0233(0.0057)	-0.0158(**0.0048**)	**-0.0010**(0.0052)
		0.95	-0.0134(**0.0045**)	-0.0258(0.0048)	0.0337(0.0057)	**0.0004**(0.0045)
		0.99	-0.0268(**0.0047**)	**-0.0112**(0.0048)	0.1371(0.0253)	**-0.0017**(0.0071)
		0.995	-0.0355(**0.0070**)	-0.0073(0.0070)	0.1744(0.0389)	**-0.0053**(0.0106)
		0.999	-0.0215(**0.0149**)	-0.0433(0.0226)	0.2591(0.0796)	**-0.0003**(0.0159)
	-0.01	0.9	-0.0024(0.0062)	**-0.0214**(0.0058)	-0.0144(**0.0049**)	0.0016(0.0054)
		0.95	-0.0169(0.0045)	**-0.0278**(0.0047)	0.0345(**0.0057**)	-0.0006(0.0044)
		0.99	-0.0268(0.0050)	**-0.0100**(0.0050)	0.1495(**0.0290**)	-0.0028(0.0072)
		0.995	-0.0238(0.0065)	**0.0073**(0.0076)	0.2096(**0.0530**)	0.0064(0.0100)
		0.999	-0.0127(0.0163)	**0.0483**(0.0242)	0.3132(**0.1125**)	0.0066(0.0172)

Table 4.12: The RBIAS and RRMSE value of the estimates of K4D obtained by four estimation methods for $k = 0.4$, $-1.2 \le h \le 1.2$ and $n = 100$. (Cont.)

k	h	$x(F)$	RBIAS(RRMSE)			
			MLE	MPLE.CD5	MPLE.MS3	LM
	0.01	0.90	-0.0029(0.0055)	-0.0221(0.0052)	-0.0157(**0.0045**)	**0.0001**(0.0048)
		0.95	-0.0159(0.0044)	-0.0276(0.0046)	0.0356(0.0056)	**0.0002**(**0.0042**)
		0.99	-0.0257(**0.0050**)	-0.0088(0.0052)	0.1553(0.0318)	**0.0026**(0.0074)
		0.995	-0.0255(**0.0062**)	0.0066(0.0073)	0.2077(0.0520)	**0.0033**(0.0093)
		0.999	**-0.0126**(**0.0157**)	0.0683(0.0313)	0.3286(0.1247)	0.0193(0.0197)
	0.2	0.90	-0.0060(0.0056)	-0.0245(0.0055)	-0.0201(**0.0048**)	**-0.0014**(0.0050)
		0.95	-0.0176(0.0046)	-0.0300(0.0048)	0.0336(0.0057)	**-0.0038**(**0.0042**)
		0.99	-0.0258(**0.0044**)	-0.0052(0.0048)	0.1763(0.0380)	**0.0051**(0.0067)
		0.995	-0.0270(**0.0069**)	0.0063(0.0079)	0.2404(0.0676)	**0.0020**(0.0089)
		0.999	**-0.0123**(0.0195)	0.0711(0.0353)	0.3783(0.1613)	0.0128(**0.0183**)
	0.4	0.90	0.0029(0.0055)	-0.0247(0.0052)	-0.0233(**0.0046**)	**-0.0009**(0.0047)
		0.95	-0.0158(0.0046)	-0.0281(0.0047)	0.0356(0.0059)	**-0.0033**(**0.0042**)
		0.99	-0.0223(**0.0048**)	**0.0012**(0.0057)	0.1967(0.0462)	0.0045(0.0070)
		0.995	-0.0177(**0.0069**)	0.0245(0.0107)	0.2678(0.0815)	**0.0065**(0.0093)
		0.999	**0.0023**(0.0217)	0.0948(0.0475)	0.4366(0.2104)	0.0123(**0.0172**)
	0.6	0.90	**0.0004**(0.0055)	-0.0229(0.0052)	-0.0262(**0.0048**)	-0.0006(0.0049)
		0.95	**-0.0154**(0.0046)	-0.0278(0.0049)	0.0316(0.0056)	**-0.0058**(**0.0043**)
		0.99	-0.0231(**0.0053**)	0.0021(0.0073)	0.2076(0.0505)	**0.0014**(0.0067)
		0.995	-0.0185(0.0095)	0.0229(0.0141)	0.2831(0.0905)	**0.0032**(**0.0091**)
0.4		0.999	**-0.0091**(0.0258)	0.0895(0.0610)	0.4810(0.2539)	0.0154(**0.0159**)
	0.8	0.90	-0.0047(0.0051)	-0.0219(0.0051)	-0.0324(0.0053)	**-0.0025**(**0.0050**)
		0.95	-0.0114(**0.0033**)	-0.0204(0.0037)	0.0316(0.0048)	**-0.0020**(0.0036)
		0.99	-0.0296(**0.0055**)	-0.0079(0.0068)	0.2060(0.0502)	**-0.0067**(0.0064)
		0.995	-0.0244(**0.0084**)	0.0112(0.0146)	0.3000(0.1005)	**0.0036**(0.0090)
		0.999	-0.0164(0.0290)	0.0562(0.0572)	0.5123(0.2883)	**0.0117**(**0.0182**)
	1.0	0.9	-0.0020(**0.0043**)	-0.0273(0.0046)	-0.0368(0.0052)	**0.0013**(0.0046)
		0.95	-0.0216(**0.0037**)	-0.0226(0.0040)	0.0273(0.0049)	**-0.0012**(0.0039)
		0.99	-0.0176(**0.0053**)	**0.0004**(0.0091)	0.2153(0.0539)	0.0009(0.0066)
		0.995	-0.0105(**0.0086**)	0.0186(0.0155)	0.3096(0.1077)	**0.0064**(0.0099)
		0.999	**0.0044**(0.0212)	0.0671(0.0721)	0.5428(0.3251)	**0.0089**(**0.0176**)
	1.2	0.9	-0.0043(**0.0030**)	-0.0239(0.0043)	-0.0439(0.0057)	**0.0009**(0.0045)
		0.95	-0.0039(**0.0026**)	-0.0164(0.0038)	0.0180(0.0046)	**-0.0024**(0.0040)
		0.99	0.0056(**0.0030**)	0.0106(0.0064)	0.2214(0.0569)	**0.0017**(0.0061)
		0.995	0.0065(**0.0038**)	0.0218(0.0129)	0.3227(0.1171)	**0.0010**(0.0088)
		0.999	0.0172(**0.0146**)	0.0681(0.0621)	0.5796(0.3647)	**0.0146**(0.0161)

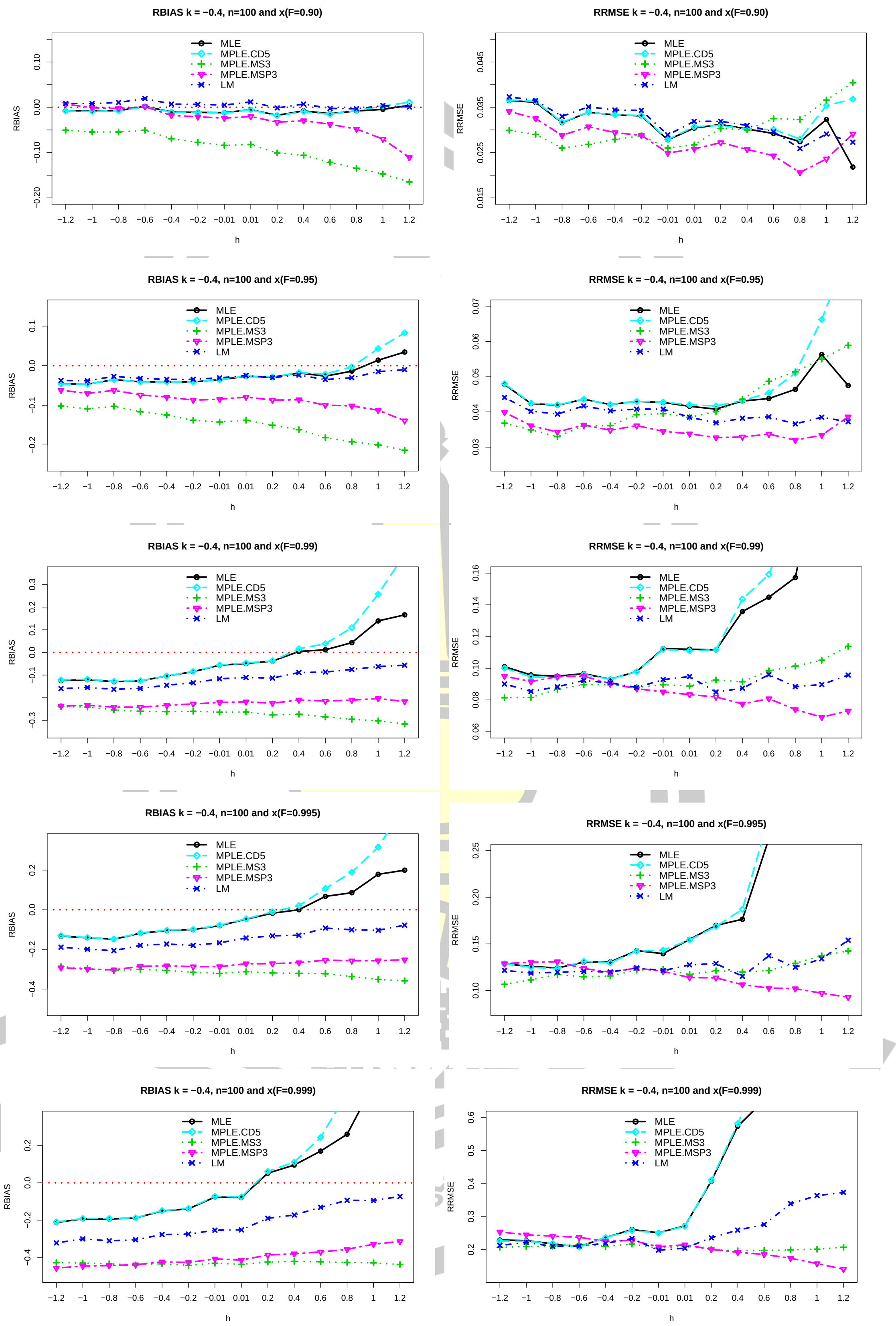


Figure 4.21: RBIAS and RRMSE of the all quantile estimators of MLE, MPLE.CD, MPLE.MS3, MPLE.MSP3 and LM for value of $k = -0.4$ and sample size $n = 100$.

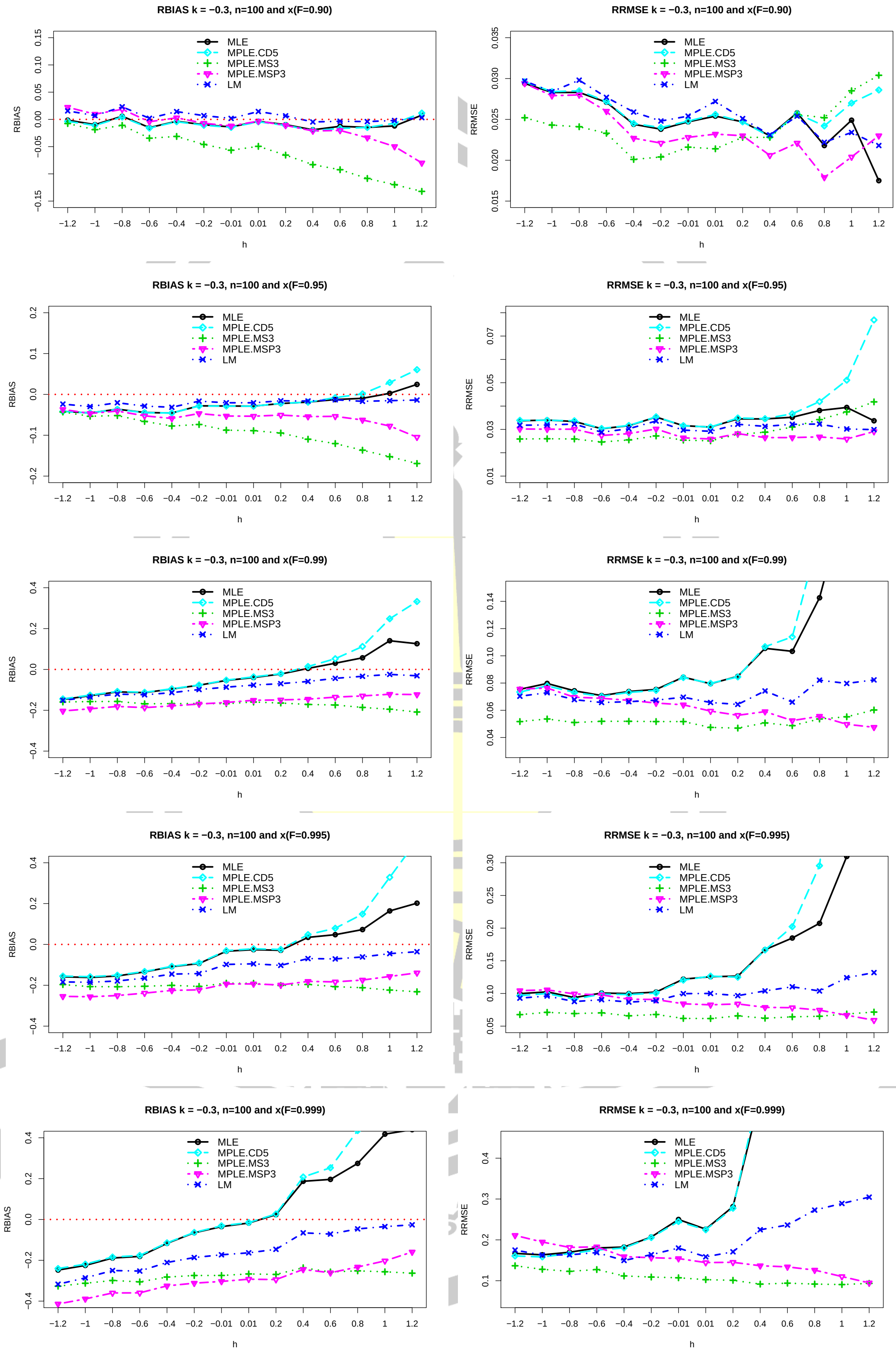


Figure 4.22: RBIAS and RRMSE of the all quantile estimators of MLE, MPLE.CD, MPLE.MS3, MPLE.MSP3 and LM for value of $k = -0.3$ and sample size $n = 100$.

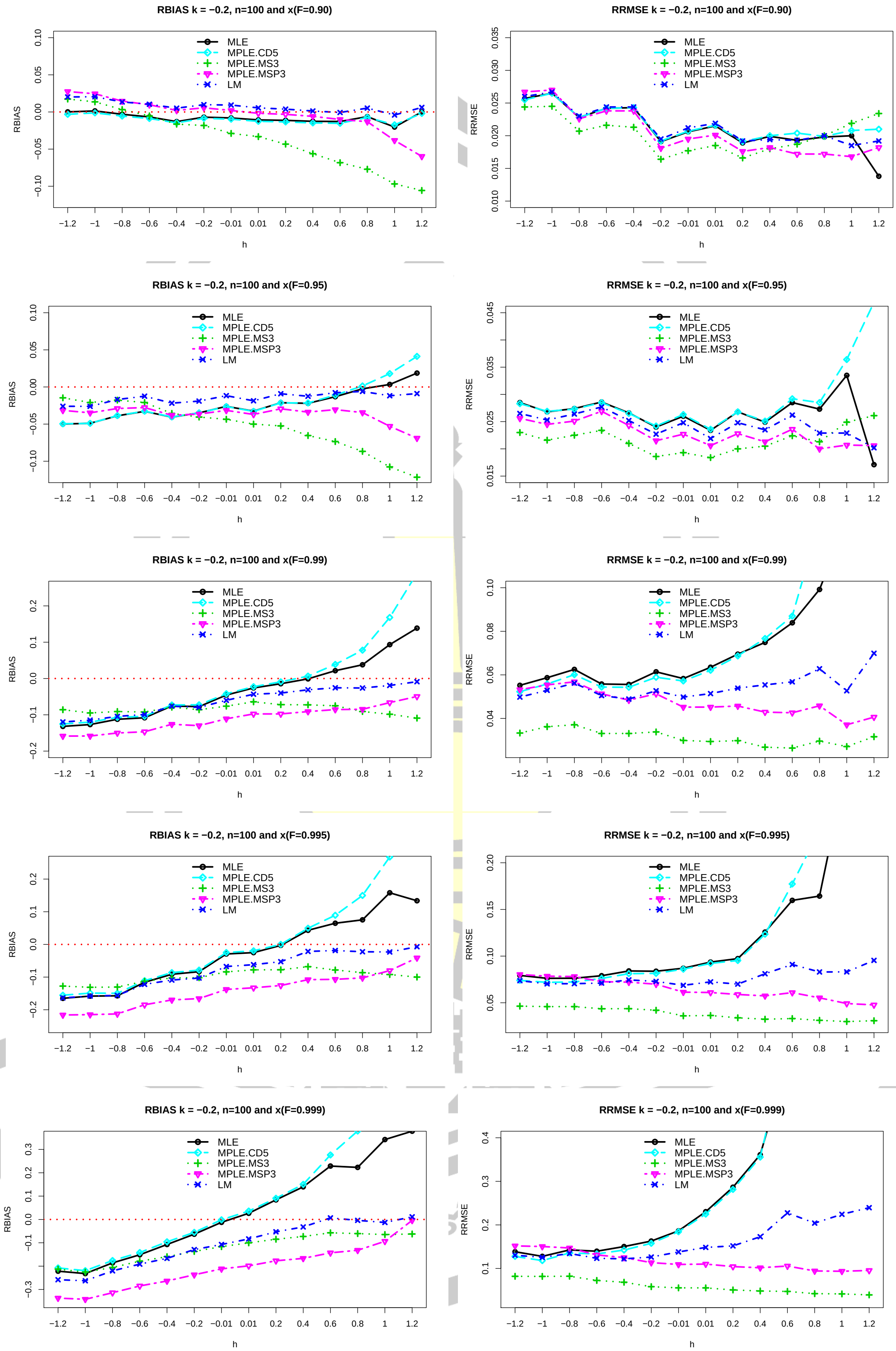


Figure 4.23: RBIAS and RRMSE of the all quantile estimators of MLE, MPLE.CD, MPLE.MS3, MPLE.MSP3 and LM for value of $k = -0.2$ and sample size $n = 100$.

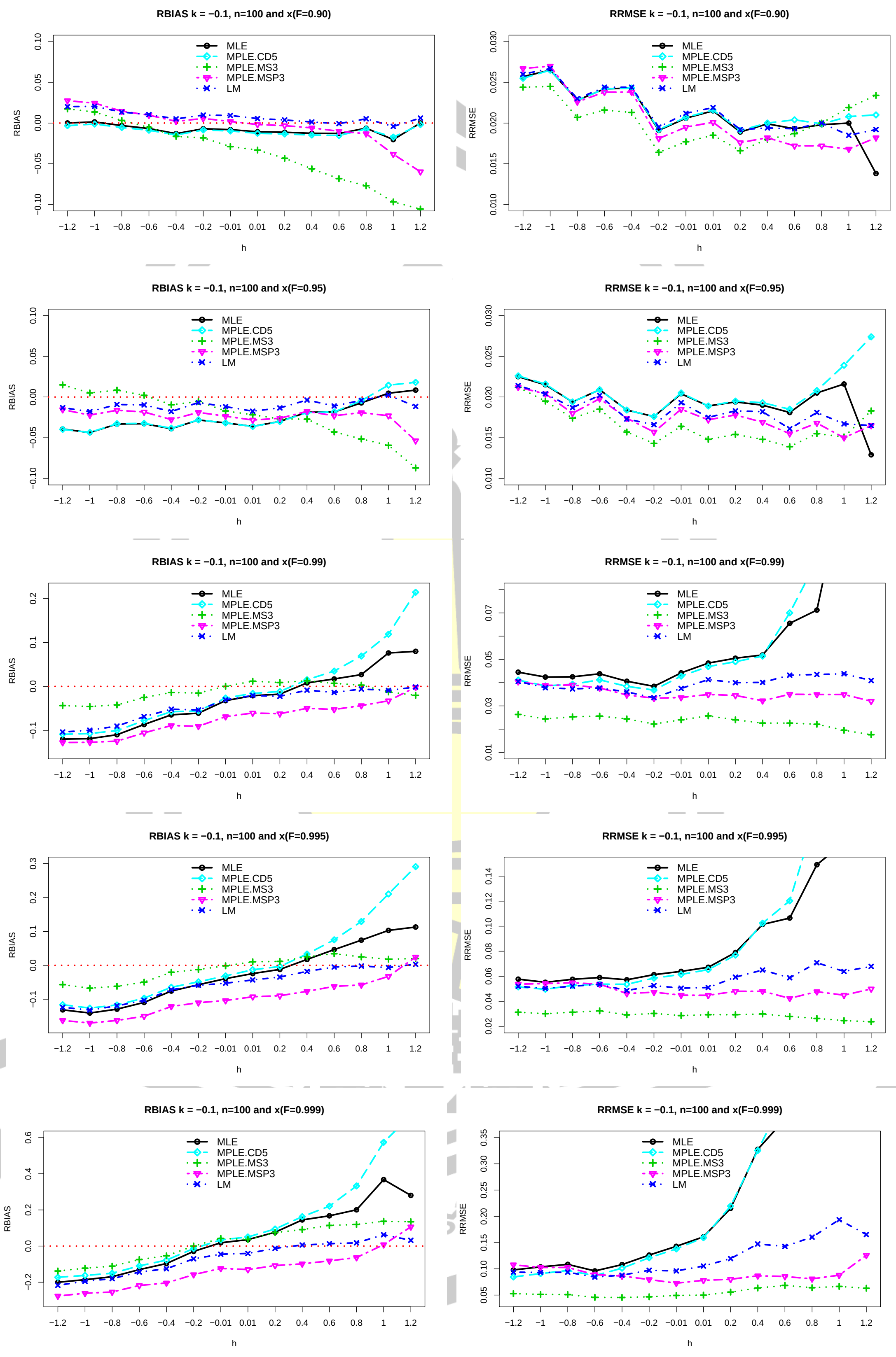


Figure 4.24: RBIAS and RRMSE of the all quantile estimators of MLE, MPLE.CD, MPLE.MS3, MPLE.MSP3 and LM for value of $k = -0.1$ and sample size $n = 100$.

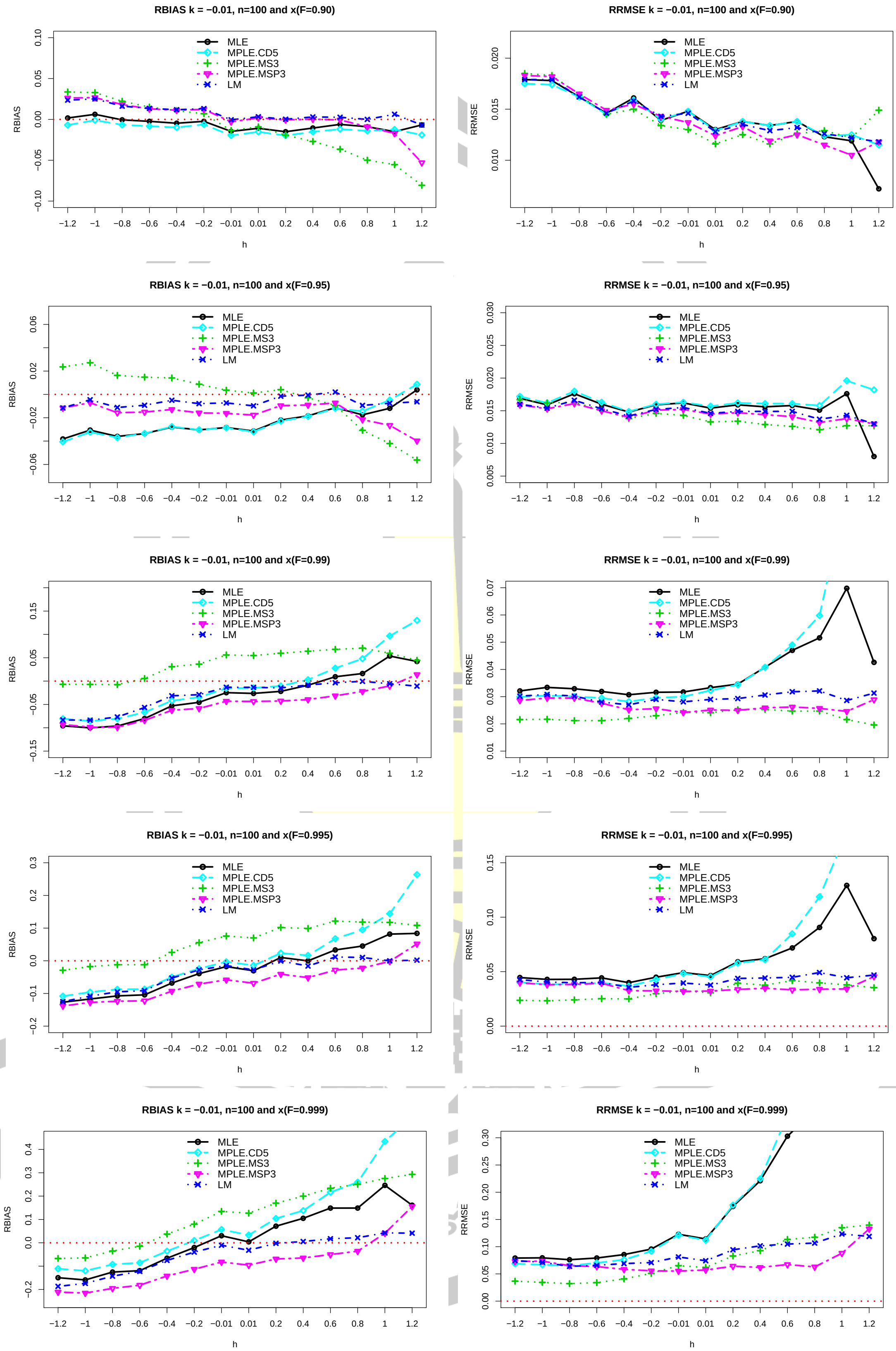


Figure 4.25: RBIAS and RRMSE of the all quantile estimators of MLE, MPLE.CD, MPLE.MS3, MPLE.MSP3 and LM for value of $k = -0.01$ and sample size $n = 100$.

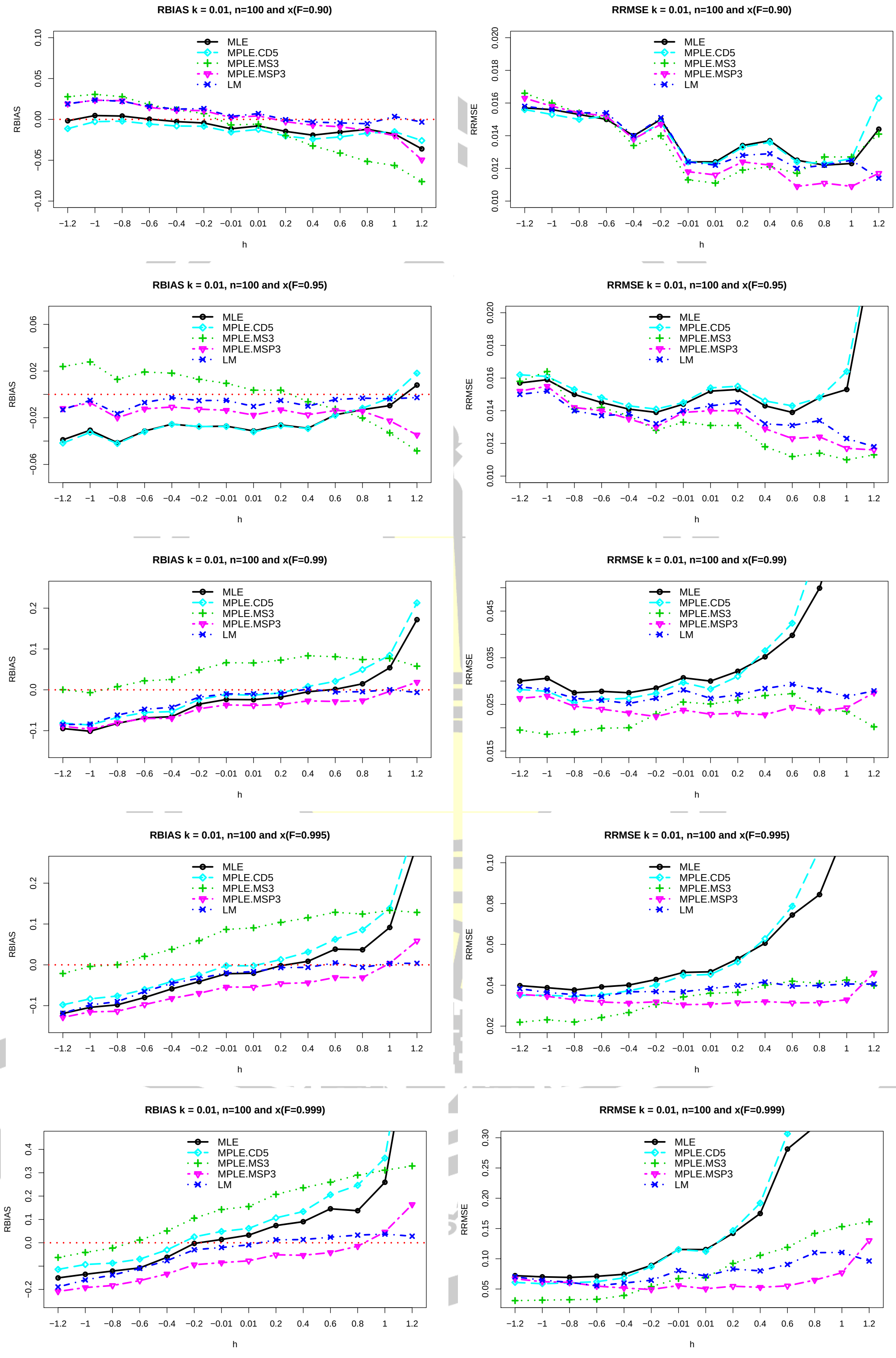


Figure 4.26: RBIAS and RRMSE of the all quantile estimators of MLE, MPLE.CD, MPLE.MSP3 and LM.MSP3 for value of $k = 0.01$ and sample size $n = 100$.

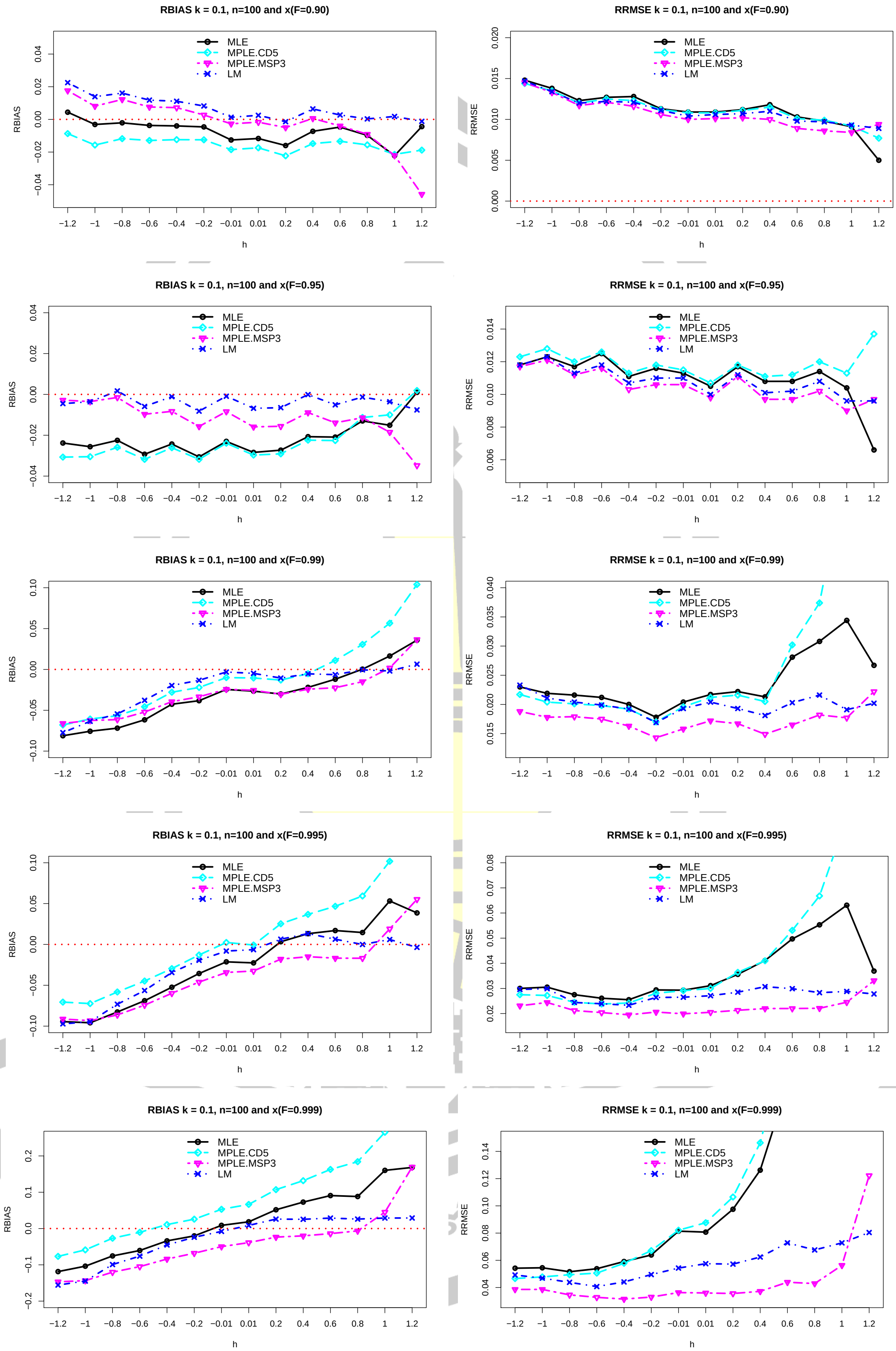


Figure 4.27: RBIAS and RRMSE of the all quantile estimators of MLE, MPLE.CD, MPLE.MSP3 and LM.MSP3 for value of $k = 0.1$ and sample size $n = 100$.

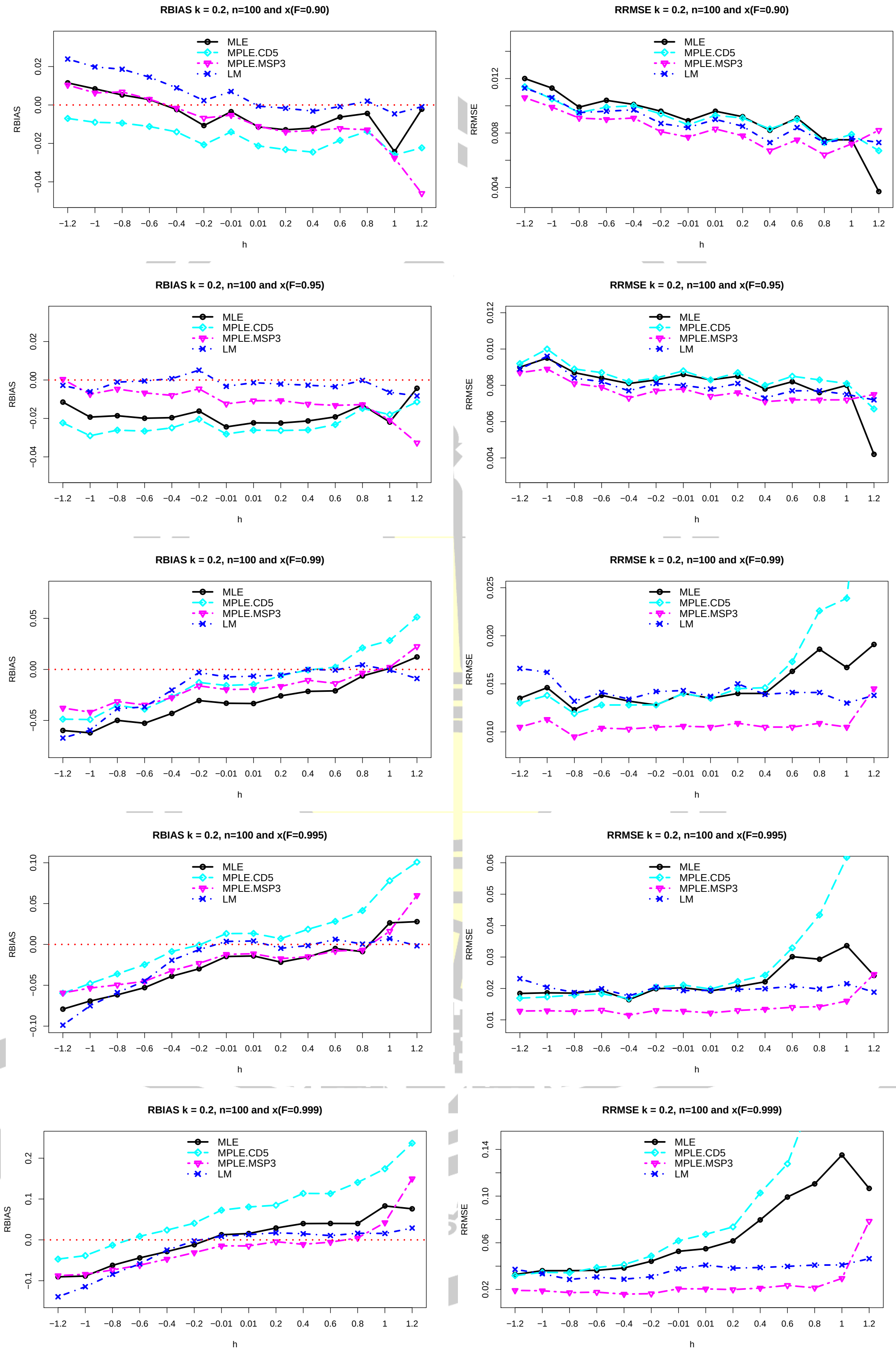


Figure 4.28: RBIAS and RRMSE of the all quantile estimators of MLE, MPLE.CD, MPLE.MSP3 and LM.MSP3 for value of $k = 0.2$ and sample size $n = 100$.

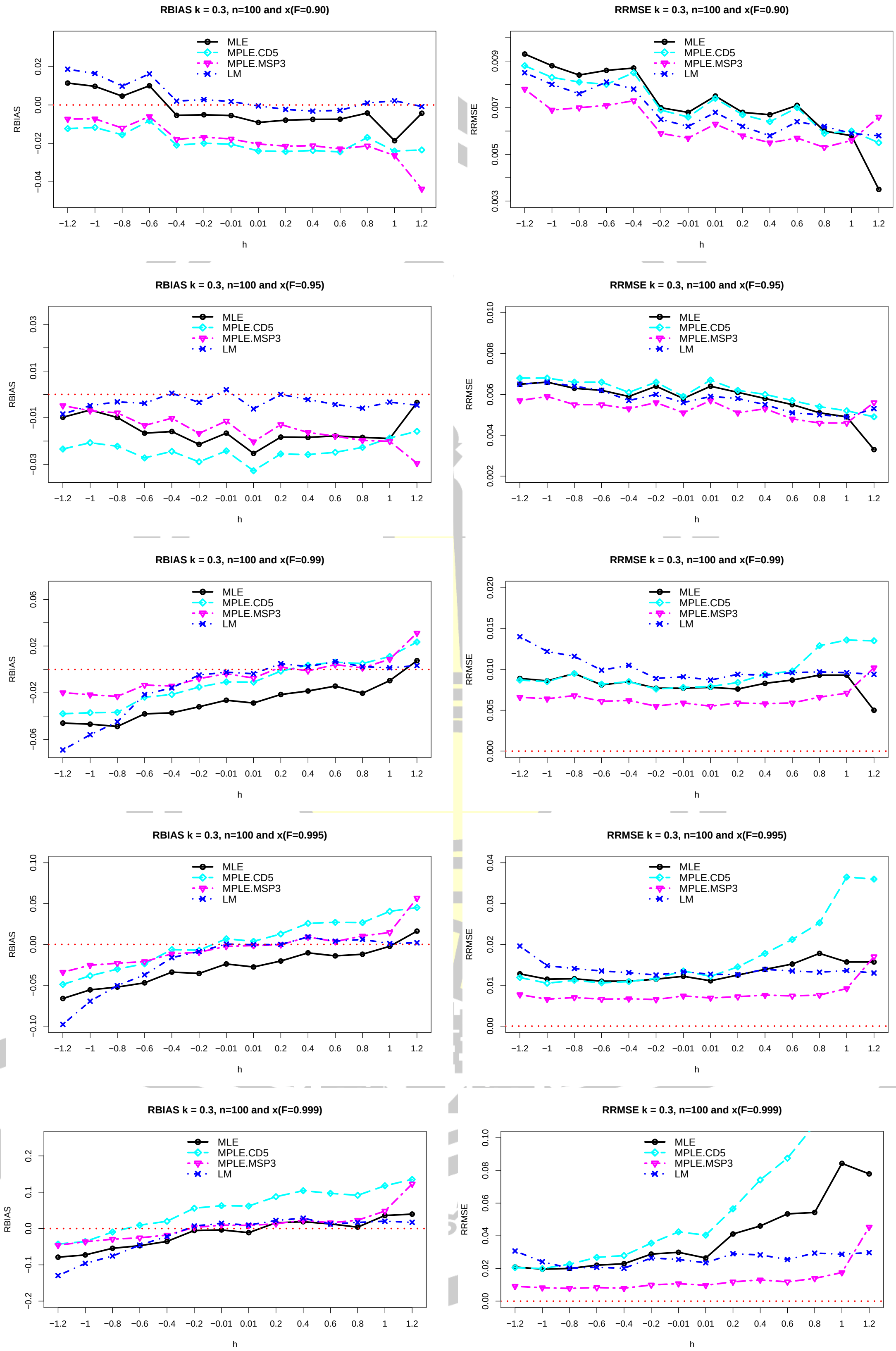


Figure 4.29: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MLE, MPLE.CD, MPLE.MSP3 and LM for value of $k = 0.3$ and sample size $n = 100$.

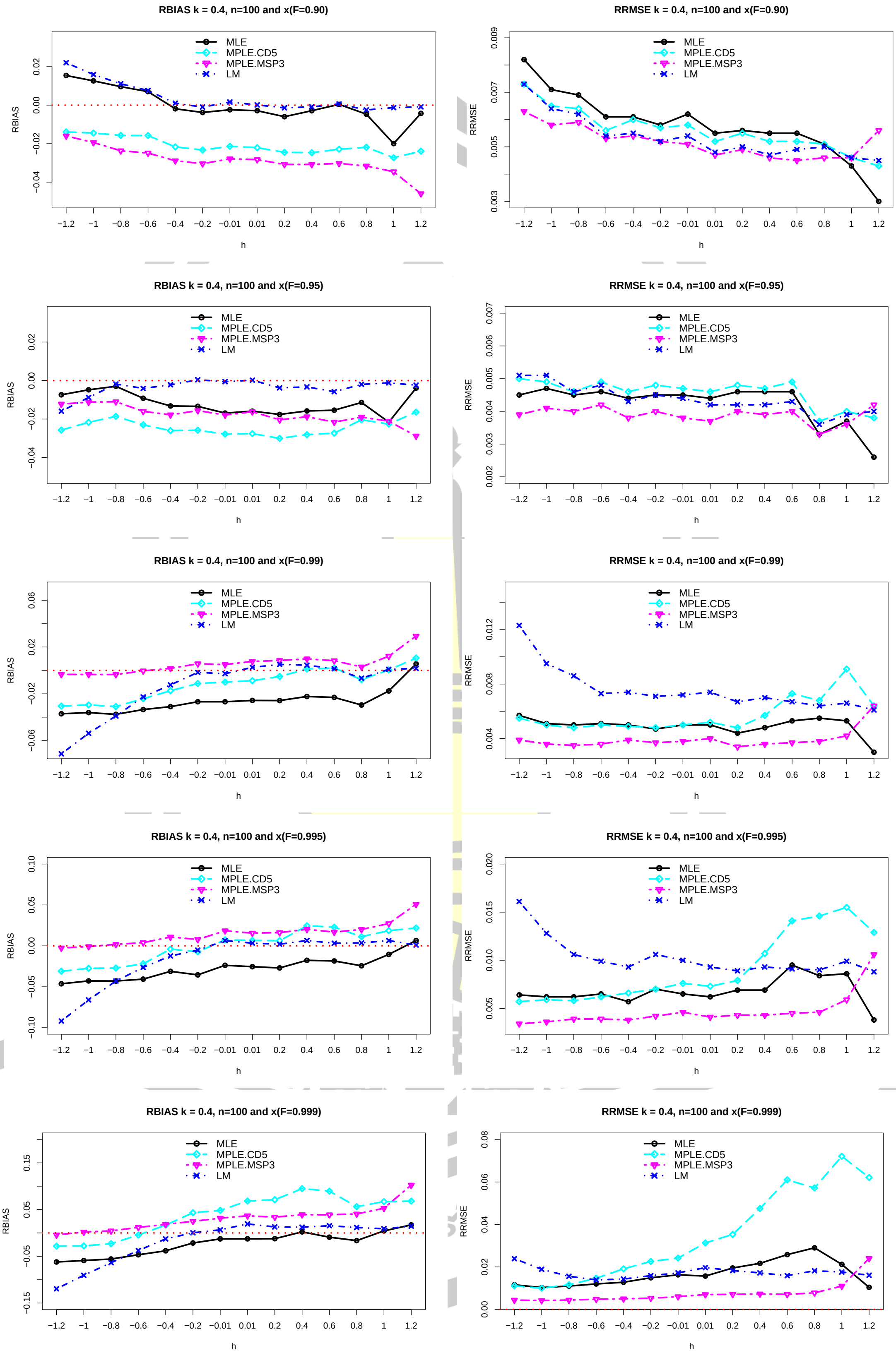


Figure 4.30: RBIAS and RRMSE of the all quantile estimators of MLE, MPLE.CD, MPLE.MSP3 and LM for value of $k = 0.4$ and sample size $n = 100$.

(iv) Comparisons the performance of maximum penalized likelihood estimator (MPLE) with maximum likelihood estimator and L-moment estimator for four parameter kappa distribution

Table 4.13: The RBIAS and RRMSE value of the estimates of quantile estimators of K4D for $k = -0.2$, $h = -1$.

n	$x(F)$	RBIAS(RRMSE)			
		MLE	MPLE.CD5	MPLE.MS3	LM
30	0.90	0.0346(0.0849)	0.0101(0.0870)	**0.0032(0.0650)**	0.0373(0.0826)
	0.95	-0.0529(0.1002)	-0.0286(0.1092)	**-0.0128(0.0610)**	-0.0036(0.0931)
	0.99	-0.2198(0.2251)	-0.1112(0.2197)	**-0.0851(0.0687)**	-0.1744(0.1541)
	0.995	-0.2655(0.3173)	**-0.1165**(0.3437)	-0.1223(**0.0739**)	-0.2290(0.1929)
	0.999	-0.3420(0.4526)	-0.1238(0.6219)	**-0.2010(0.1074)**	-0.3359(0.3126)
50	0.90	**0.0012**(0.0517)	-0.0122(0.0508)	-0.0023(**0.0424**)	0.0183(0.0506)
	0.95	-0.0616(0.0512)	-0.0610(0.0525)	**-0.0193(0.0382)**	-0.0286(0.0489)
	0.99	-0.1827(0.1083)	-0.1571(0.0977)	**-0.0883(0.0499)**	-0.1511(0.0919)
	0.995	-0.2229(0.1561)	-0.1840(0.1428)	**-0.1227(0.0613)**	-0.1994(0.1294)
	0.999	-0.2803(0.2510)	-0.2282(0.2154)	**-0.1952(0.0917)**	-0.2864(0.2082)
100	0.90	0.0014(0.0265)	**-0.0012**(0.0265)	0.0136(**0.0245**)	-0.0205(0.0267)
	0.95	-0.0485(0.0268)	0.0490(0.0269)	**-0.0209(0.0216)**	0.0261(0.0253)
	0.99	-0.1269(0.0587)	-0.1202(0.0558)	**-0.0949(0.0362)**	-0.1147(0.0529)
	0.995	-0.1581(0.0762)	-0.1490(0.0719)	**-0.1311(0.0459)**	-0.1580(0.0703)
	0.999	-0.2319(0.1278)	**0.2195**(0.1181)	-0.2255(**0.0818**)	-0.2627(0.1255)

In Tables 4.13 and 4.14 illustrate the RBIAS and RRMSE when the k value was positive. Tables 4.15 and 4.16 show the RBIAS and RRMSE when the k value was negative. Overall, it was found that both RBIAS and RRMSE of four parameter estimation methods declined when the size of the samples increased.

Table 4.14: The RBIAS and RRMSE value of the estimates of quantile estimators of K4D for $k = -0.2,\ h = 0.2$.

n	$x(F)$	RBIAS(RRMSE)			
		MLE	MPLE.CD5	MPLE.MS3	LM
30	0.90	**0.0028**(0.0741)	0.0153(0.0972)	-0.1057(**0.0539**)	0.0164(0.0733)
	0.95	-0.0402(0.1008)	**0.0050**(0.1256)	-0.1311(**0.0570**)	-0.0299(0.0797)
	0.99	**-0.0662**(0.2664)	0.1395(0.6414)	-0.1668(**0.0696**)	-0.1002(0.1614)
	0.995	**-0.0452**(0.6573)	0.2328(1.1579)	-0.1715(**0.0779**)	-0.1298(0.1923)
	0.999	**0.0588**(1.3336)	0.5955(4.8252)	-0.1812(**0.0956**)	-0.1290(0.4368)
50	0.90	**0.0050**(0.0394)	-0.0103(0.0404)	-0.0808(**0.0334**)	0.0095(0.0411)
	0.95	-0.0247(0.0558)	-0.0163(0.0591)	-0.0883(**0.0363**)	**-0.0100**(0.0505)
	0.99	-0.0531(0.1434)	**-0.0111**(0.1510)	-0.1186(**0.0468**)	-0.0755(0.0974)
	0.995	**-0.0202**(0.2269)	0.0447(0.2604)	-0.1097(**0.0525**)	-0.0682(0.1385)
	0.999	**-0.0213**(0.4795)	0.1193(0.5315)	-0.1413(**0.0697**)	-0.1155(0.2520)
100	0.90	-0.0113(0.0189)	-0.0134(0.0191)	-0.0433(**0.0166**)	**0.0038**(0.0192)
	0.95	-0.0212(0.0268)	-0.0214(0.0268)	-0.0524(**0.0200**)	**0.0091**(0.0248)
	0.99	-0.0144(0.0695)	**-0.0092**(0.0687)	-0.0720(**0.0298**)	-0.0401(0.0539)
	0.995	-0.0029(0.0973)	**-0.0000**(0.0953)	-0.0775(**0.0339**)	-0.0527(0.0700)
	0.999	0.0839(0.2865)	0.0908(0.2815)	-0.0848(**0.0507**)	**-0.0529**(0.1581)

Table 4.15: The RBIAS and RRMSE value of the estimates of quantile estimators of K4D for $k = 0.3,\ h = -0.4$.

n	$x(F)$	RBIAS(RRMSE)			
		MLE	MPLE.CD5	MPLE.MSP3	LM
30	0.90	0.0311(0.0287)	-0.0232(0.0250)	-0.0244(**0.0225**)	**0.0226**(0.0242)
	0.95	-0.0324(0.0224)	-0.0454(0.0231)	-0.0106(**0.0206**)	**-0.0036**(0.0223)
	0.99	-0.1146(0.0333)	-0.0213(0.0317)	0.0216(**0.0219**)	**-0.0332**(0.0314)
	0.995	-0.1340(0.0475)	**-0.0075**(0.0597)	0.0278(**0.0287**)	-0.0483(0.0453)
	0.999	-0.1667(0.10741)	0.0927(0.1181)	**0.0372(0.0350)**	-0.0671(0.0694)
50	0.90	**0.0069**(0.0157)	-0.0264(0.0142)	-0.0220(**0.0127**)	0.0012(0.0138)
	0.95	-0.0226(0.0122)	-0.0370(0.0126)	-0.0120(**0.0110**)	**0.0018**(0.0121)
	0.99	-0.0828(0.0206)	-0.0370(0.0185)	**-0.0140(0.0133)**	-0.0361(0.0211)
	0.995	-0.0828(0.0238)	-0.0087(0.0202)	**-0.0036(0.0123)**	-0.0337(0.0231)
	0.999	-0.0899(0.0446)	0.0421(0.0446)	**-0.0063(0.0162)**	-0.0579(0.0359)
100	0.90	-0.0054(0.0087)	-0.0209(0.0085)	-0.0179(**0.0073**)	**-0.0020**(0.0078)
	0.95	-0.0159(0.0059)	-0.0244(0.0061)	-0.0102(**0.0053**)	**0.0000**(0.0057)
	0.99	-0.0372(0.0085)	-0.0215(0.0085)	**-0.0142(0.0062)**	-0.0158(0.0105)
	0.995	-0.0339(0.0110)	**-0.0063**(0.0109)	-0.0116(**0.0067**)	-0.0162(0.0131)
	0.999	-0.0355(0.0229)	0.0200(0.0279)	**-0.0181(0.0079)**	-0.0217(0.0201)

Table 4.16: The RBIAS and RRMSE value of the estimates of quantile estimators of K4D for $k = 0.3,\ h = 0.4$.

n	$x(F)$	RBIAS(RRMSE)			
		MLE	MPLE.CD5	MPLE.MSP3	LM
30	0.90	-0.0114(0.0216)	-0.0357(0.0223)	-0.0447(**0.0194**)	**0.0018**(0.0208)
	0.95	-0.0420(0.0206)	-0.0105(0.0287)	-0.0168(**0.0189**)	**-0.0031**(0.0202)
	0.99	-0.0806(0.0362)	0.0972(0.0761)	0.0335(**0.0246**)	**-0.0078**(0.0348)
	0.995	-0.0957(0.0502)	0.1935(0.1698)	0.0606(**0.0324**)	**0.0049**(0.0479)
	0.999	-0.0756(0.1998)	0.4695(0.8015)	0.1003(**0.0499**)	**0.0212**(0.0957)
50	0.90	-0.0071(0.0153)	-0.0386(0.0147)	-0.0345(**0.0125**)	**0.0020**(0.0138)
	0.95	-0.0222(0.0116)	-0.0299(0.0125)	-0.0149(**0.0103**)	**0.0010**(0.0112)
	0.99	-0.0524(0.0207)	0.0196(0.0268)	-0.0071(**0.0134**)	**-0.0039**(0.0196)
	0.995	-0.0418(0.0312)	0.0775(0.0546)	0.0302(**0.0170**)	**0.0094**(0.0289)
	0.999	**-0.0040**(0.0783)	0.2344(0.1853)	0.0619(**0.0274**)	0.0362(0.0614)
100	0.90	-0.0075(0.0067)	-0.0237(0.0064)	-0.0215(**0.0055**)	**-0.0032**(0.0058)
	0.95	-0.0184(0.0058)	-0.0258(0.0060)	-0.0163(**0.0053**)	**-0.0022**(0.0055)
	0.99	-0.0185(0.0083)	0.0037(0.0094)	**-0.0012**(**0.0058**)	0.0024(0.0093)
	0.995	-0.0103(0.0139)	0.0259(0.0179)	0.0091(**0.0076**)	**0.0092**(0.0139)
	0.999	**-0.0187**(0.0460)	0.1042(0.0742)	0.0229(**0.0130**)	0.0285(0.0283)

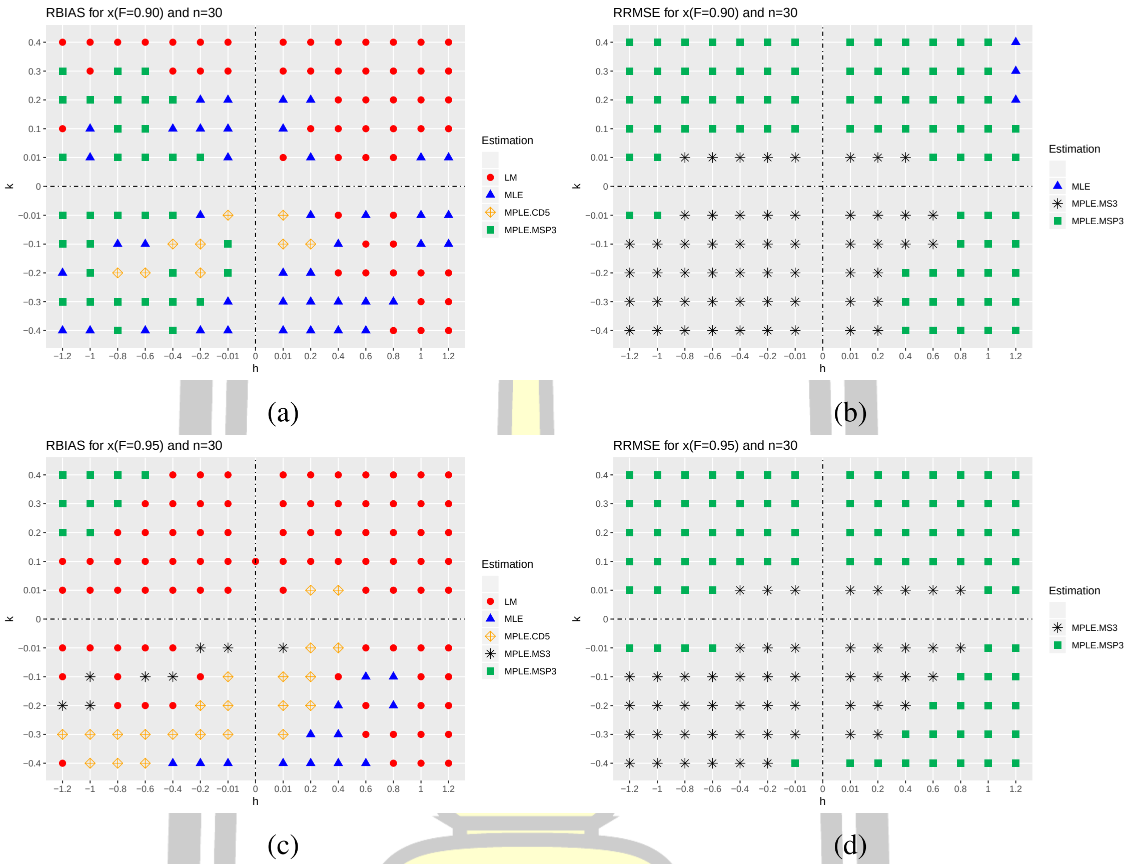


Figure 4.31: The best of estimation for area of parameter space when sample size $n = 30$ for 0.90 and 0.95 quantile estimate

The comparisons of the efficiency of parameter estimation were performed by considering the RBIAS or RRMSE, meaning that the least of RBIAS or RRMSE determined the most efficient value.

In Figure 4.31a, 4.31b, 4.31c and 4.31d indicate the best scales of the parameter estimation methods being studied, regarding the RBIAS and RRMSE in the quantiles that were 0.90 and 0.95 respectively. Figure 4.32a, 4.32b, 4.32c, 4.32d,4.32e and 4.32f show the best scales of the parameter estimation methods in the study in case that the RBIAS and RRMSE of the quantiles were 0.99, 0.995, and 0.95 respectively. When the k value was positive, the MPLE.MSP3 was the most efficient, compared to that of the MLE, MPLE.MS3, and LM methods when the k value was negative. The MPLE.MS3 and MPLE.MSP3 were more efficient than MLE and LM methods.

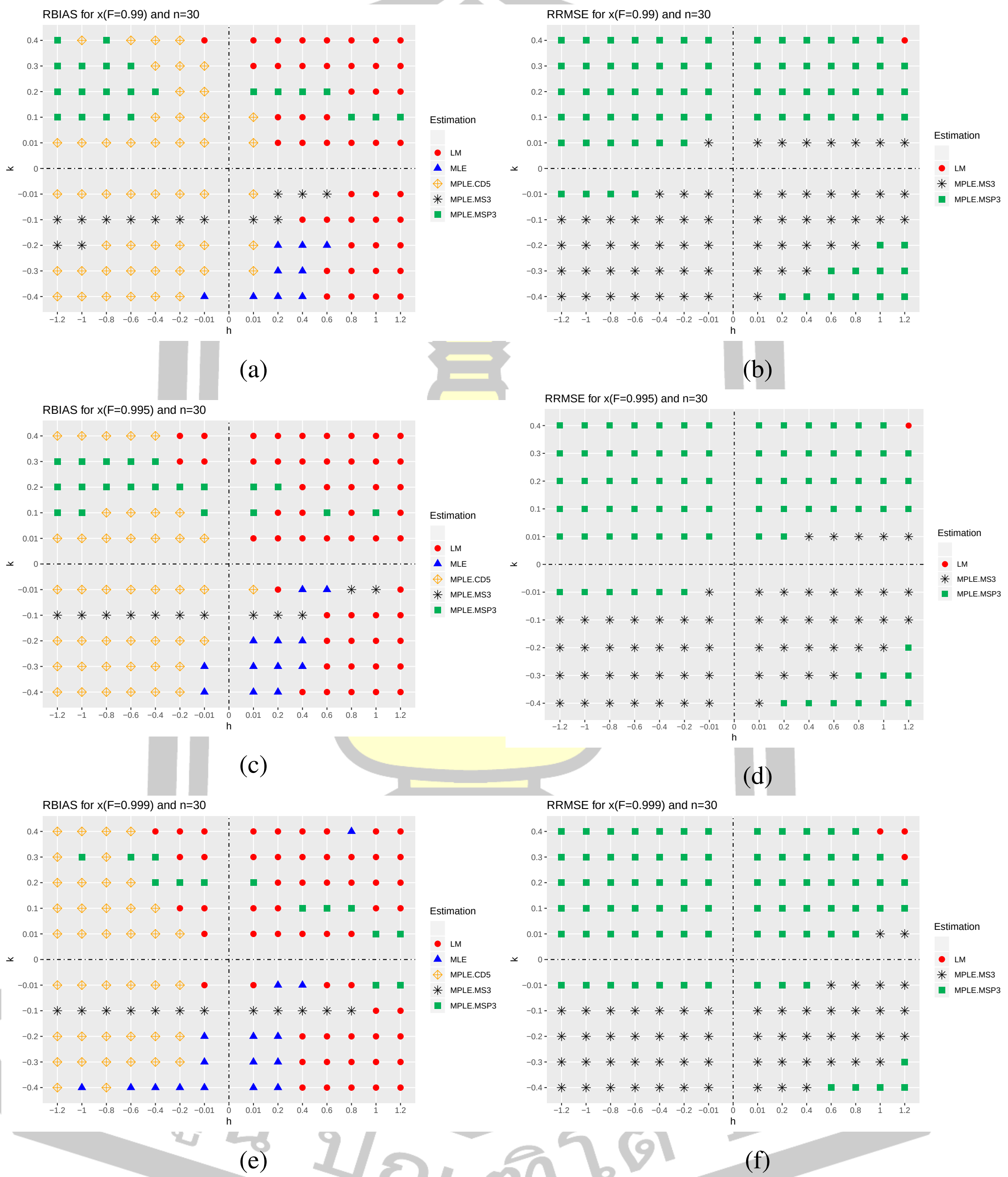
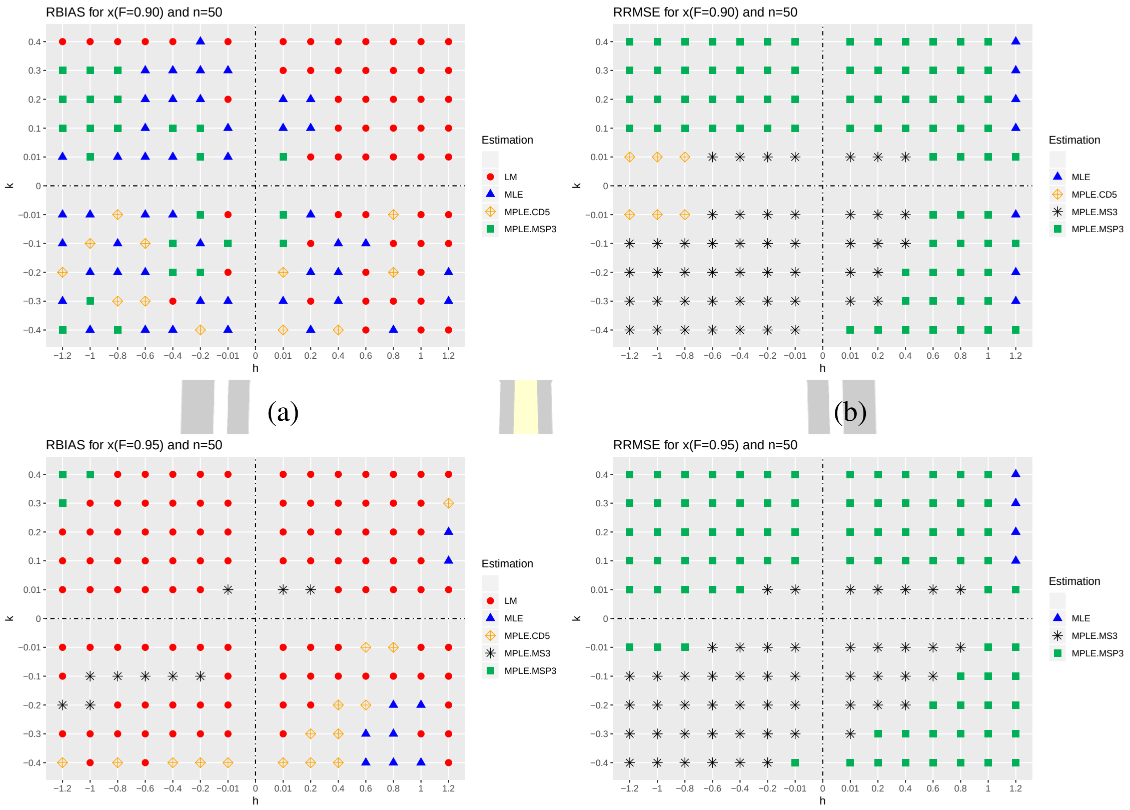


Figure 4.32: The best of estimation for area of parameter space when sample size $n = 30$ for 0.99 0.995 and 0.999 quantile estimate



(a) (b) (c) (d)

Figure 4.33: The best of estimation for area of parameter space when sample size $n = 50$ for 0.90 and 0.95 quantile estimate

In Figure 4.33a, 4.33b, 4.33c and 4.33d sindicate the best scales of the parameter estimation methods being studied, regarding the RBIAS and RRMSE in the quantiles that were 0.90 and 0.95 respectively. Figure 4.34a, 4.34b, 4.34c, 4.34d,4.34e and 4.34f show the best scales of the parameter estimation methods in the study in case that the RBIAS and RRMSE of the quantiles were 0.99, 0.995, and 0.95 respectively. When the k value was positive, the MPLE.MSP3 was the most efficient, compared to that of the MLE, MPLE.MS3, and LM methods (except when h = -1.2) when the k value was negative. The MPLE.MS3 and MPLE.MSP3 were more efficient than MLE and LM methods.

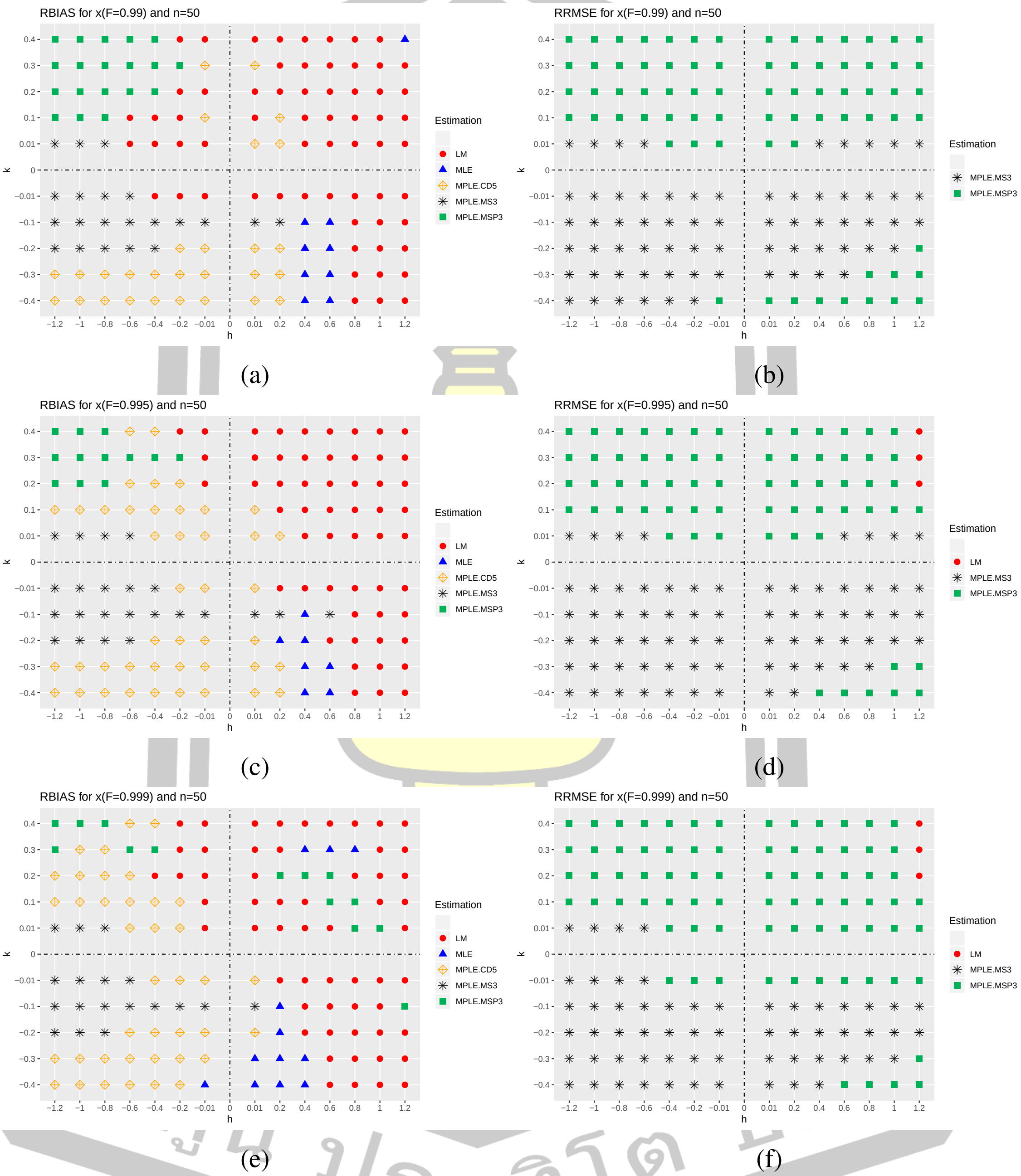


Figure 4.34: The best of estimation for area of parameter space when sample size $n = 50$ for 0.99 0.995 and 0.999 quantile estimate

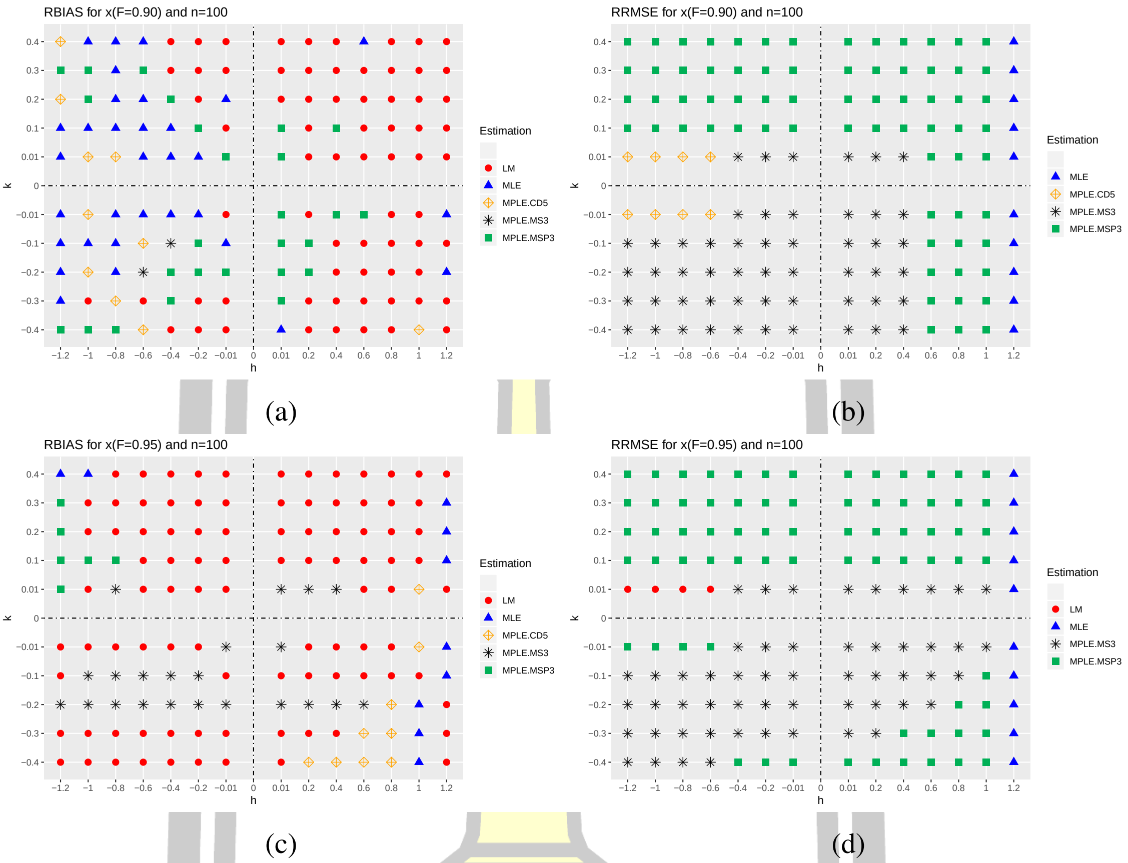


Figure 4.35: The best of estimation for area of parameter space when sample size $n = 100$ for 0.90 and 0.95 quantile estimate

In Figure 4.35a, 4.35b, 4.35c and 4.35d indicate the best scales of the parameter estimation methods being studied, regarding the RBIAS and RRMSE in the quantiles that were 0.90 and 0.95 respectively. Figure 4.36a, 4.36b, 4.36c, 4.36d,4.36e and 4.36f show the best scales of the parameter estimation methods in the study in case that the RBIAS and RRMSE of the quantiles were 0.99, 0.995, and 0.95 respectively. When the k value was positive, the MPLE.MSP3 was the most efficient, compared to that of the MLE, MPLE.MS3, and LM methods (except when h = -1.2) when the k value was negative. The MPLE.MS3 and MPLE.MSP3 were more efficient than MLE and LM methods.

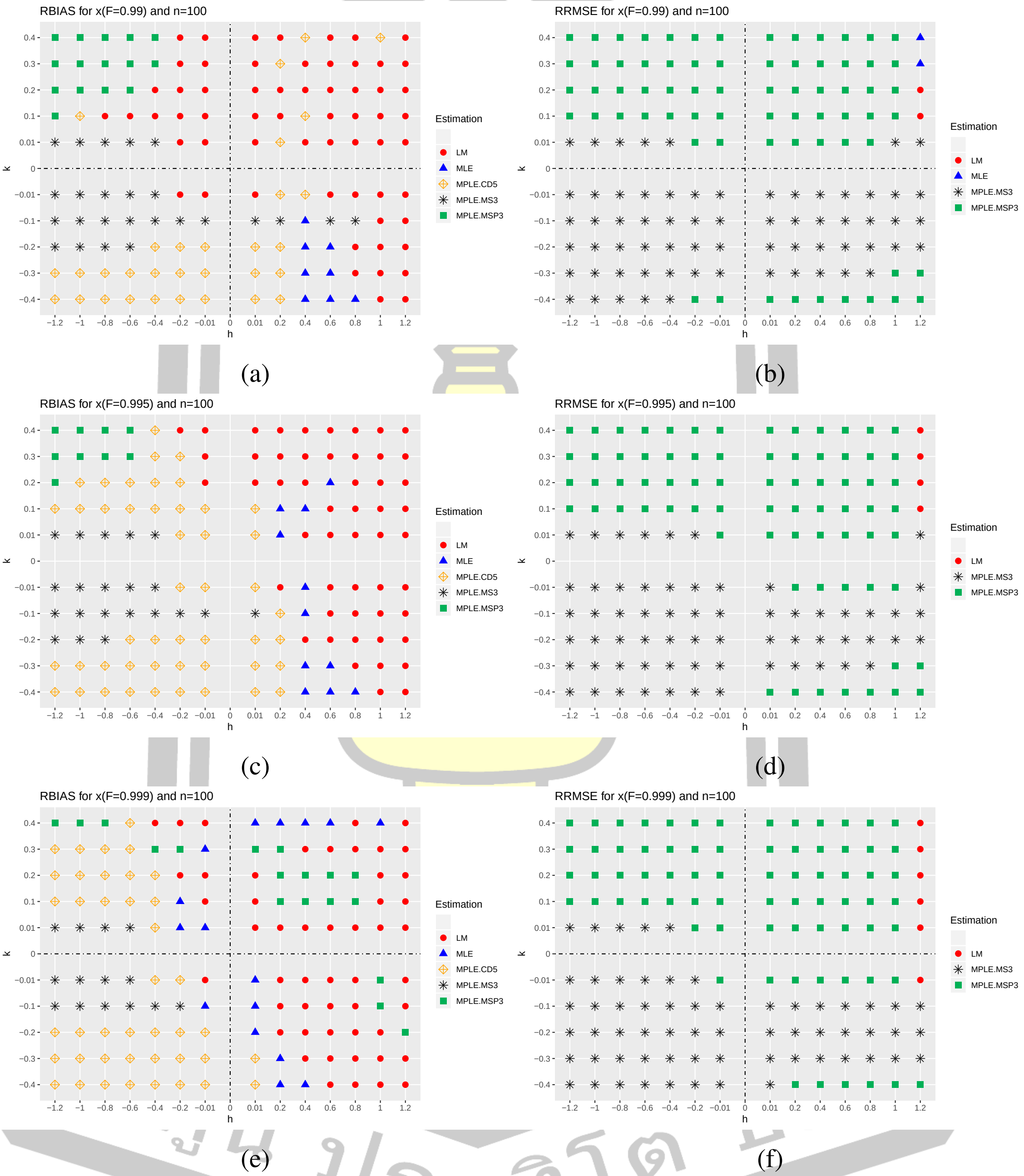


Figure 4.36: The best of estimation for area of parameter space when sample size $n = 100$ for 0.99 0.995 and 0.999 quantile estimate

4.1.2 Application to Hydrological Data

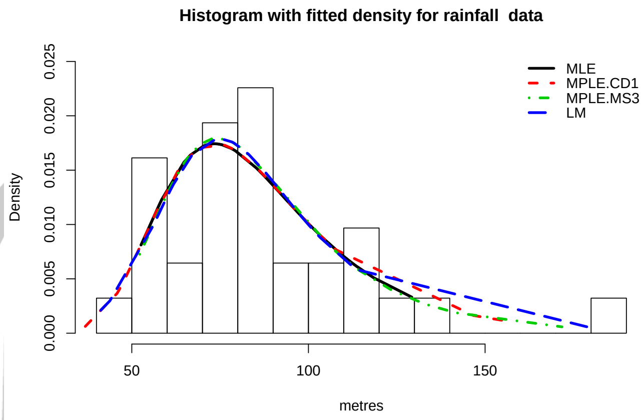
1. Rainfall data

The annual maximum rainfall data (mm) was measured at Pattaya of Thailand, recorded over the period of 1984 to 2014 for 31 year. Anderson Darling (AD) goodness-of-fit test criteria are used to observe the significance of the methods in extreme event analysis. Modified prediction absolute error (MPAE) is used to assess goodness-of-fit for the above $50-th$ percentile of the distribution for different methods. Table 4.17 presents the estimates of the K4D with MPAE, and AD goodness-of-fit criteria for a different method. The results obtained from MPAE, AD criteria show minimal differences in the performance among a different method presented in Table 4.17 for these data set the MPLE.MS3 has small value of MAPE and AD are of a good fit. The model diagnostic plots (histogram with fitted density and qq-plot) fitted to the rainfall using different estimates method are presented in Figure 4.37a and 4.37b, the histogram with a fitted density plot suggests that the MPLE.CD5, MPLE.MS3 and MPLE.MSP3 fits the rainfall data reasonable well with K4D.

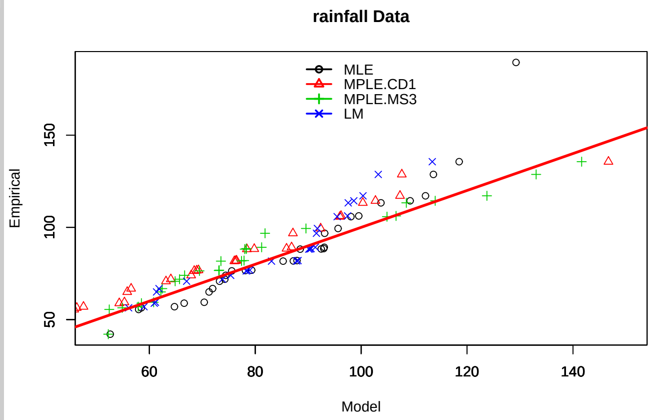
We can obtain 95% confidence interval for a 20-year return level by profile likelihood method can be obtained explicitly by a reparameterization of the K4D model. In Figure 4.37c shows the profile log-likelihood for a 20-year return level of the the annual maximum rainfall at Pattaya, Thailand. A 95% confidence interval for a 20-year return level of [120.5551, 207.6853] with 20-year return level is 142.5551 mm.

Table 4.17: Four estimates of K4D fitted to rainfall data with corresponding MPAE and AD

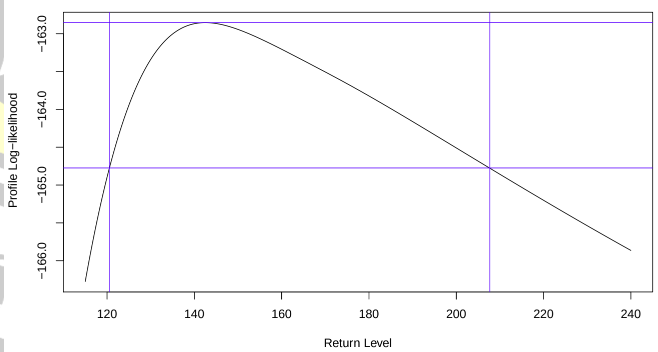
Methods	$\hat{\mu}$	$\hat{\sigma}$	$\hat{k}$	$\hat{h}$	MPAE	AD	$p-value$
MLE	74.5110	20.9023	-0.0548	-0.0227	2.4540	0.1517	0.9985
MPLE.CD1	74.6148	20.8036	-0.0570	-0.0303	2.4521	0.1515	0.9986
MPLE.CD2	74.4304	21.1771	-0.0491	-0.0080	2.5641	0.1523	0.9985
MPLE.CD3	74.5872	20.9103	-0.0548	-0.0245	2.4521	0.1516	0.9985
MPLE.CD4	74.5846	20.7918	-0.0572	-0.0299	2.4301	0.1515	0.9985
MPLE.CD5	74.3193	21.2931	-0.0477	-0.00006	2.4501	0.1527	0.9985
MPLE.MS1	75.3850	19.6777	-0.0994	-0.1174	2.5536	0.1409	0.9986
MPLE.MS2	76.7957	18.3922	-0.1171	-0.2493	2.3547	0.1489	0.9987
MPLE.MS3	76.8616	18.3379	-0.1185	-0.2558	**2.3492**	**0.1489**	**0.9987**
MPLE.MSP1	74.1789	21.4727	-0.0356	0.0141	2.4664	0.1532	0.9984
MPLE.MSP2	74.1789	21.4727	-0.0356	0.0141	2.4624	0.1532	0.9984
MPLE.MSP3	74.1736	21.4700	-0.0356	0.0142	2.4802	0.1532	0.9984
LM	78.7127	16.8276	-0.1669	-0.5120	2.4802	0.1469	0.9989



(a) Histogram with fitted density for rainfall data



(b) Q–Q plot for rainfall data



(c) Profile likelihood for 20-year return level for rainfall data

Figure 4.37: Histogram, Q–Q plot and Profile likelihood for 20-year return level of the K4D for annual maximum rainfall data

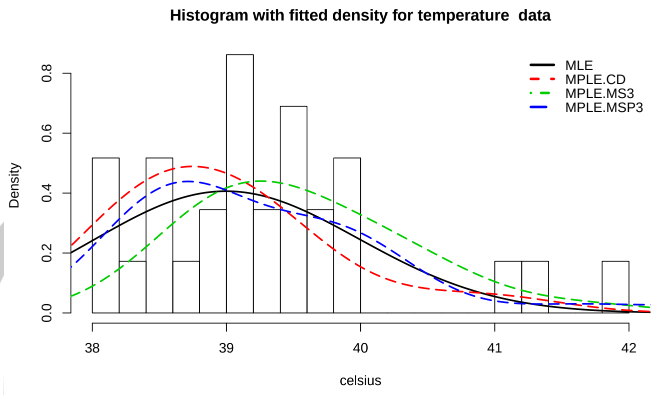
2. Temperature Data

The annual maximum temperature data was measured at Surin of Thailand, recorded over the period of 1990 to 2018 for 29 year. Table 4.18 presents all estimates method for K4D fitted to the temperature data MPAE and AD criteria. The result from the MPAE and AD Based on the goodness-of-fit criteria are approximately identical for different method of all estimates methods in , therefore, the MPLE.MS3 method are as effective for fitting the temperature data. But in this case LM estimate cannot compute, MPLE.MS2 and MPLE.MSP1 estimate can be used alternatively. There are all possible reasons for the identical results in Table 4.38a. Furthermore, Figure 4.38a and 4.38b presents histogram with fitted density and qq-plot seems consistent for the data for four estimates method. Figure 4.38c shows the profile log-likelihood for a 20-year return level of the the temperature data. This leads to a 95% confidence interval for a 20-year return level of [40.6546, 41.6757] with 20-year return level is 40.6546 celsius.

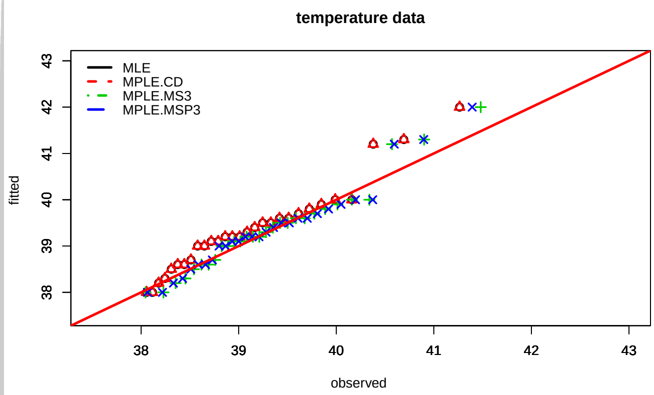
Table 4.18: Estimates of K4D fitted to temperature data with corresponding MPAE and AD

Methods	$\hat{\mu}$	$\hat{\sigma}$	$\hat{k}$	$\hat{h}$	MPAE	AD	$p-value$
MLE	39.4574	0.4805	-0.2436	-4.3756	0.4805	2.0163	0.0904
MPLE.CD1	39.4574	0.4805	-0.2436	-4.3756	0.4805	2.0163	0.0904
MPLE.CD2	39.5727	0.3699	-0.2329	-4.4154	0.3699	1.7931	0.1201
MPLE.CD3	39.4574	0.4805	-0.2436	-4.3756	0.4805	1.8863	0.1065
MPLE.CD4	39.4574	0.4805	-0.2436	-4.3756	0.4805	1.8863	0.1065
MPLE.CD5	39.4574	0.4805	-0.2436	-4.3756	0.4805	2.0163	0.0904
MPLE.MS1	38.9802	0.2784	-0.0713	-0.1212	0.2784	0.3606	0.8858
MPLE.MS2	39.0457	0.2793	-0.0907	-0.2884	0.2793	0.3317	0.9119
MPLE.MS3	39.0489	0.2794	-0.0916	-0.2971	0.2794	0.3305	0.9129
MPLE.MSP1	38.9361	0.2914	0.0147	0.0263	0.2914	0.3615	0.8850
MPLE.MSP2	38.8923	0.2944	0.0370	0.1153	0.2944	0.3798	0.8675
MPLE.MSP3	38.8864	0.2948	0.0399	0.1265	0.2948	0.3824	0.8650
LM	*	*	*	*	*	*	*

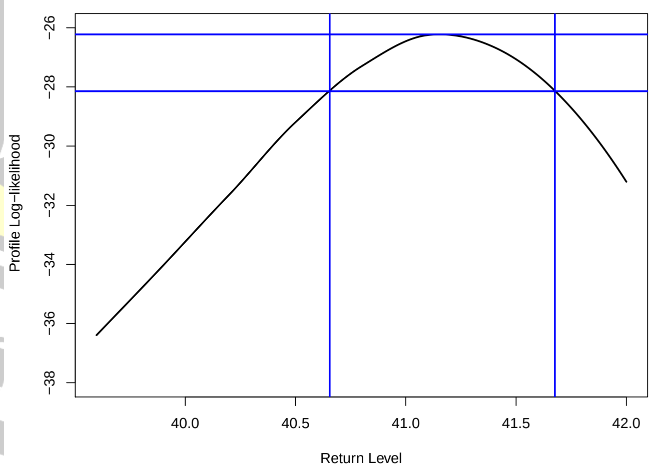
*LM Cannot be computed



(a) Histogram with fitted density for temperature data



(b) Q–Q plot for temperature data



(c) Profile likelihood for 20-year return level for temperature data

Figure 4.38: Histogram, Q–Q plot and Profile likelihood for 20-year return level of the K4D for temperature data

4.2 The result of studying r-largest order statistics for four parameter kappa distribution (r-K4D)

4.2.1 Studying of the r-largest order statistics model

Smith (1986) presents a family of statistics distributions for extreme values base on a fixed number $r >= 1$ of the largest events. The idea is as follows. Let $Y_1, Y_2, ...$ denote a sequence of independent and identically distribution random variables and let $M_n = \max(Y_1, ...Y_n)$. Suppose that normalizing constants a_n and b_n such that $(M_n - b_n)/a_n$ converges in distribution can be found. That is,

$$P\left(\frac{M_n - b_n}{a_n} \leq x\right) \rightarrow G(x) \text{ as } n \rightarrow \infty \tag{4.1}$$

for some proper distribution G. Then, the only nondegenerate form for G is the GEV. The basis of Smith's sample of size n (r fixed, $n \rightarrow \infty$). Let $Y_1, Y_2, ...$ be i.i.d., as before, and suppose that, for each n, the value $Y_1, ..., Y_n$ are ordered as $Y_{n,1} \geqslant ... \geqslant Y_{n,n}$. Let a_n and b_n be as before and consider, for fixed r, the joint distribution of

$$\left(\frac{Y_{n,1} - b_n}{a_n}, \frac{Y_{n,2} - b_n}{a_n}, ..., \frac{Y_{n,r} - b_n}{a_n}\right). \tag{4.2}$$

Then, under precisely the same condition to Equation 4.1, this joint distribution converges to that of a vector $(X_1, ..., X_n)$ for which an explicit formula is available. The joint density of $(X_1, ..., X_n)$ is

$$\begin{aligned} f(x_1, ..., x_r) &= \exp\left\{-\left[1 - k\left(\frac{x_r - \mu}{\sigma}\right)\right]^{1/k}\right\} \\ &\times \prod_{j=1}^{r} \sigma^{-1}\left[1 - k\left(\frac{x_j - \mu}{\sigma}\right)\right]^{(1/k)-1}, \end{aligned} \tag{4.3}$$

where $-\infty < \mu < \infty, \sigma > 0$ and $-\infty < k < \infty$; $x_r \leq x_{r-1} \leq ... \leq x_1$ and $x_j : \left[1 - k\left(\frac{x_j - \mu}{\sigma}\right)\right] > 0$ for $j = 1, 2, ..., r$.

4.2.2 The r-largest order statistics of K4D in case of r=2

The four parameter kappa distribution introduced by Hosking (1994) is a genealized form of the GEV when $h = 0$. It is a candidate for being fitted to data when these three parameter distribution give an inadequate fit, or when the experimenter doese not want to be committed to the use of a particular three parameter distribution. Park suggestion the joint probability density function is

$$f(x_1, ..., x_r) = C_r \times [F(x_r)]^{1-rh} \times \prod_{j=1}^{r} \sigma^{-1} \left[1 - k\left(\frac{x_j - \mu}{\sigma}\right)\right]^{\frac{1}{k}-1}, \tag{4.4}$$

where $F(x)$ is the cdf of the K4D with cdf is

$$F(x) = \left\{1 - h\left[1 - k\left(\frac{x-\mu}{\sigma}\right)\right]^{\frac{1}{k}}\right\}^{\frac{1}{h}},$$

and

$$C_r = \begin{cases} 1 & \text{if } r = 1, \\ \prod_{i=1}^{r}[1-(r-i)h] & \text{if } r \leq 2, \end{cases}$$

with $-\infty < \mu < \infty$, $\sigma > 0$, $-\infty < k, h < \infty$; $x_{r-1} \leq \ldots \leq x_1$ and x_j : $1 - h\left[1 - k\left(\frac{x_j - \mu}{\sigma}\right)\right]^{\frac{1}{k}} > 0$ for $j = 1, 2, \ldots, r$.
In case $r = 1$ reduces to the K4D of the density function.

$$f(x_1) = [F(x_1)]^{1-h} \times \sigma^{-1}\left[1 - k\left(\frac{x_1 - \mu}{\sigma}\right)\right]^{\frac{1}{k}-1}. \tag{4.5}$$

The case $h \to 0$, the joint probability density function in Equation 4.4 reduces to the r-largest order statistics model of GEV in Equation 4.3.
In this study we focus an r-largest order statistics model with r=2 of the K4D. Thus, the joint probability density function of X_1 and X_2 is

$$\begin{aligned} f(x_1, x_2) &= [1-h] \times [F(x_2)]^{1-2h} \times \sigma^{-1}\left[1 - k\left(\frac{x_1 - \mu}{\sigma}\right)\right]^{\frac{1}{k}-1} \\ &\quad \times \sigma^{-1}\left[1 - k\left(\frac{x_2 - \mu}{\sigma}\right)\right]^{\frac{1}{k}-1}. \end{aligned} \tag{4.6}$$

The marginal pdf of X_1 is derived to the following by integral of $f(x_1, ..., x_r)$ with respect to x_2

$$\begin{aligned}
f(x_1) &= \int_{-\infty}^{\infty} f(x_1, x_2)dx_2 \qquad ; \mu + \frac{\sigma}{k}(1-h^{-k}) < x_2 < x_1 < \mu + \frac{\sigma}{k} \\
&= \int_{\mu+\frac{\sigma}{k}(1-h^{-k})}^{x_1} [1-h] \times [F(x_2)]^{1-2h} \times \sigma^{-1}\left[1-k\left(\frac{x_1-\mu}{\sigma}\right)\right]^{\frac{1}{k}-1} \\
&\quad \times \sigma^{-1}\left[1-k\left(\frac{x_2-\mu}{\sigma}\right)\right]^{\frac{1}{k}-1} dx_2 \\
&= \sigma^{-1}\left[1-k\left(\frac{x_1-\mu}{\sigma}\right)\right]^{\frac{1}{k}-1} \int_{\mu+\frac{\sigma}{k}(1-h^{-k})}^{x_1} [1-h]\left[\left\{1-h\left[1-k\left(\frac{x_2-\mu}{\sigma}\right)\right]^{\frac{1}{k}}\right\}^{\frac{1}{h}}\right]^{1-2h} \\
&\quad \times \sigma^{-1}\left[1-k\left(\frac{x_2-\mu}{\sigma}\right)\right]^{\frac{1}{k}-1} dx_2.
\end{aligned}$$

Where $y = \left[1-k\left(\frac{x_2-\mu}{\sigma}\right)\right]$,

so that $\frac{dy}{dx_2} = \frac{-k}{\sigma}$ and $dx_2 = \left(\frac{-\sigma}{k}\right)dy.$

$$f(x_1) = \sigma^{-1}\left[1-k\left(\frac{x_1-\mu}{\sigma}\right)\right]^{\frac{1}{k}-1} \int_{\mu+\frac{\sigma}{k}(1-h^{-k})}^{\left[1-k\left(\frac{x_2-\mu}{\sigma}\right)\right]} [1-h]\left[\left\{1-hy^{\frac{1}{k}}\right\}^{\frac{1}{h}}\right]^{1-2h}\left(\frac{-1}{k}\right) y^{\frac{1}{k}-1}dy,$$

Give $v = 1-hy^{\frac{1}{k}}$, $\frac{dv}{dy} = \left(\frac{-h}{k}\right)y^{\frac{1}{k}-1}$, $dy = \left(\frac{-k}{h}\right)\left(\frac{1}{y^{\frac{1}{k}-1}}\right)dv$,

$$\begin{aligned}
f(x_1) &= \sigma^{-1}\left[1-k\left(\frac{x_1-\mu}{\sigma}\right)\right]^{\frac{1}{k}-1} \int_{\mu+\frac{\sigma}{k}(1-h^{-k})}^{1-hy^{\frac{1}{k}}} [1-h]v^{\frac{1}{h}-2}\left(\frac{-y^{\frac{1}{k}-1}}{k}\right)\left(\frac{-k}{hy^{\frac{1}{k}-1}}\right)dv \\
&= \sigma^{-1}\left[1-k\left(\frac{x_1-\mu}{\sigma}\right)\right]^{\frac{1}{k}-1} \int_{\mu+\frac{\sigma}{k}(1-h^{-k})}^{1-hy^{\frac{1}{k}}} [1-h]v^{\frac{1}{h}-2}\left(\frac{1}{h}\right)dv \\
&= \sigma^{-1}\left[1-k\left(\frac{x_1-\mu}{\sigma}\right)\right]^{\frac{1}{k}-1} \times \left(\frac{1}{h}-1\right)\left[\left(\frac{v^{\frac{1}{h}-1}}{\frac{1}{h}-1}\right)\Bigg|_{\mu+\frac{\sigma}{k}(1-h^{-k})}^{1-hy^{\frac{1}{k}}}\right] \\
&= \sigma^{-1}\left[1-k\left(\frac{x_1-\mu}{\sigma}\right)\right]^{\frac{1}{k}-1} \times \left[v^{\frac{1}{h}-1}\Bigg|_{\mu+\frac{\sigma}{k}(1-h^{-k})}^{1-hy^{\frac{1}{k}}}\right]
\end{aligned}$$

$$= \sigma^{-1}\left[1-k\left(\frac{x_1-\mu}{\sigma}\right)\right]^{\frac{1}{k}-1} \times \left[\left\{1-h\left[1-k\left(\frac{x_2-\mu}{\sigma}\right)\right]^{\frac{1}{k}}\right\}^{\frac{1}{h}-1} \Bigg|_{\mu+\frac{\sigma}{k}(1-h^{-k})}^{x_1}\right],$$

consider term of

$\left\{1-h\left[1-k\left(\frac{\mu+\frac{\sigma}{k}(1-h^{-k})-\mu}{\sigma}\right)\right]^{\frac{1}{k}}\right\}^{\frac{1}{h}-1}$

$$\begin{aligned}
\left\{1-h\left[1-k\left(\frac{\mu+\frac{\sigma}{k}(1-h^{-k})-\mu}{\sigma}\right)\right]^{\frac{1}{k}}\right\}^{\frac{1}{h}-1} &= \left\{1-h\left[1-(1-h^{-k})\right]^{\frac{1}{k}}\right\}^{\frac{1}{h}-1} \\
&= \left\{1-h\left(h^{-k}\right)^{\frac{1}{k}}\right\}^{\frac{1}{h}-1} \\
&= (1-1)^{\frac{1}{h}-1} \\
&= 0.
\end{aligned}$$

So that

$$\begin{aligned}
f(x_1) &= \sigma^{-1}\left[1-k\left(\frac{x_1-\mu}{\sigma}\right)\right]^{\frac{1}{k}-1} \times \left[\left\{1-h\left[1-k\left(\frac{x_1-\mu}{\sigma}\right)\right]^{\frac{1}{k}}\right\}^{\frac{1}{h}-1} - 0\right] \\
&= \left[\left\{1-h\left[1-k\left(\frac{x_1-\mu}{\sigma}\right)\right]^{\frac{1}{k}}\right\}^{\frac{1}{h}}\right]^{1-h} \times \sigma^{-1}\left[1-k\left(\frac{x_1-\mu}{\sigma}\right)\right]^{\frac{1}{k}-1} \\
&= [F(x_1)]^{1-h} \times \sigma^{-1}\left[1-k\left(\frac{x_1-\mu}{\sigma}\right)\right]^{\frac{1}{k}-1}. \qquad (4.7)
\end{aligned}$$

The marginal pdf of X_2 is derived to the following by integral of $f(x_1, ..., x_r)$ with respect to x_1

$$\begin{aligned}
f(x_2) &= \int_{-\infty}^{\infty} f(x_1, x_2)dx_1 \qquad ; \mu + \frac{\sigma}{k}(1-h^{-k}) < x_2 < x_1 < \mu + \frac{\sigma}{k} \\
&= \int_{x_2}^{\mu+\frac{\sigma}{k}} [1-h] \times F(x_2)^{1-2h} \times \sigma^{-1}\left[1-k\left(\frac{x_1-\mu}{\sigma}\right)\right]^{\frac{1}{k}-1} \\
&\quad \times \sigma^{-1}\left[1-k\left(\frac{x_2-\mu}{\sigma}\right)\right]^{\frac{1}{k}-1} dx_1 \\
&= F(x_2)^{1-2h} \times \sigma^{-1}\left[1-k\left(\frac{x_2-\mu}{\sigma}\right)\right]^{\frac{1}{k}-1} \int_{x_2}^{\mu+\frac{\sigma}{k}} [1-h]\sigma^{-1}\left[1-k\left(\frac{x_1-\mu}{\sigma}\right)\right]^{\frac{1}{k}-1} dx_1,
\end{aligned}$$

Give $y = \left[1-k\left(\frac{x_1-\mu}{\sigma}\right)\right]$, $\quad \frac{dy}{dx_1} = \left(\frac{-k}{\sigma}\right)$ and $\quad dx_1 = \left(\frac{-\sigma}{k}\right) dy,$

$$
\begin{aligned}
f(x_2) &= F(x_2)^{1-2h} \times \sigma^{-1}\left[1-k\left(\frac{x_2-\mu}{\sigma}\right)\right]^{\frac{1}{k}-1} \int_{\left[1-k\left(\frac{x_1-\mu}{\sigma}\right)\right]}^{\mu+\frac{\sigma}{k}} [1-h]\sigma^{-1} y^{\frac{1}{k}-1}\left(\frac{-\sigma dy}{k}\right) \\
&= F(x_2)^{1-2h} \times \sigma^{-1}\left[1-k\left(\frac{x_2-\mu}{\sigma}\right)\right]^{\frac{1}{k}-1} \int_{\left[1-k\left(\frac{x_1-\mu}{\sigma}\right)\right]}^{\mu+\frac{\sigma}{k}} [1-h]\left(\frac{-y^{\frac{1}{k}-1}}{k}\right) dy \\
&= F(x_2)^{1-2h} \times \sigma^{-1}\left[1-k\left(\frac{x_2-\mu}{\sigma}\right)\right]^{\frac{1}{k}-1} \times [1-h]\left(\frac{-y^{\frac{1}{k}}}{\frac{k}{k}}\right) \Bigg|_{\left[1-k\left(\frac{x_1-\mu}{\sigma}\right)\right]}^{\mu+\frac{\sigma}{k}} \\
&= F(x_2)^{1-2h} \times \sigma^{-1}\left[1-k\left(\frac{x_2-\mu}{\sigma}\right)\right]^{\frac{1}{k}-1} \times -[1-h]\left[y^{\frac{1}{k}} \Bigg|_{\left[1-k\left(\frac{x_1-\mu}{\sigma}\right)\right]}^{\mu+\frac{\sigma}{k}}\right] \\
&= F(x_2)^{1-2h} \times \sigma^{-1}\left[1-k\left(\frac{x_2-\mu}{\sigma}\right)\right]^{\frac{1}{k}-1} \times -[1-h]\left[\left[1-k\left(\frac{x_1-\mu}{\sigma}\right)\right]^{\frac{1}{k}} \Bigg|_{x_2}^{\mu+\frac{\sigma}{k}},\right]
\end{aligned}
$$

consider in term of $\left[\left[1-k\left(\frac{x_1-\mu}{\sigma}\right)\right]^{\frac{1}{k}} \Bigg|_{x_2}^{\mu+\frac{\sigma}{k}}\right]$

$$
\begin{aligned}
\left[\left[1-k\left(\frac{x_1-\mu}{\sigma}\right)\right]^{\frac{1}{k}} \Bigg|_{x_2}^{\mu+\frac{\sigma}{k}}\right] &= \left[\left[1-k\left(\frac{\mu+\frac{\sigma}{k}-\mu}{\sigma}\right)\right]^{\frac{1}{k}} - \left[1-k\left(\frac{x_2-\mu}{\sigma}\right)\right]^{\frac{1}{k}}\right] \\
&= \left[0-\left[1-k\left(\frac{x_2-\mu}{\sigma}\right)\right]^{\frac{1}{k}}\right] \\
&= -\left[1-k\left(\frac{x_2-\mu}{\sigma}\right)\right]^{\frac{1}{k}}. \qquad (4.8)
\end{aligned}
$$

So that

$$
\begin{aligned}
f(x_2) &= F(x_2)^{1-2h} \times \sigma^{-1}\left[1-k\left(\frac{x_2-\mu}{\sigma}\right)\right]^{\frac{1}{k}-1} \times -[1-h] \times -\left[1-k\left(\frac{x_2-\mu}{\sigma}\right)\right]^{\frac{1}{k}} \\
&= [1-h] \times F(x_2)^{1-2h} \times \sigma^{-1}\left[1-k\left(\frac{x_2-\mu}{\sigma}\right)\right]^{\frac{2}{k}-1}.
\end{aligned}
$$

(4.9)

The r-largest order statistics of K4D in case of r=2

$$f(x_1,x_2) = [1-h] \times [F(x_2)]^{1-2h} \times \sigma^{-1}\left[1-k\left(\frac{x_1-\mu}{\sigma}\right)\right]^{\frac{1}{k}-1} \times\sigma^{-1}\left[1-k\left(\frac{x_2-\mu}{\sigma}\right)\right]^{\frac{1}{k}-1}.$$

The marginal pdf of X_1 is

$$f(x_1) = [F(x_1)]^{1-h} \times \sigma^{-1}\left[1-k\left(\frac{x_1-\mu}{\sigma}\right)\right]^{\frac{1}{k}-1}.$$

Finally, The marginal pdf of X_2 is

$$f(x_2) = [1-h] \times F(x_2)^{1-2h} \times \sigma^{-1}\left[1-k\left(\frac{x_2-\mu}{\sigma}\right)\right]^{\frac{2}{k}-1}.$$

CHAPTER 5

CONCLUSIONS

In this thesis to study and compare performance of MLE, MPLE.CD1, MPLE.CD2, MPLE.CD3, MPLE.CD4, MPLE.CD5, MPLE.MS1, MPLE.MS2, MPLE.MS3, MPLE. MSP1, MPLE.MSP2, MPLE.MSP3 and LM method for four parameter kappa distribution (K4D) using Monte Carlo simulations are conducted to investigate the effect of RBIAS and RRMSE on heavy upper quantiles estimated using different methods. On the simulation study include the R program for evaluation and to study the r-largest order statistics for four-parameter kappa distribution with r=2. Finally, recommendation are present for future research. This chapter is organized as follows. 5.1 Performance of MLE, all MPLE and LM method 5.2 Application to Hydrological Data 5.3 The r-largest order statistics (r=2) for four-parameter kappa distribution 5.4 Recommendation

5.1 Performance of MLE, all MPLE and LM method

The compare performance of MLE, MPLE.CD1, MPLE.CD2, MPLE.CD3, MPLE.CD4, MPLE.CD5, MPLE.MS1, MPLE.MS2, MPLE.MS3, MPLE.MSP1, MPLE. MSP2, MPLE.MSP3 and LM method for four parameter kappa distribution (K4D) RBIAS and RRMSE criteria. First, for positive value of k The compare performance of MLE, the best of MPLE.CD (MPLE.CD5), the best of MPLE.MS and MPLE.MSP (MPLE.MSP3) and LM method for all quantiles estimated, for all sample size (n = 30,50 and 100) the MPLE.MSP3 does better than the MLE, MPLE,CD5, MPLE.MS3 and LM estimation in term RRMSE for all quantiles except case $h = -1.2$. Second, for negative value and close to 0 of k The compare performance of MLE, the best of MPLE.CD (MPLE.CD5), the best of MPLE.MS and MPLE.MSP (MPLE.MS3) and LM method for all quantiles estimated, for all sample size (n = 30,50 and 100) the MPLE.MSP3 and the MPLE.MS3 does better than the MLE, MPLE,CD5 and LM estimation in term RRMSE for all quantiles

5.2 Application to Hydrological Data

For application with hydrology data, Anderson Darling (AD) goodness-of-fit test and modified from the expected prediction squared error (MPAE) are used a criteria to select the optimal model of hydrology data. In the application with two hydrology data sets, maximum rainfall data was measured in Pattaya of Thailand and the annual maximum temperature data was measured in Surin province, Thailand. In this rainfall data suggests that the MPLE.CD5, MPLE.MS3 and MPLE.MSP3 fits the rainfall data reasonable well with K4D. We can obtain 95% confidence interval for a 20-year return level by profile likelihood method for a 20-year return level of [120.5551, 207.6853] with 20-year return level is 142.5551 mm. The annual maximum temperature data the MPLE.MS3 method are as effective for fitting the temperature data with a 95% confidence interval for a 20-year return level of [40.6546, 41.6757] with 20-year return level is 40.6546 celsius.

5.3 The r-largest order statistics of K4D in case of r=2

In the study to r-largest order statistics with r=2 for four-parameter kappa distribution.

The joint probability density function of X_1 and X_2 is

$$\begin{aligned} f(x_1, x_2) &= [1-h] \times [F(x_2)]^{1-2h} \times \sigma^{-1}\left[1-k\left(\frac{x_1-\mu}{\sigma}\right)\right]^{\frac{1}{k}-1} \\ &\quad \times\sigma^{-1}\left[1-k\left(\frac{x_2-\mu}{\sigma}\right)\right]^{\frac{1}{k}-1}. \end{aligned} \tag{5.1}$$

The marginal pdf of X_1 is

$$f(x_1) = [F(x_1)]^{1-h} \times \sigma^{-1}\left[1-k\left(\frac{x_1-\mu}{\sigma}\right)\right]^{\frac{1}{k}-1}. \tag{5.2}$$

Finally, The marginal pdf of X_2 is

$$f(x_2) = [1-h] \times F(x_2)^{1-2h} \times \sigma^{-1}\left[1-k\left(\frac{x_2-\mu}{\sigma}\right)\right]^{\frac{2}{k}-1}. \tag{5.3}$$

5.4 Recommendation

For future research are as follows

- In this study we were considered K4D is four parameter, location, scale and two shape parameter can be extended to Weakby distribution (five parameter) location, scale and tree shape parameter.

- Simulation shows good performance via estimations when citerr. Moreover, MPLE.MSP3 estimations suggests that could be a generalized distribution for the heavy tail that contains four parameters.

- We encountered the convergence failures in applying the method of L-moment estimation, which is a popular estimation method in hydrology (Hosking and Wallis (1993)). For example, (Park et al., 2001) found that the L-moments estimation method fails to give feasible estimates because of convergence failure for some station's extreme rainfall of Korean extreme rainfall data. Therefore, MPLE.CD and MPLE.MS estimations perhaps alternatively characterize those data.

- For the r-largest order statistics for four-parameter kappa distribution in case of r=2 with can be applied MLE, MPLE and LM parameter estimation method for hydrological data.

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APPENDIXES

APPENDIXES A

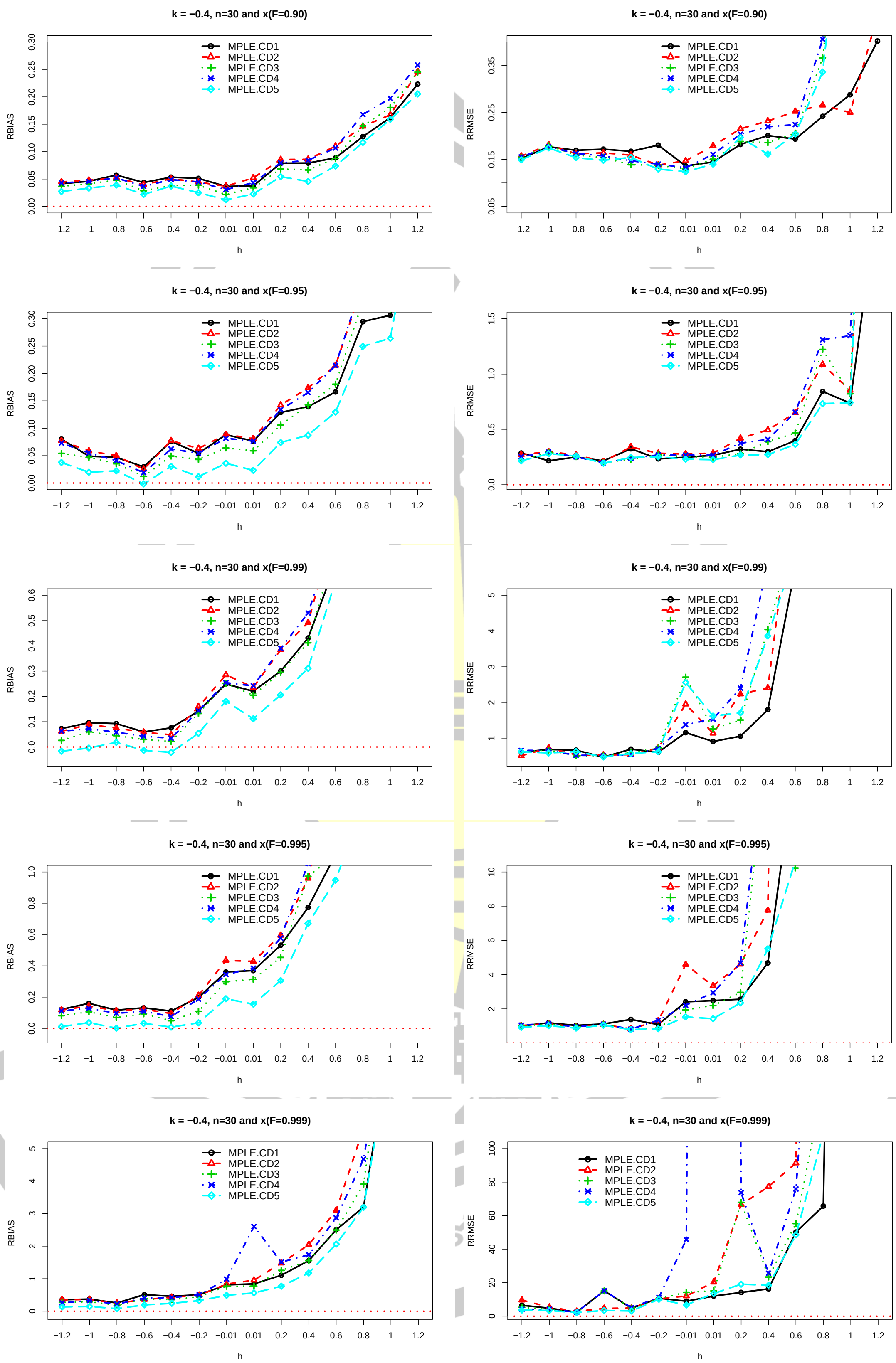

Figure A.1: RBIAS and RRMSE of the all quantile estimators of MPLE.CD1 to MPLE.CD5 for value of $k = -0.4$ and sample size $n = 30$.

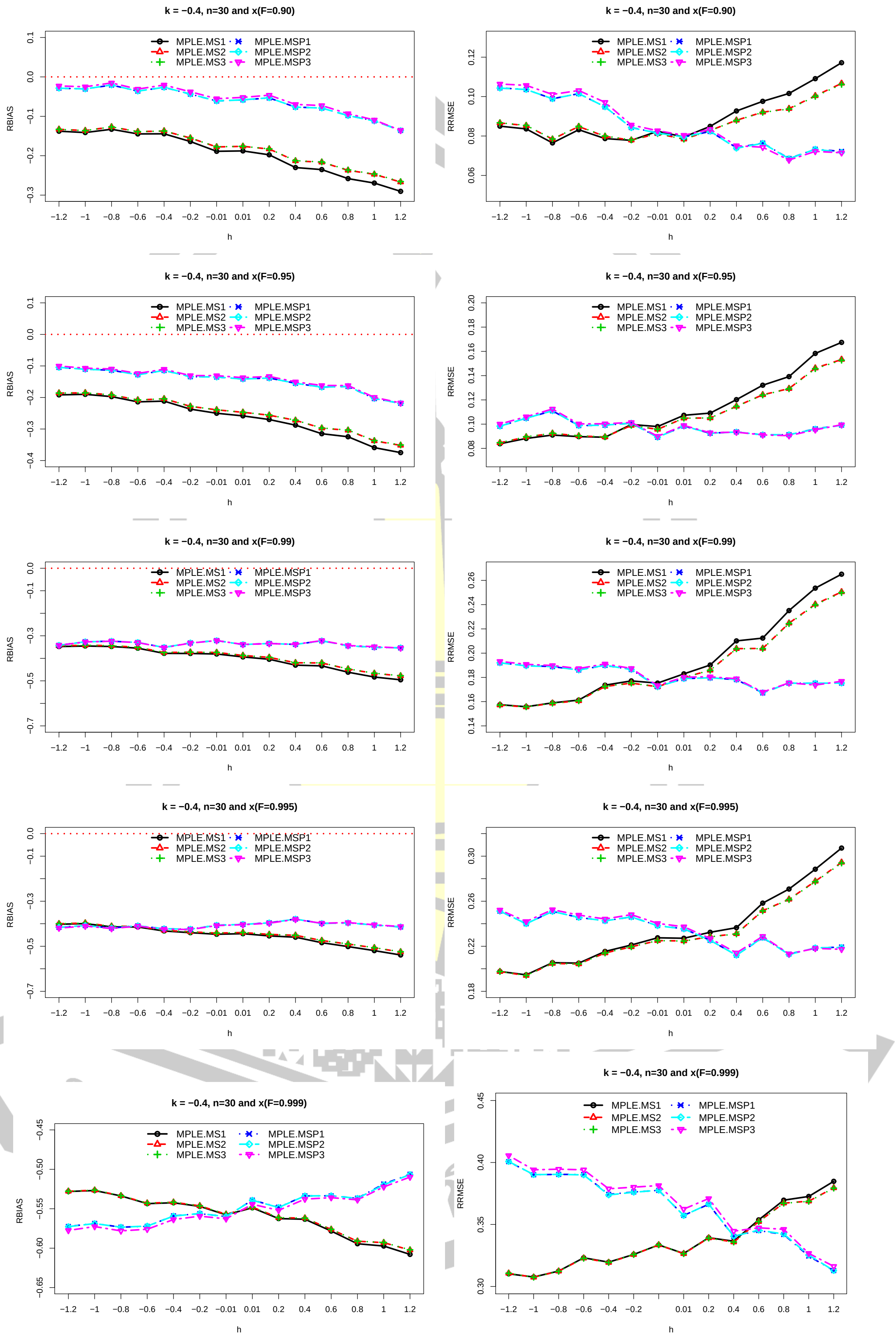

Figure A.2: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MPLE.MSP3 for value of $k = -0.4$ and sample size $n = 30$.

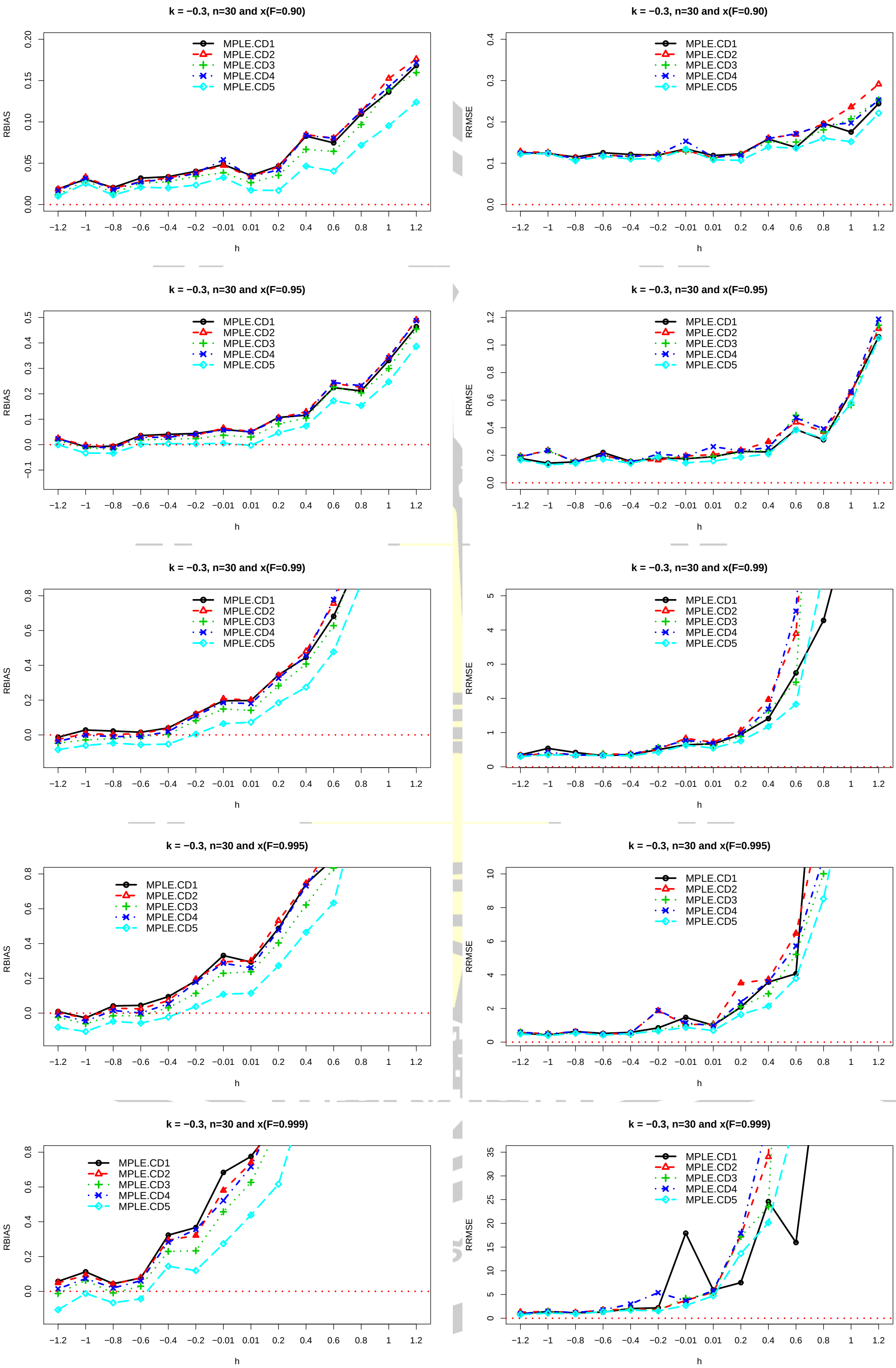

Figure A.3: RBIAS and RRMSE of the all quantile estimators of MPLE.CD1 to MPLE.CD5 for value of $k = -0.3$ and sample size $n = 30$.

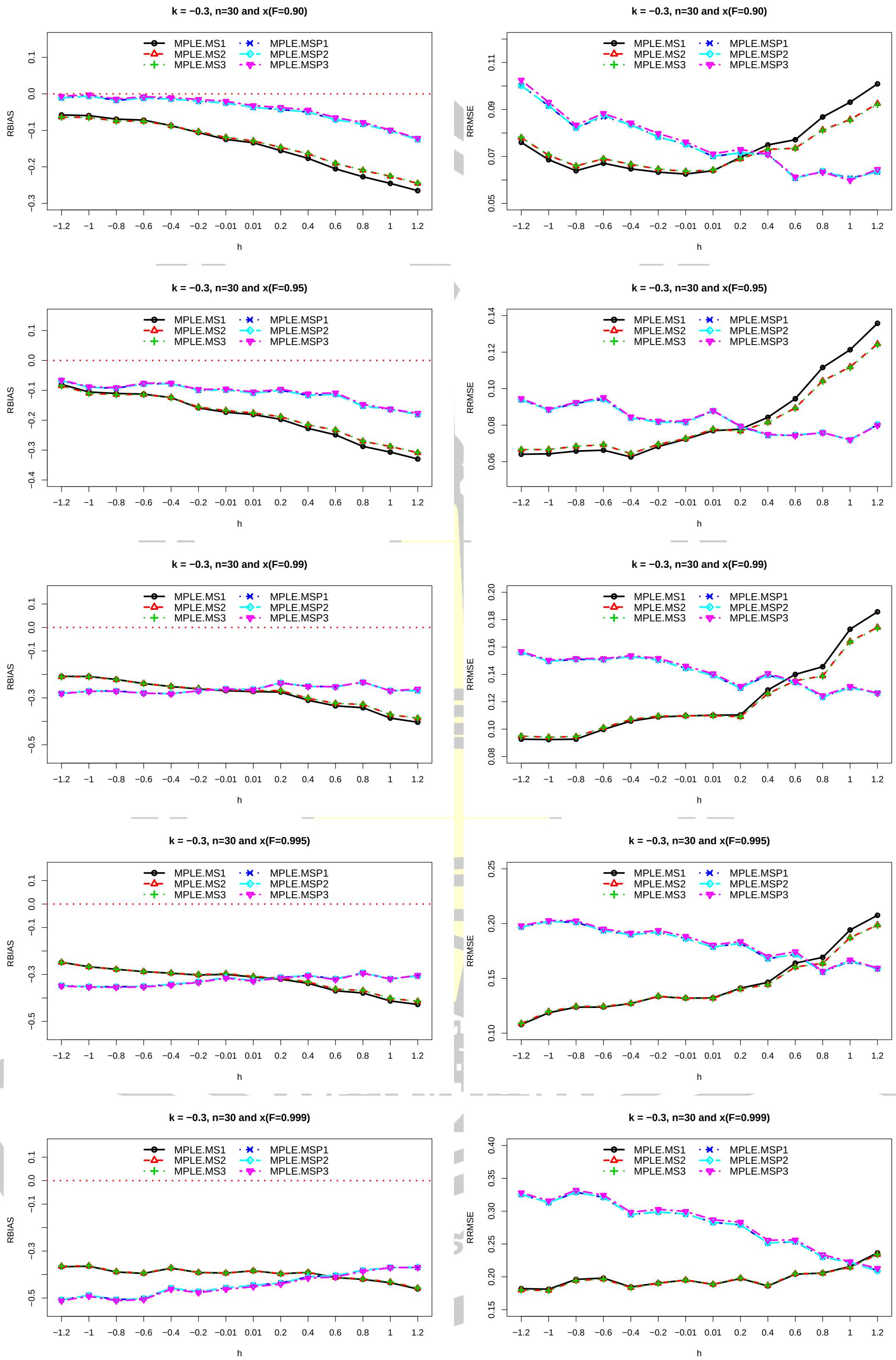

Figure A.4: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MPLE.MSP3 for value of $k = -0.3$ and sample size $n = 30$.

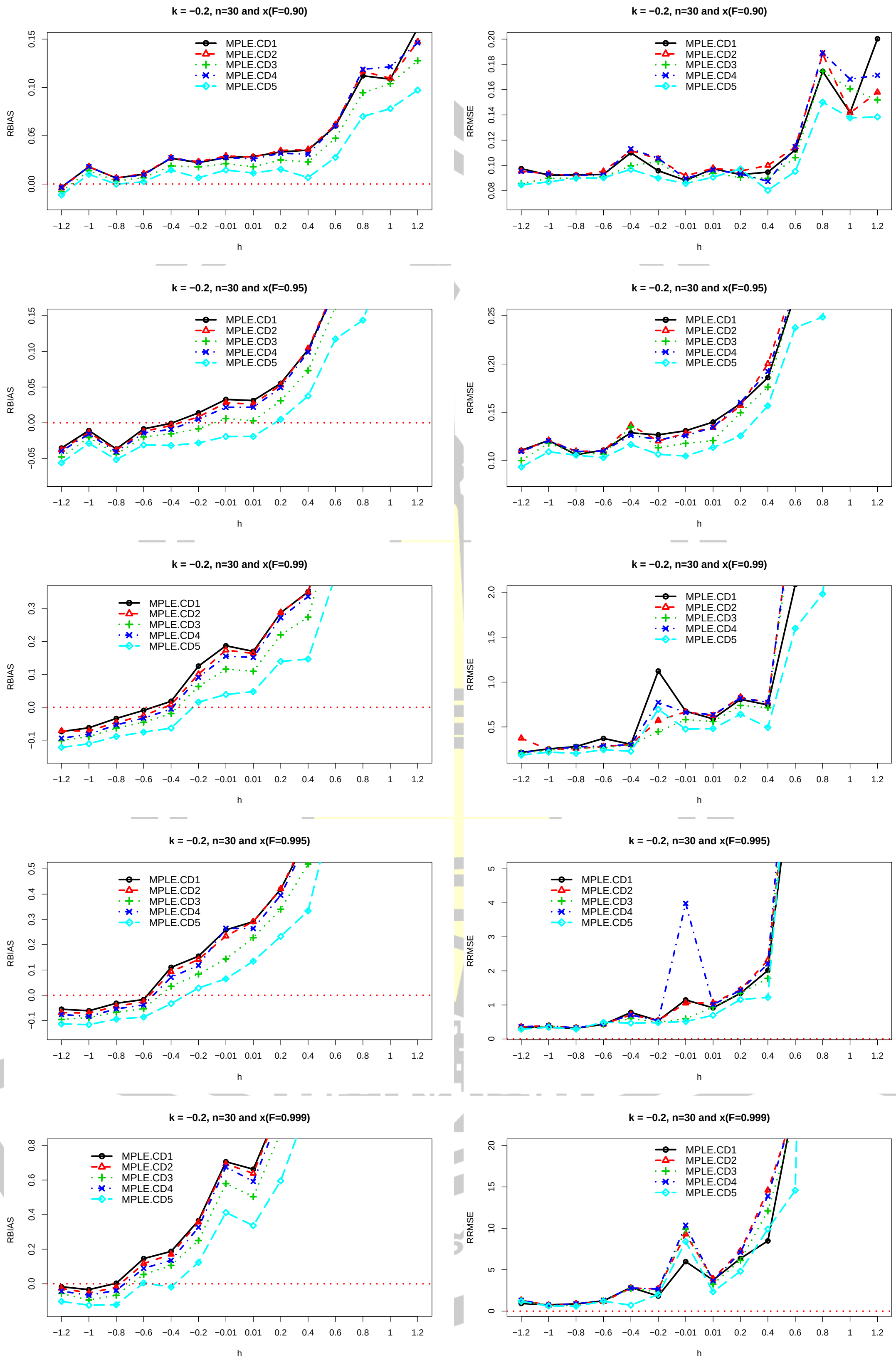

Figure A.5: RBIAS and RRMSE of the all quantile estimators of MPLE.CD1 to MPLE.CD5 for value of $k = -0.2$ and sample size $n = 30$.

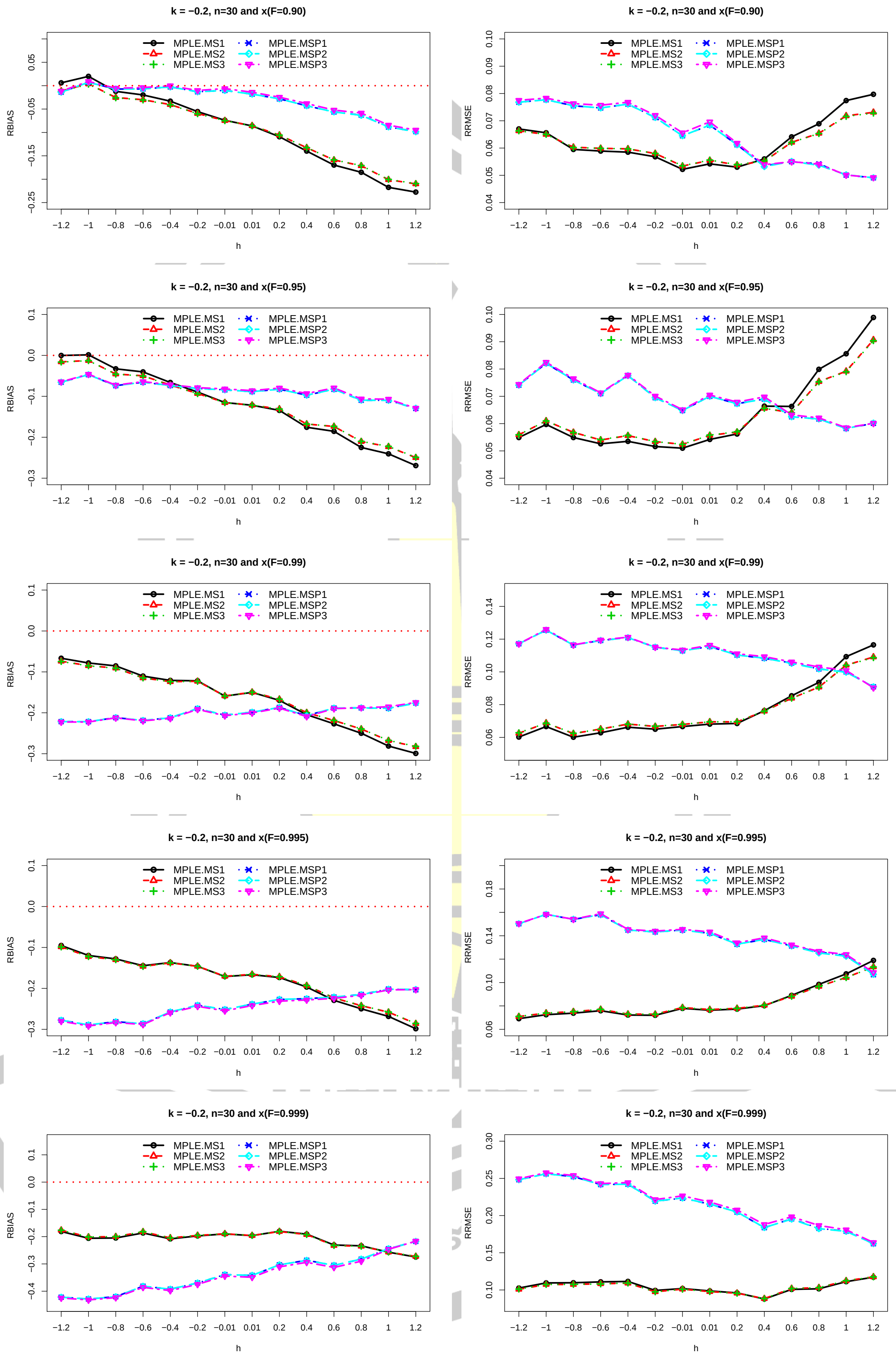

Figure A.6: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MPLE.MSP3 for value of $k = -0.2$ and sample size $n = 30$.

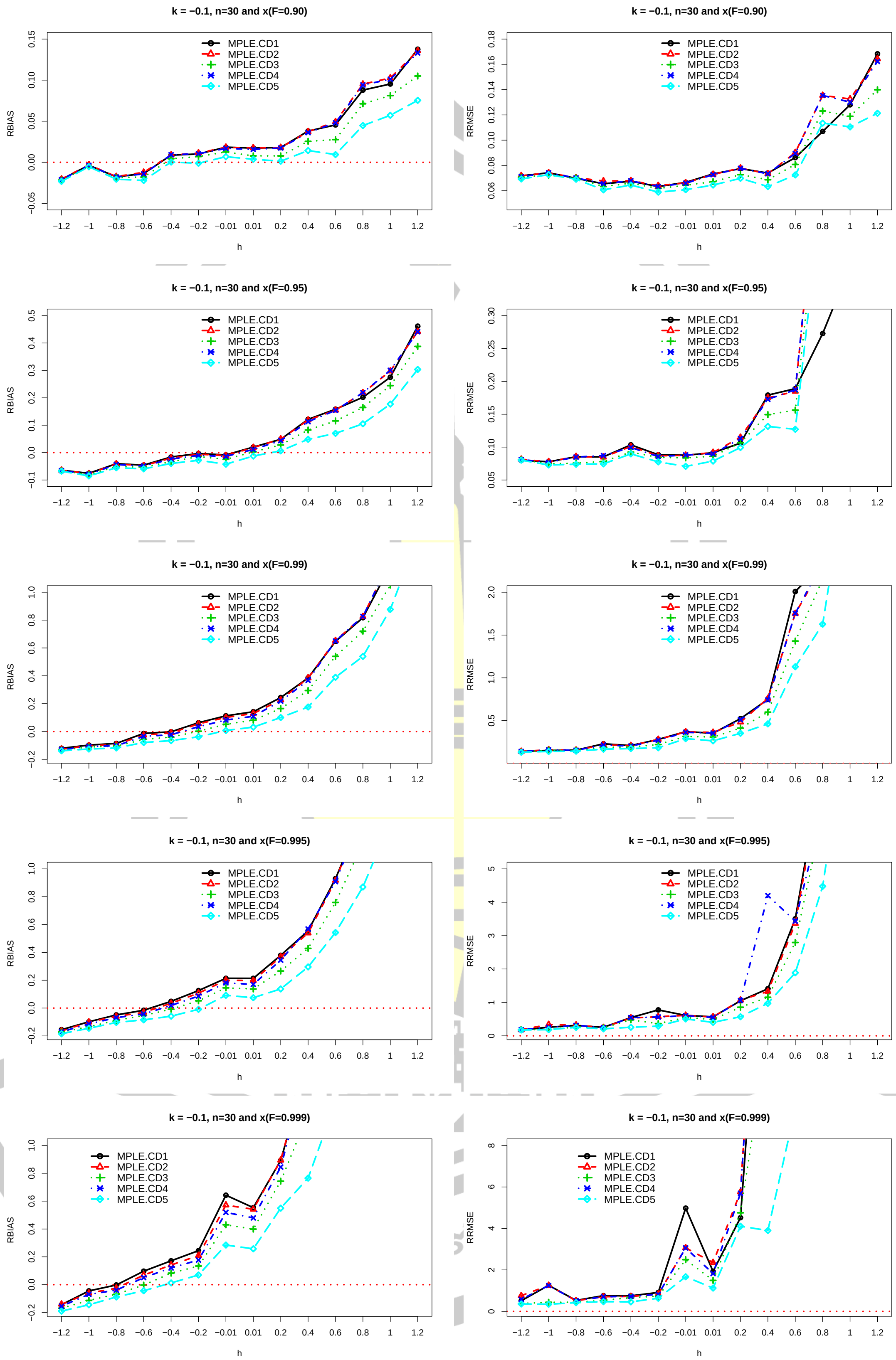

Figure A.7: RBIAS and RRMSE of the all quantile estimators of MPLE.CD1 to MPLE.CD5 for value of $k = -0.1$ and sample size $n = 30$.

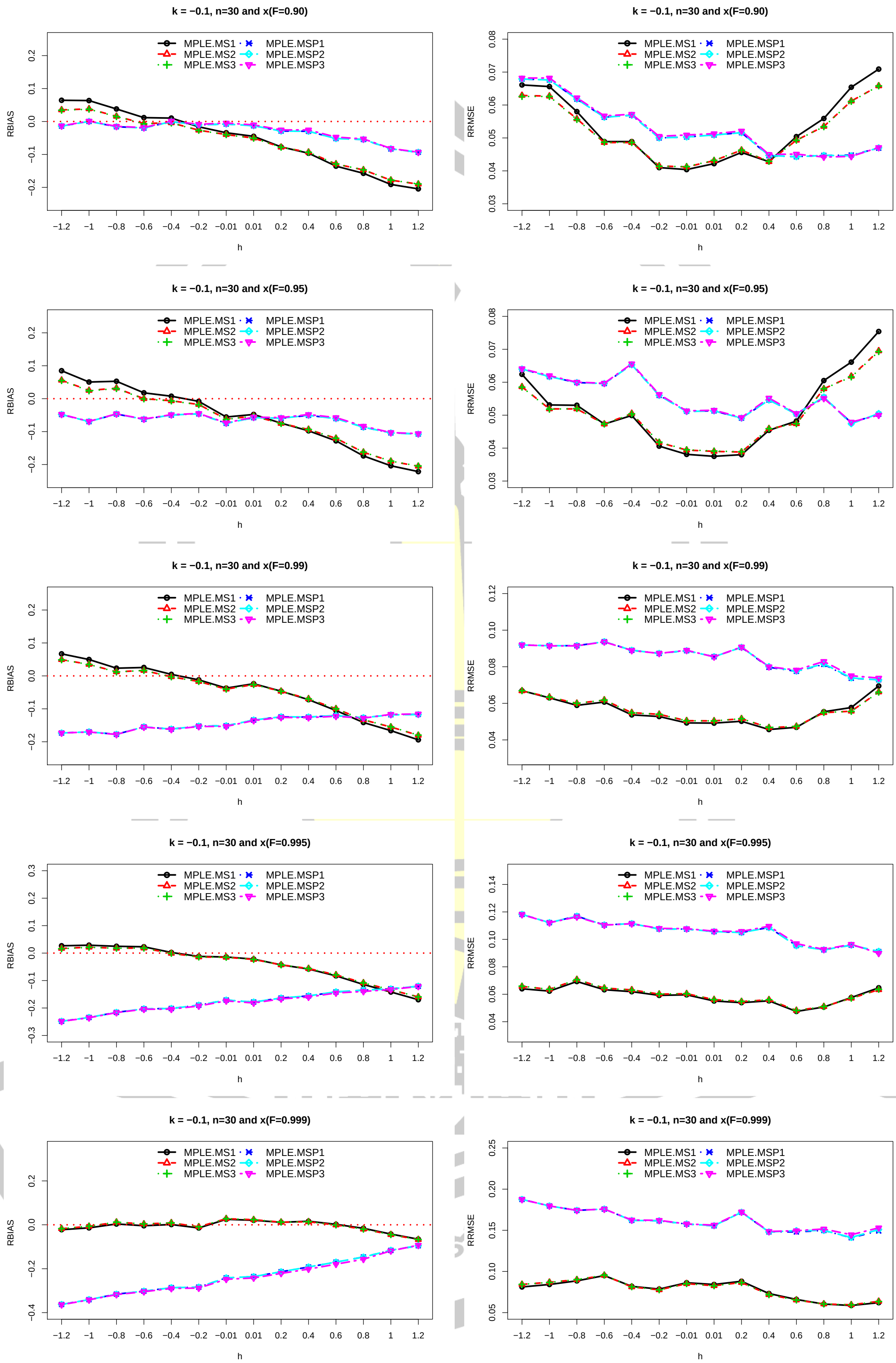

Figure A.8: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MPLE.MSP3 for value of $k = -0.1$ and sample size $n = 30$.

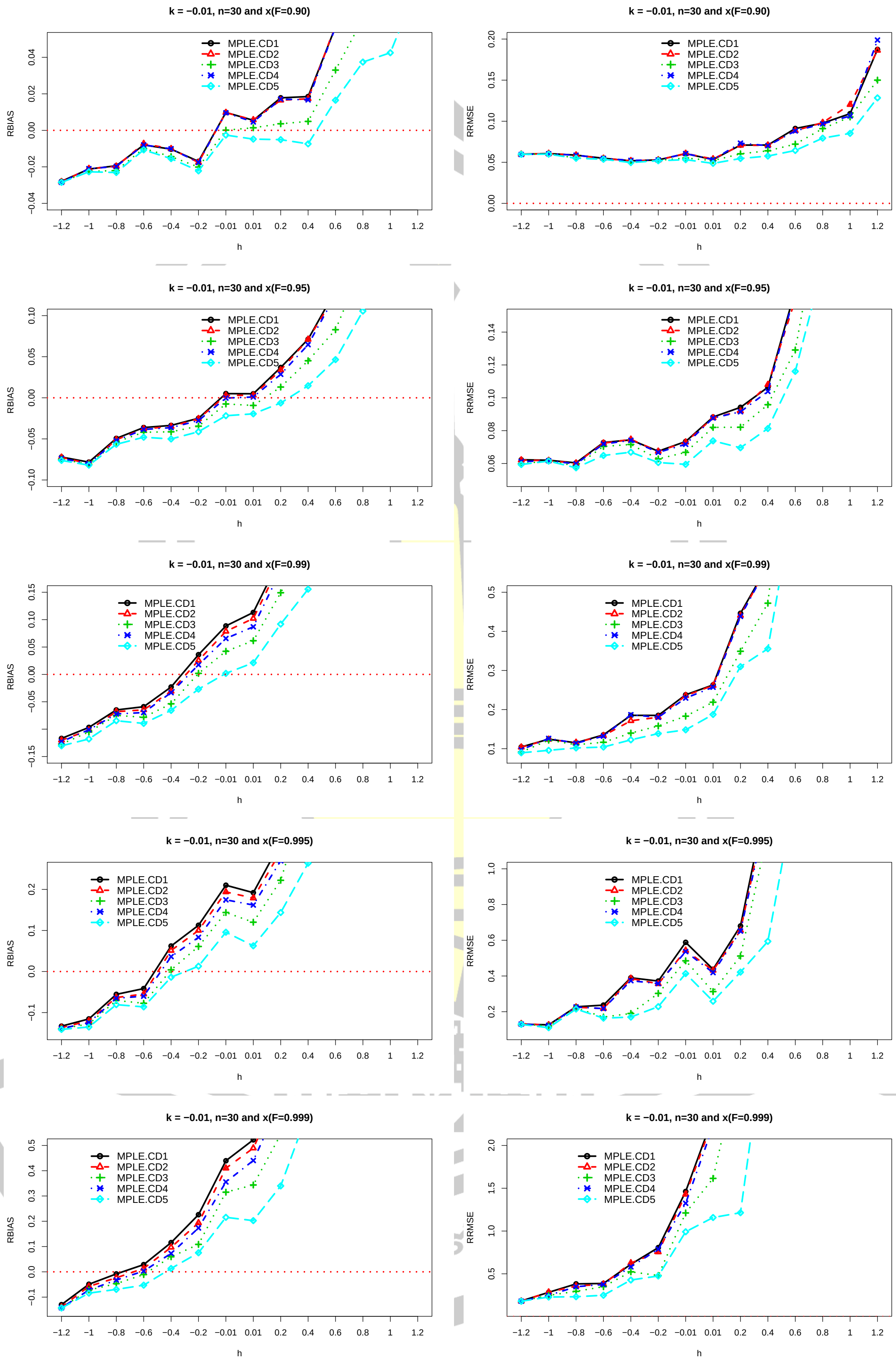

Figure A.9: RBIAS and RRMSE of the all quantile estimators of MPLE.CD1 to MPLE.CD5 for value of $k = -0.01$ and sample size $n = 30$.

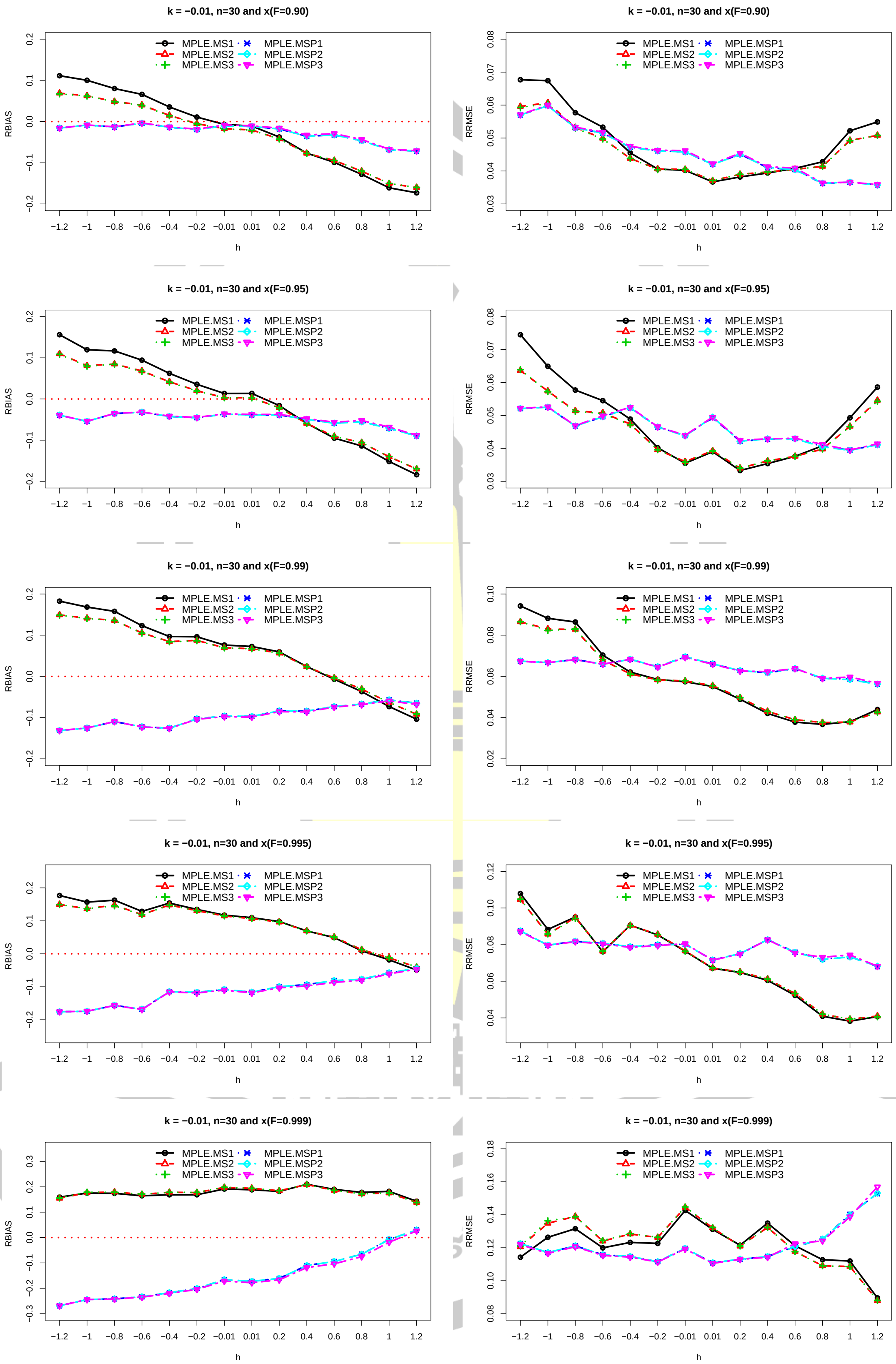

Figure A.10: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MPLE.MSP3 for value of $k = -0.01$ and sample size $n = 30$.

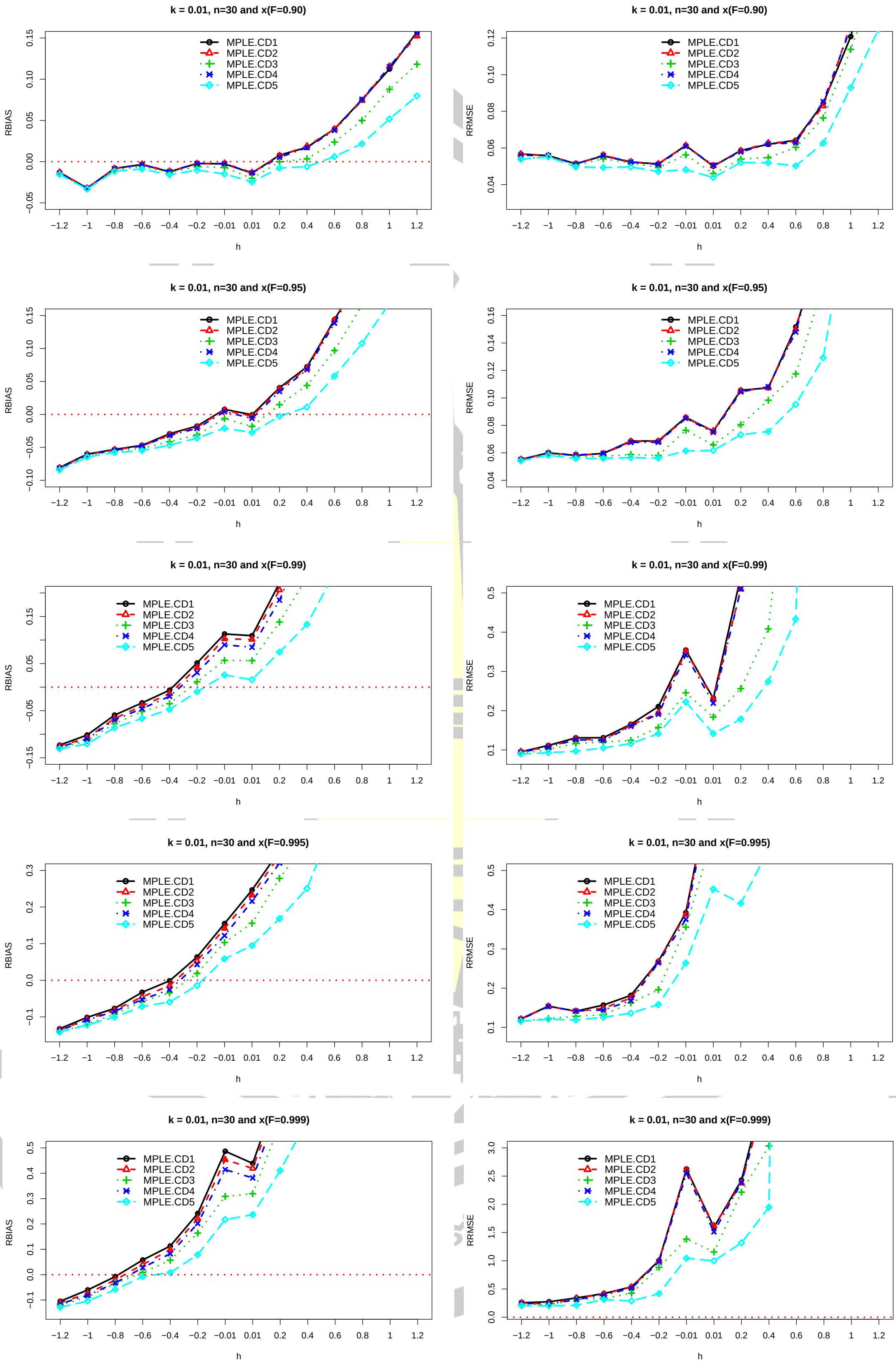

Figure A.11: RBIAS and RRMSE of the all quantile estimators of MPLE.CD1 to MPLE.CD5 for value of $k = 0.01$ and sample size $n = 30$.

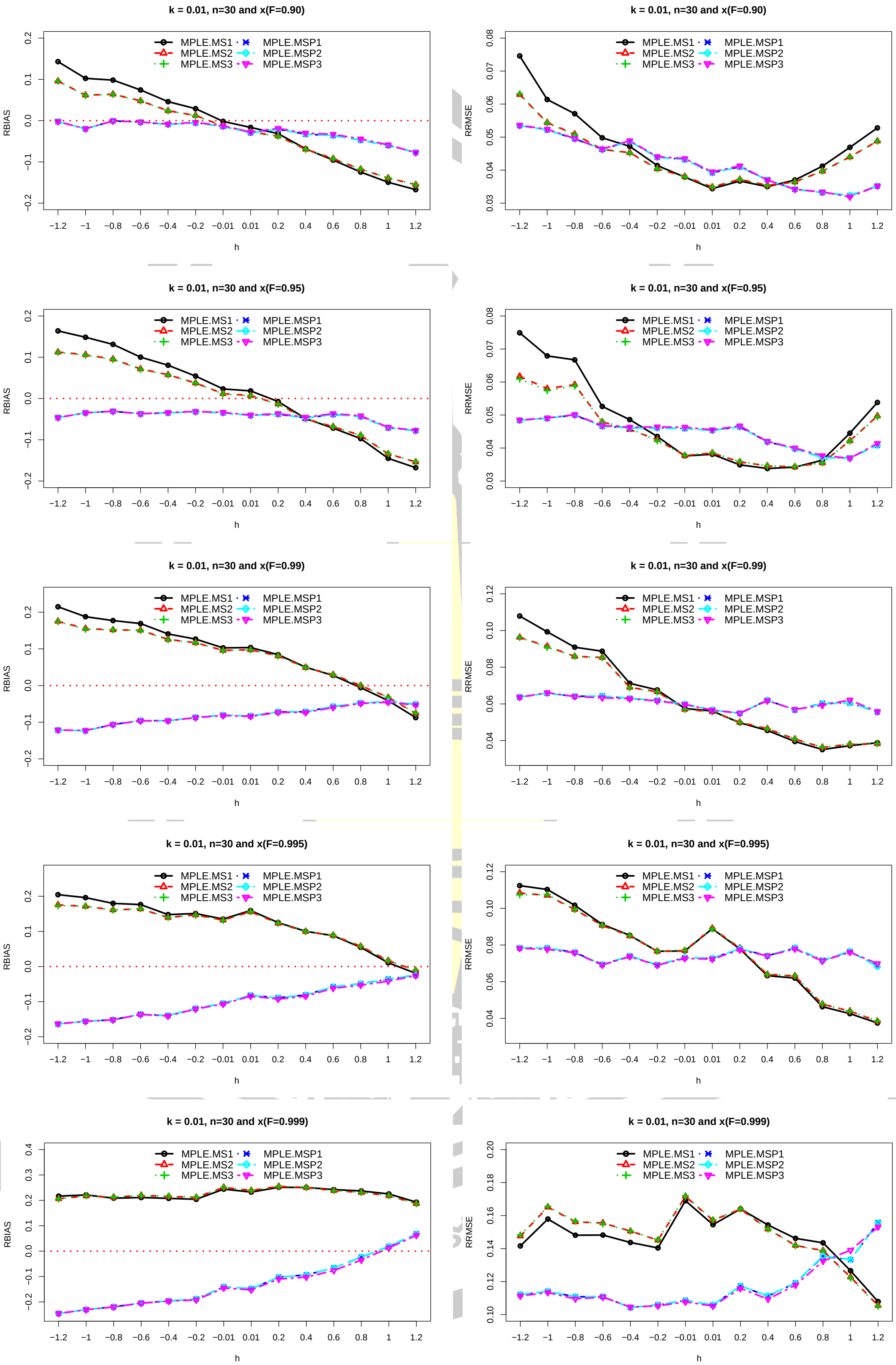

Figure A.12: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MPLE.MSP3 for value of $k = 0.01$ and sample size $n = 30$.

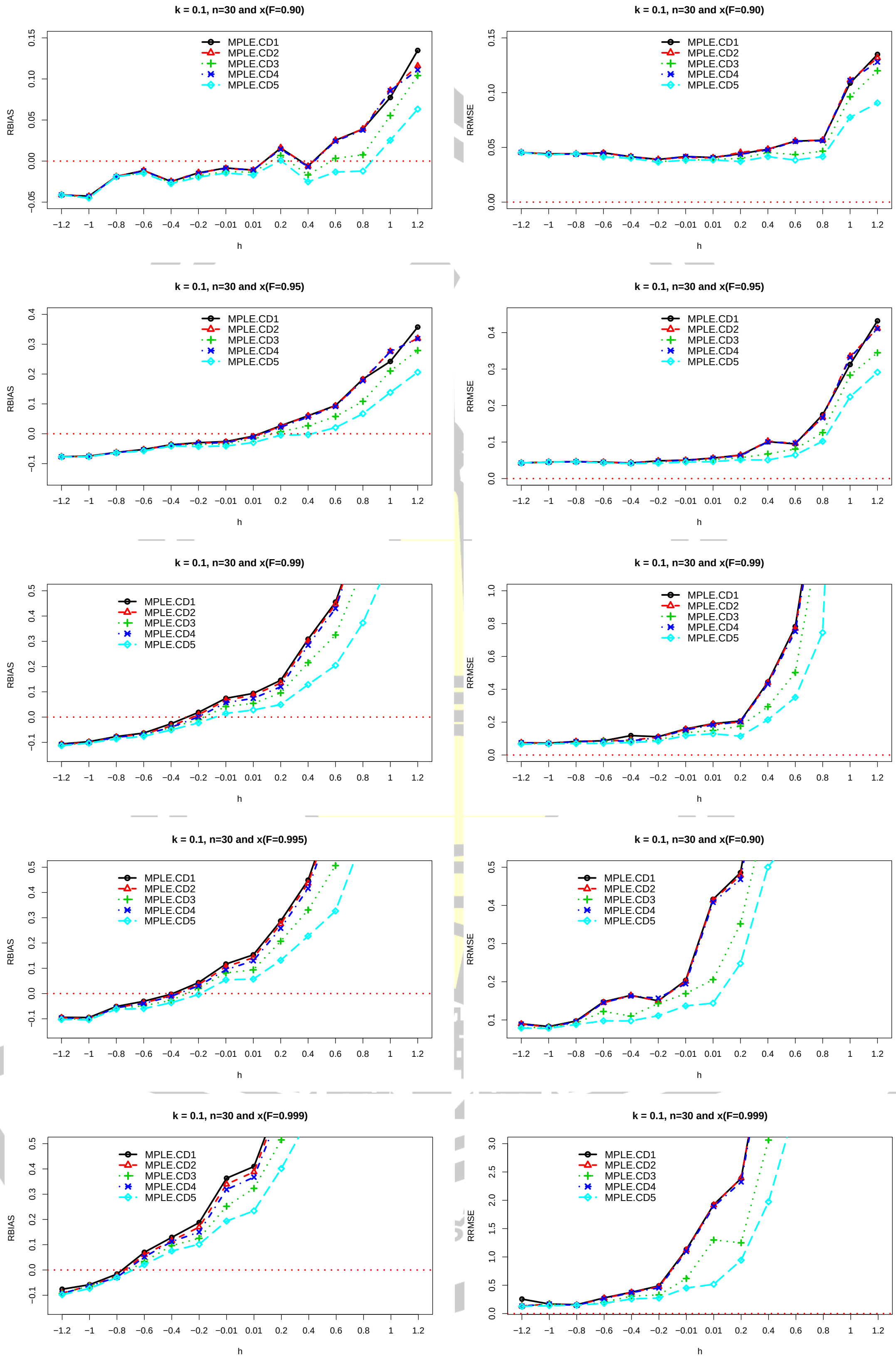

Figure A.13: RBIAS and RRMSE of the all quantile estimators of MPLE.CD1 to MPLE.CD5 for value of $k = 0.1$ and sample size $n = 30$.

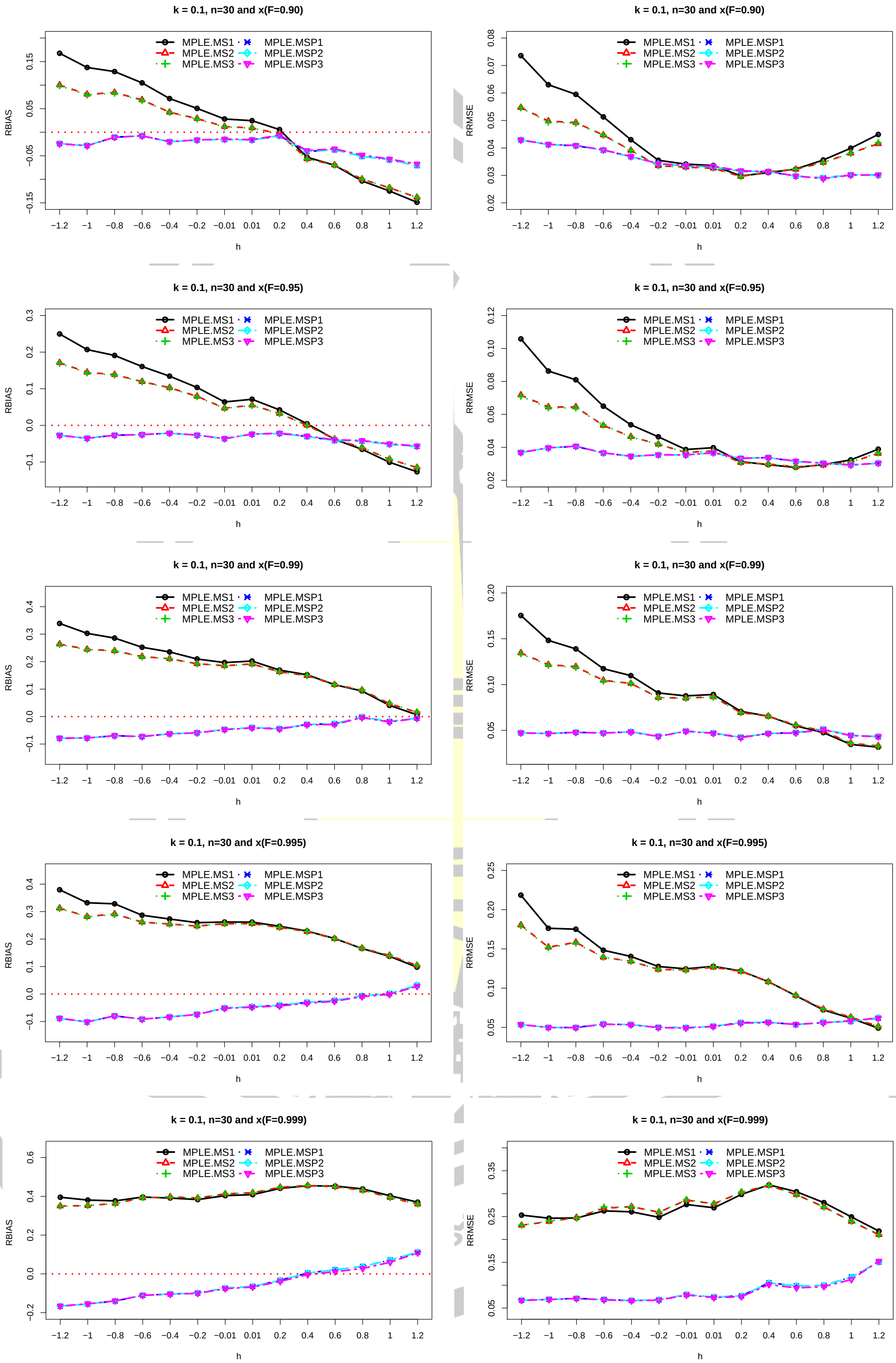

Figure A.14: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MPLE.MSP3 for value of $k = 0.1$ and sample size $n = 30$.

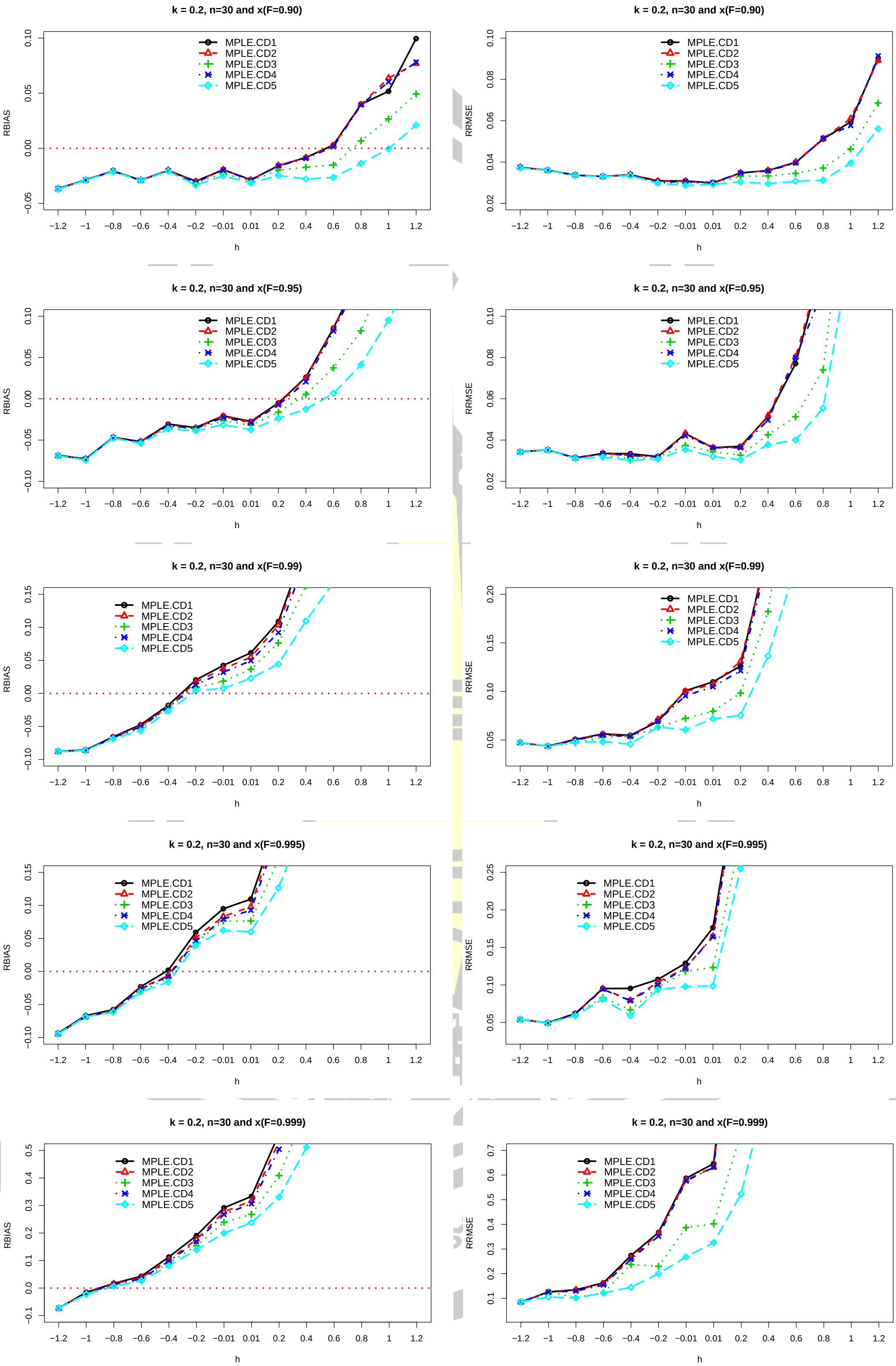

Figure A.15: RBIAS and RRMSE of the all quantile estimators of MPLE.CD1 to MPLE.CD5 for value of $k = 0.2$ and sample size $n = 30$.

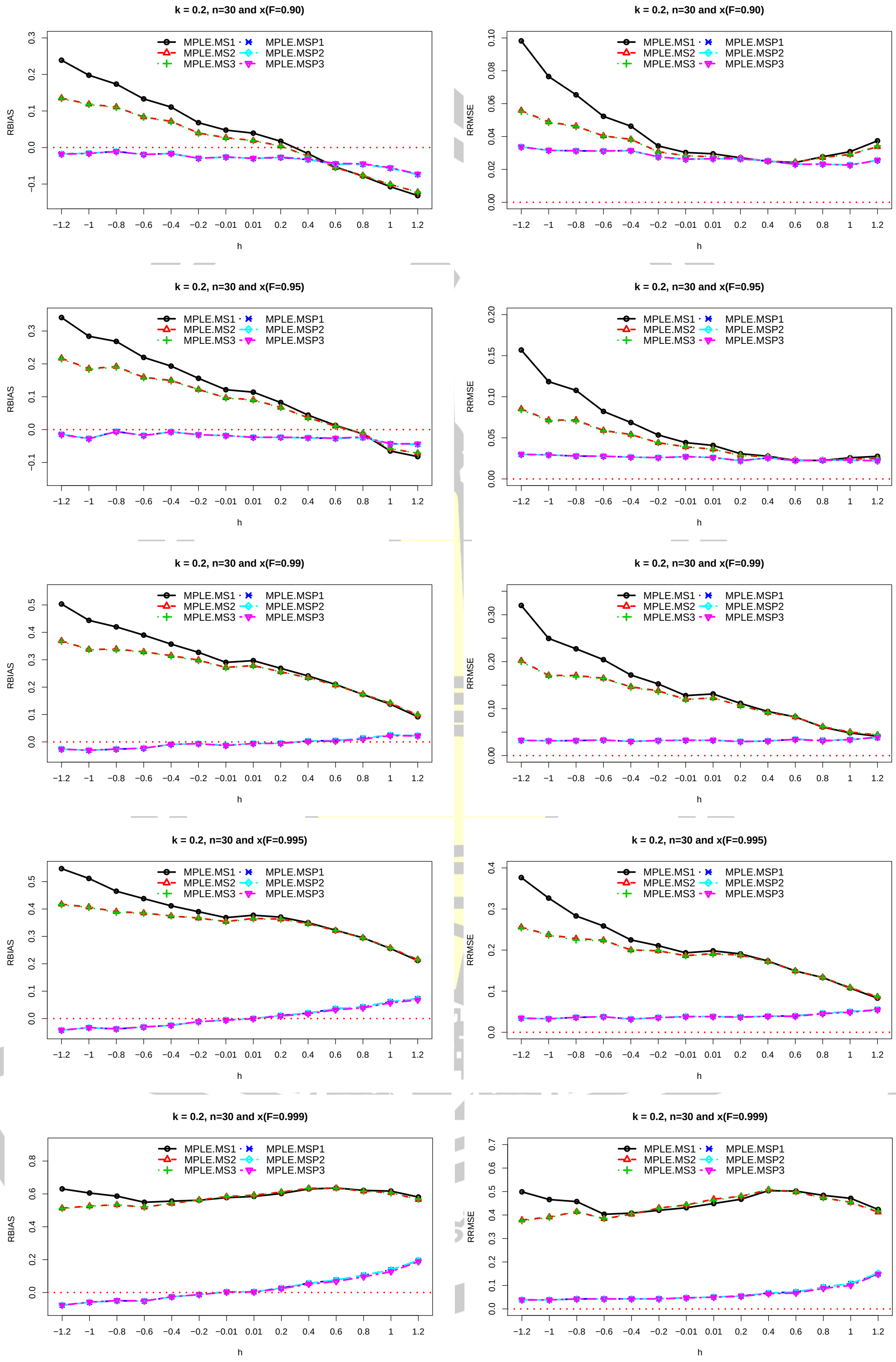

Figure A.16: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MPLE.MSP3 for value of $k = 0.2$ and sample size $n = 30$.

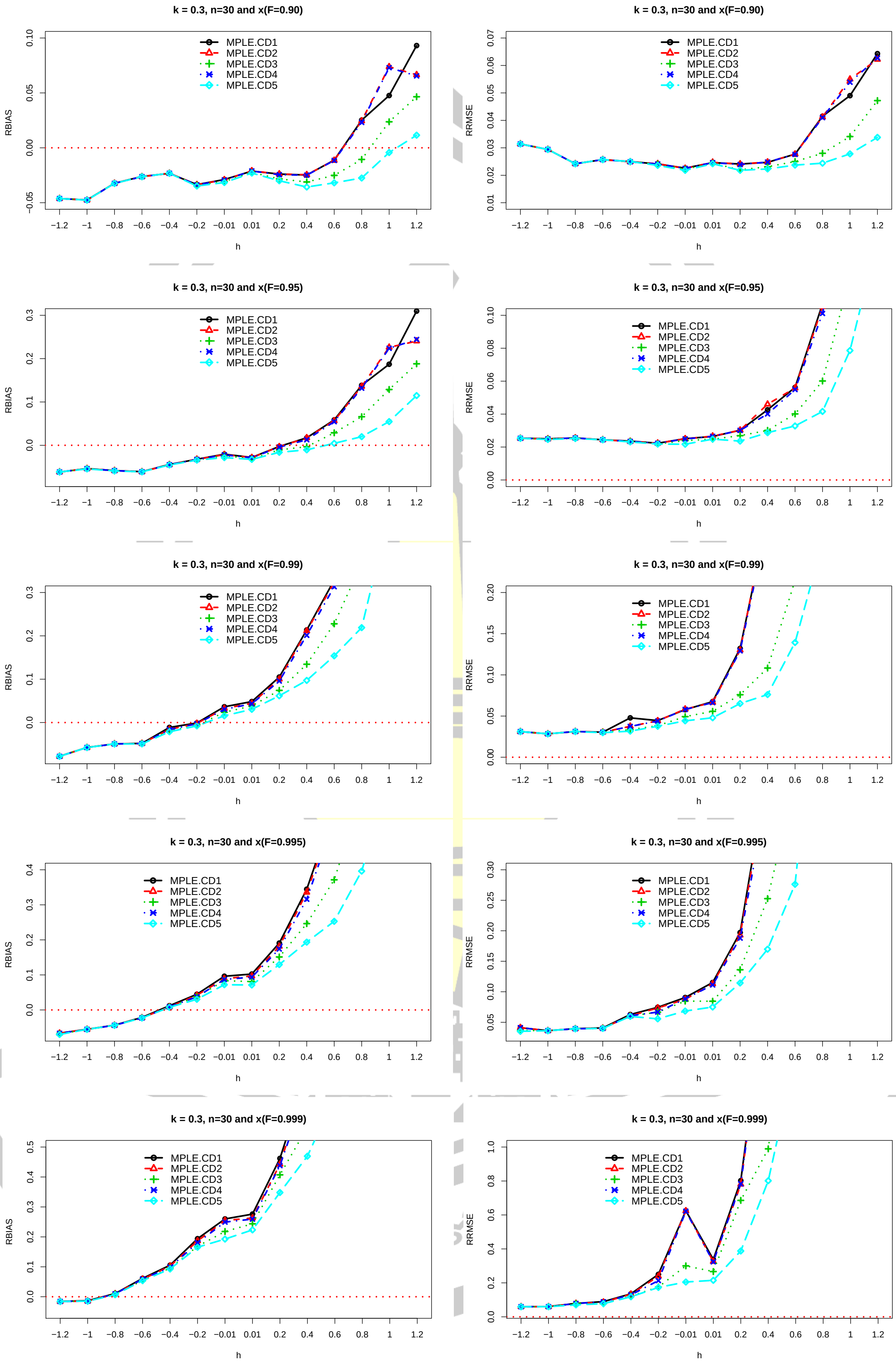

Figure A.17: RBIAS and RRMSE of the all quantile estimators of MPLE.CD1 to MPLE.CD5 for value of $k = 0.3$ and sample size $n = 30$.

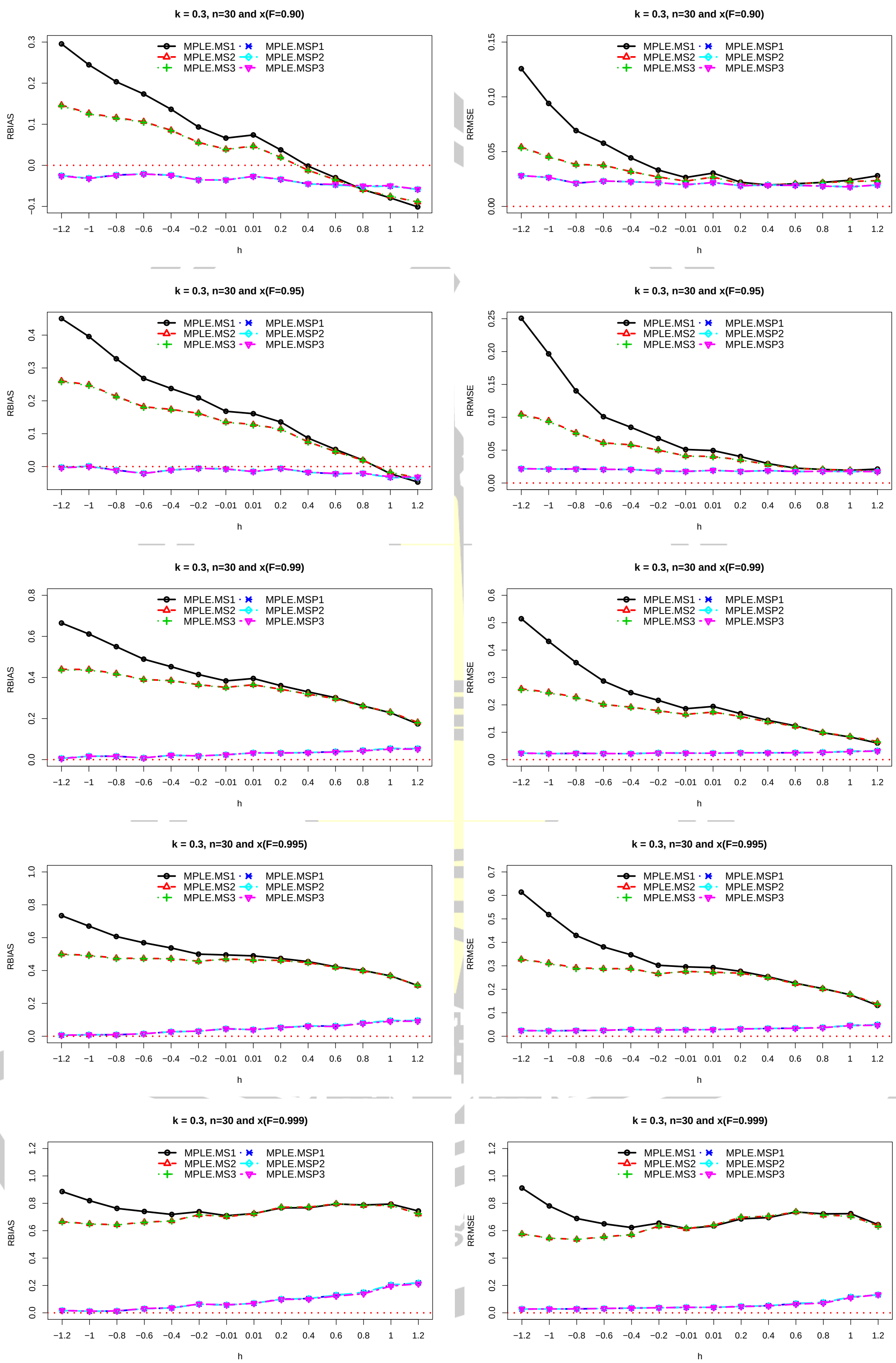

Figure A.18: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MPLE.MSP3 for value of $k = 0.3$ and sample size $n = 30$.

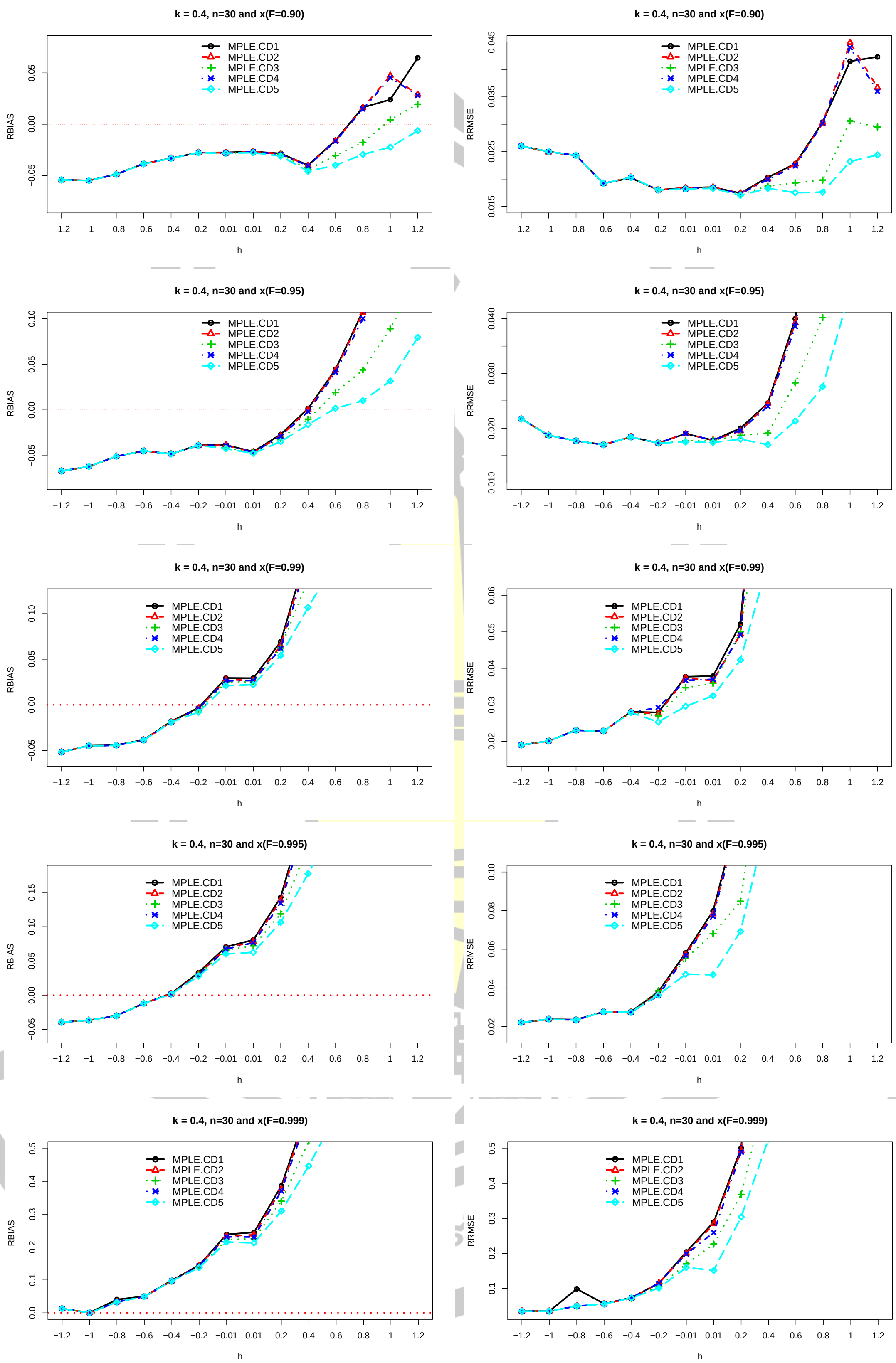

Figure A.19: RBIAS and RRMSE of the all quantile estimators of MPLE.CD1 to MPLE.CD5 for value of $k = 0.4$ and sample size $n = 30$.

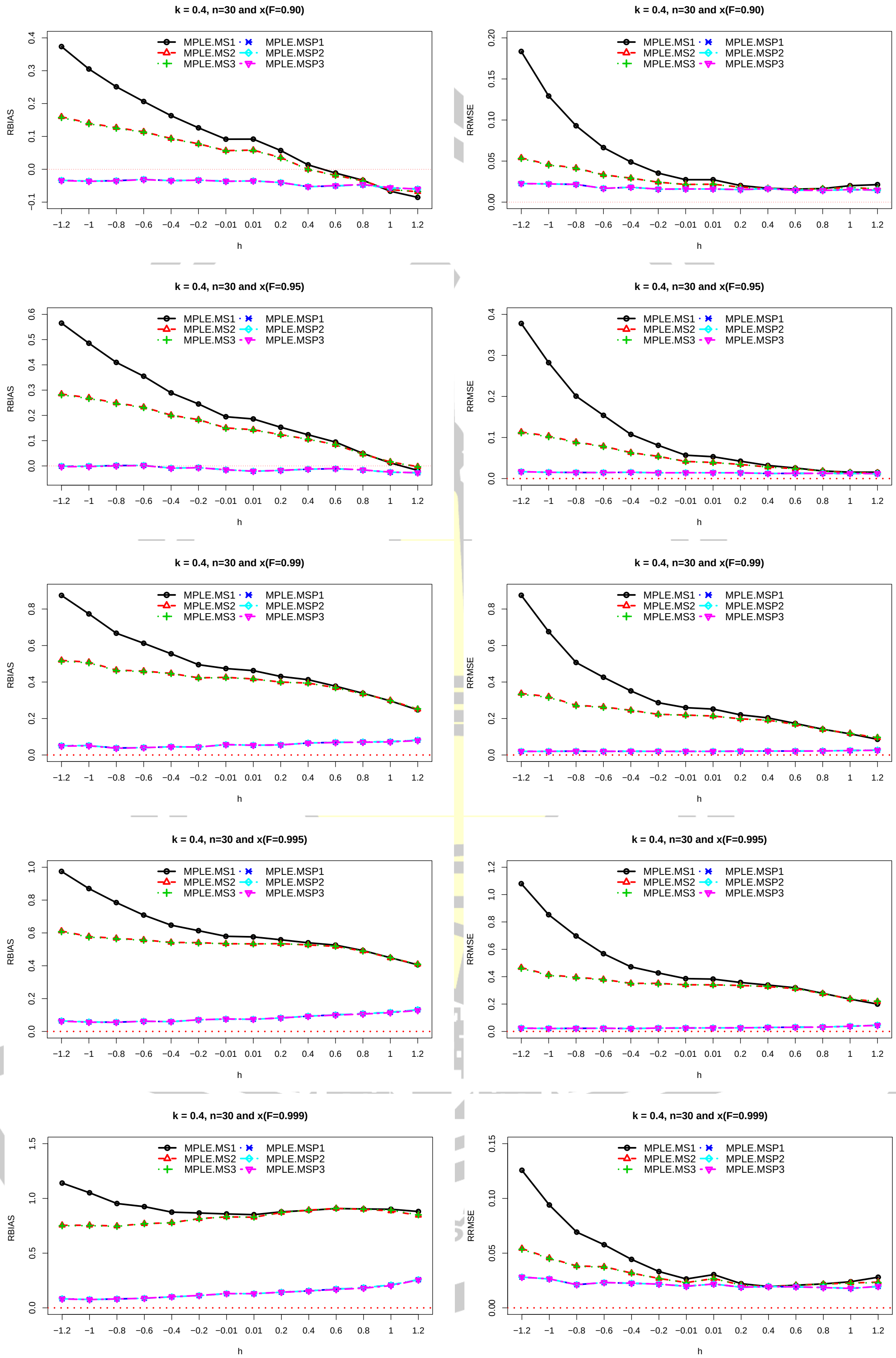

Figure A.20: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MPLE.MSP3 for value of $k = 0.4$ and sample size $n = 30$.

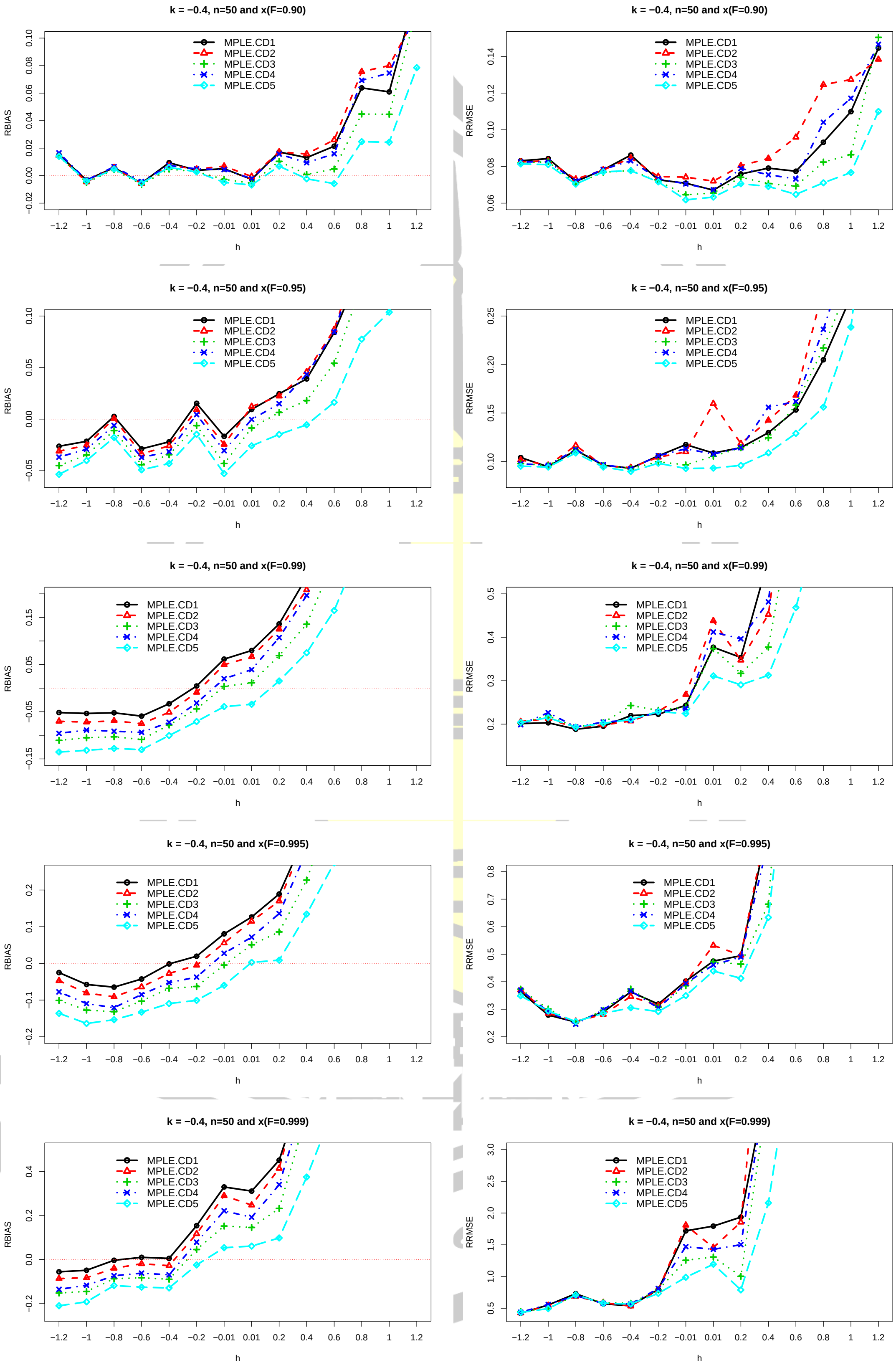

Figure A.21: RBIAS and RRMSE of the all quantile estimators of MPLE.CD1 to MPLE.CD5 for value of $k = -0.4$ and sample size $n = 50$.

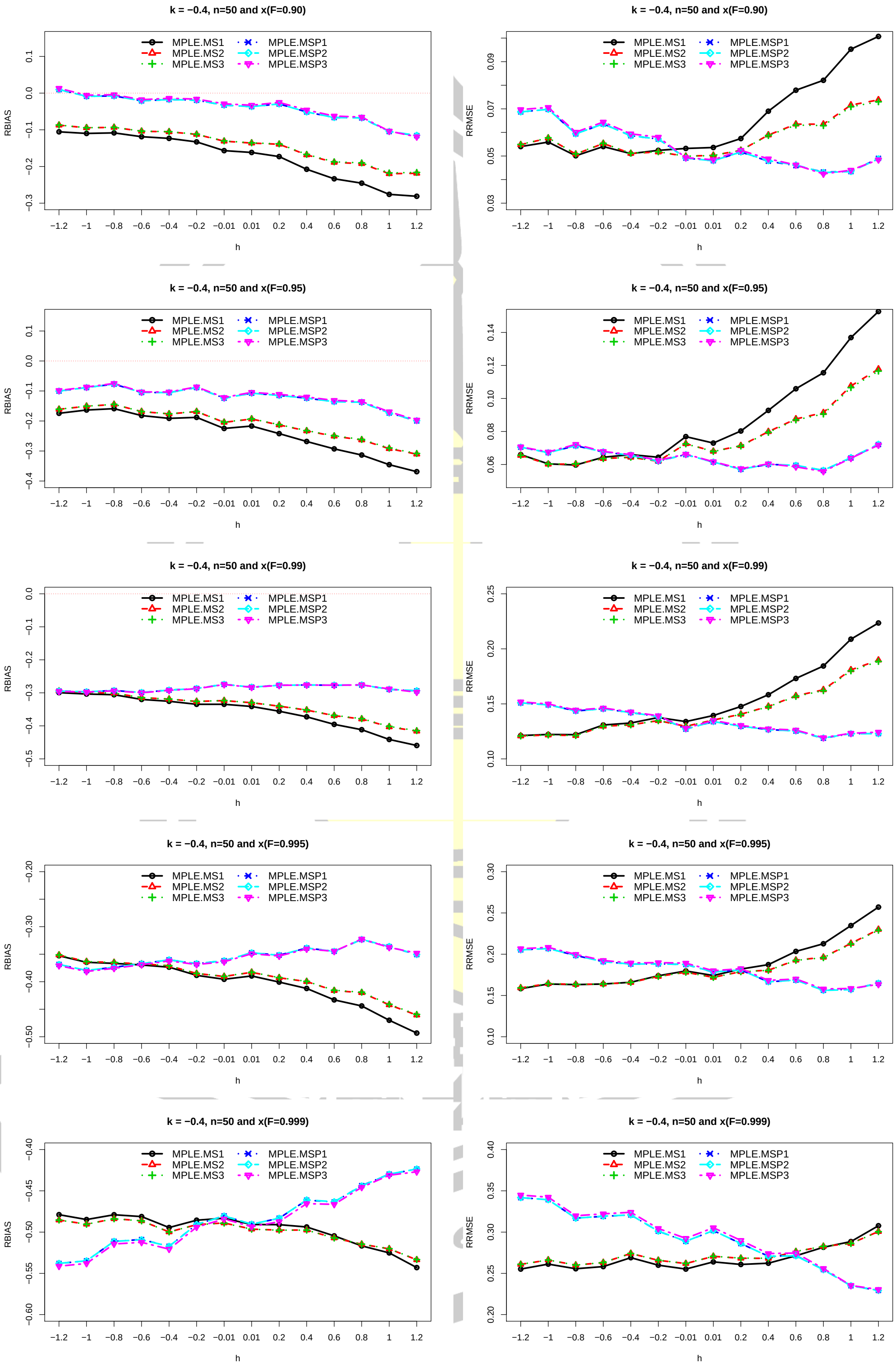

Figure A.22: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MPLE.MSP3 for value of $k = -0.4$ and sample size $n = 50$.

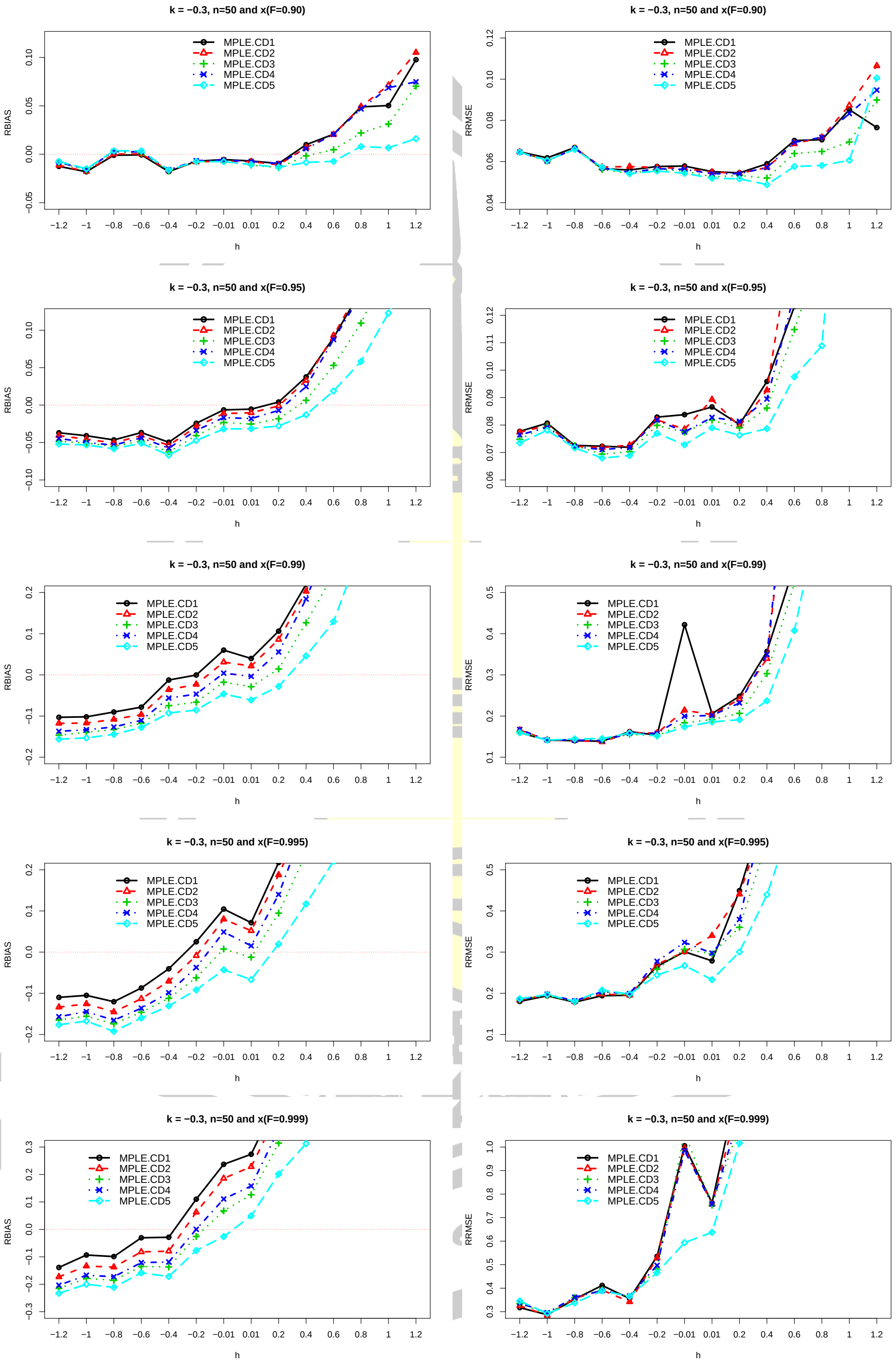

Figure A.23: RBIAS and RRMSE of the all quantile estimators of MPLE.CD1 to MPLE.CD5 for value of $k = -0.3$ and sample size $n = 50$.

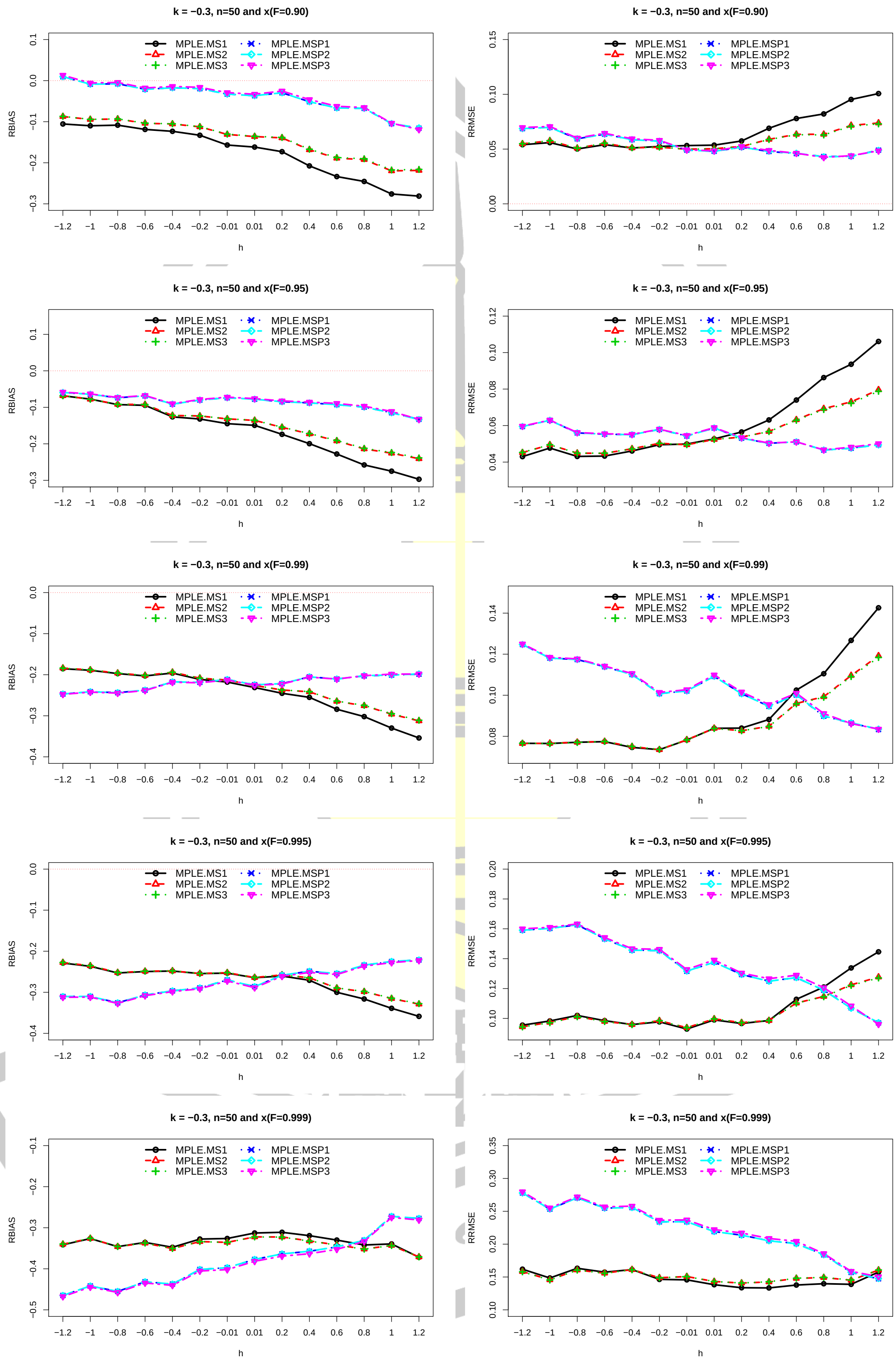

Figure A.24: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MPLE.MSP3 for value of $k = -0.3$ and sample size $n = 50$.

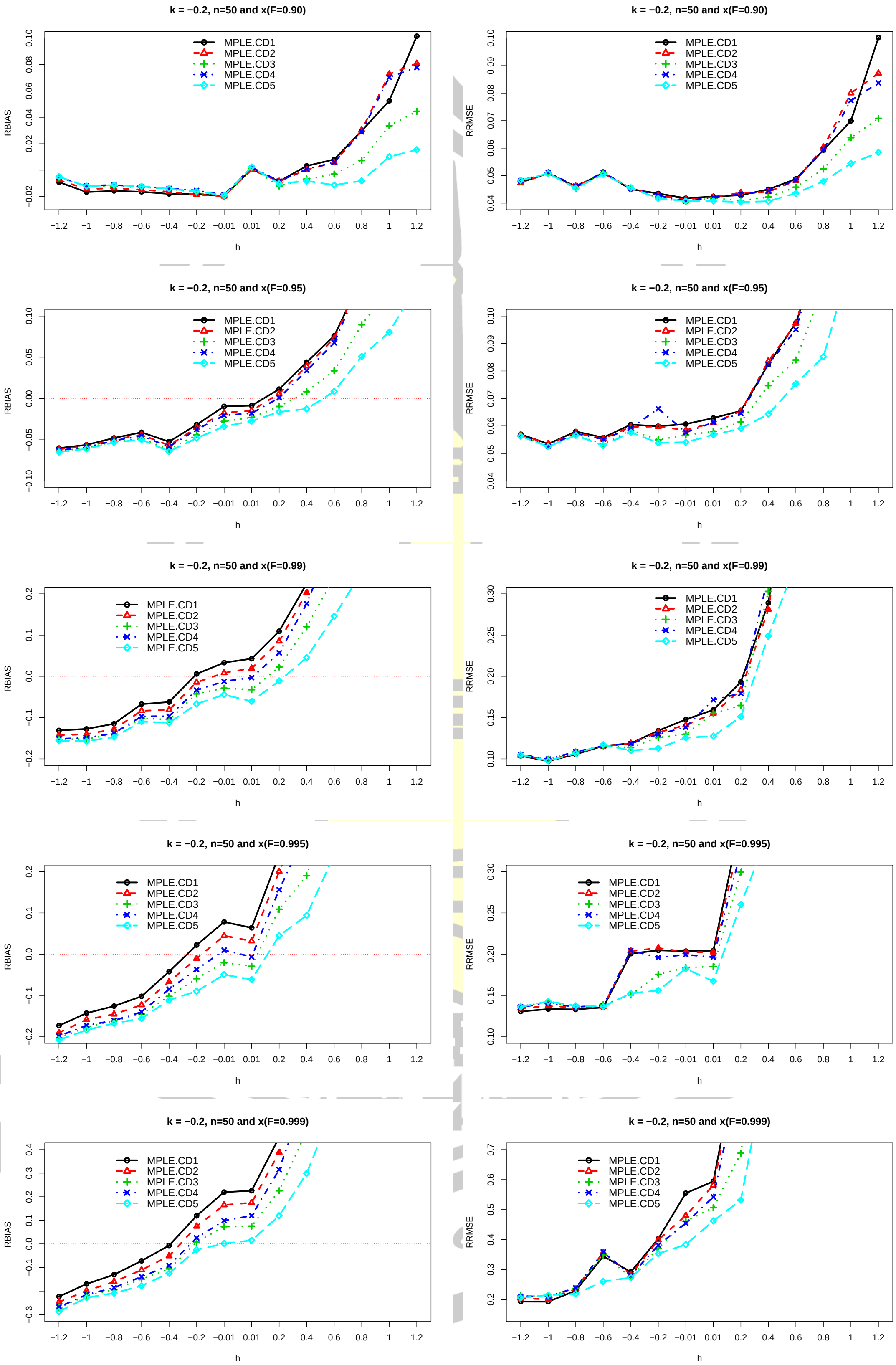

Figure A.25: RBIAS and RRMSE of the all quantile estimators of MPLE.CD1 to MPLE.CD5 for value of $k = -0.2$ and sample size $n = 50$.

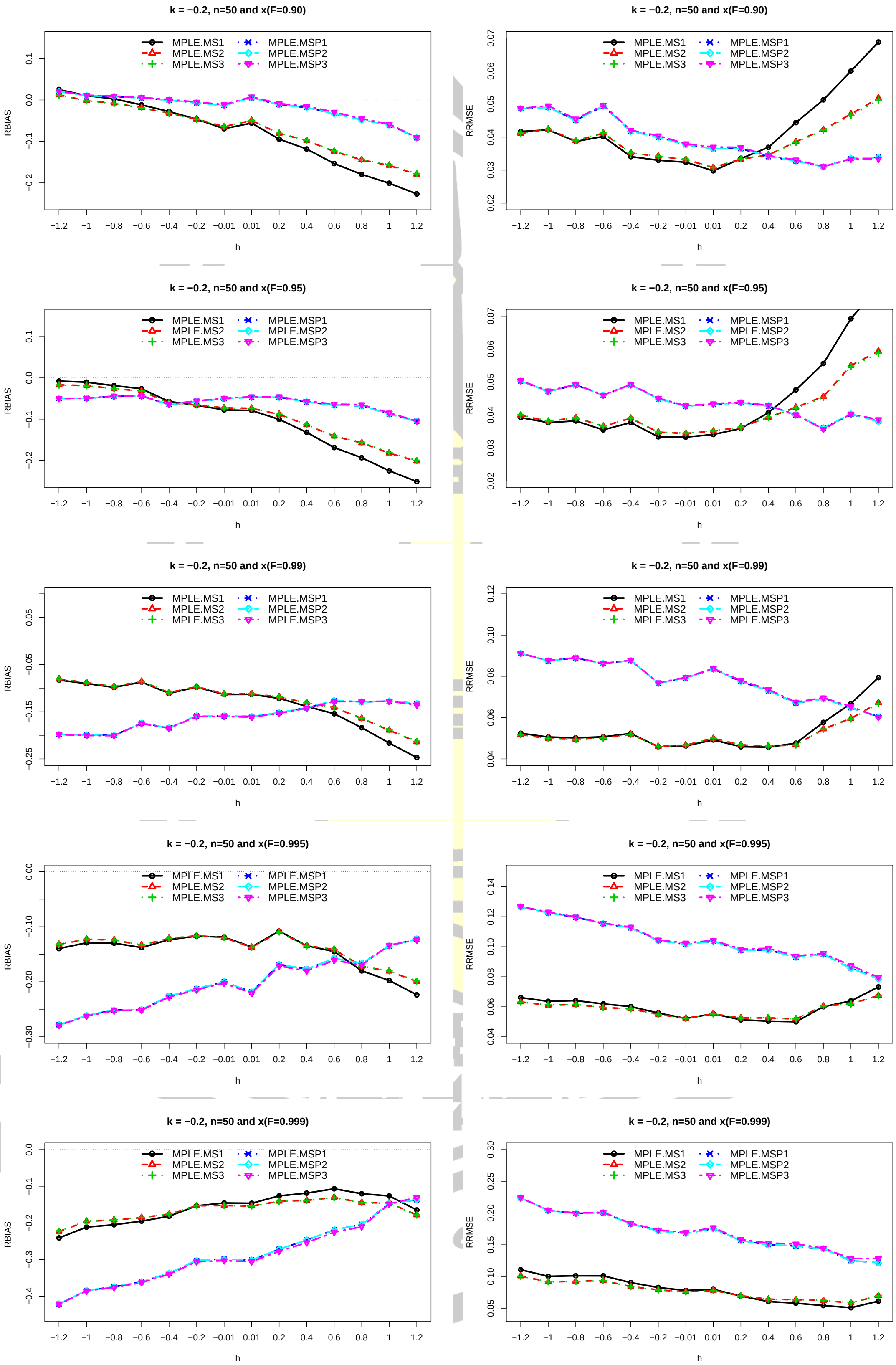

Figure A.26: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MPLE.MSP3 for value of $k = -0.2$ and sample size $n = 50$.

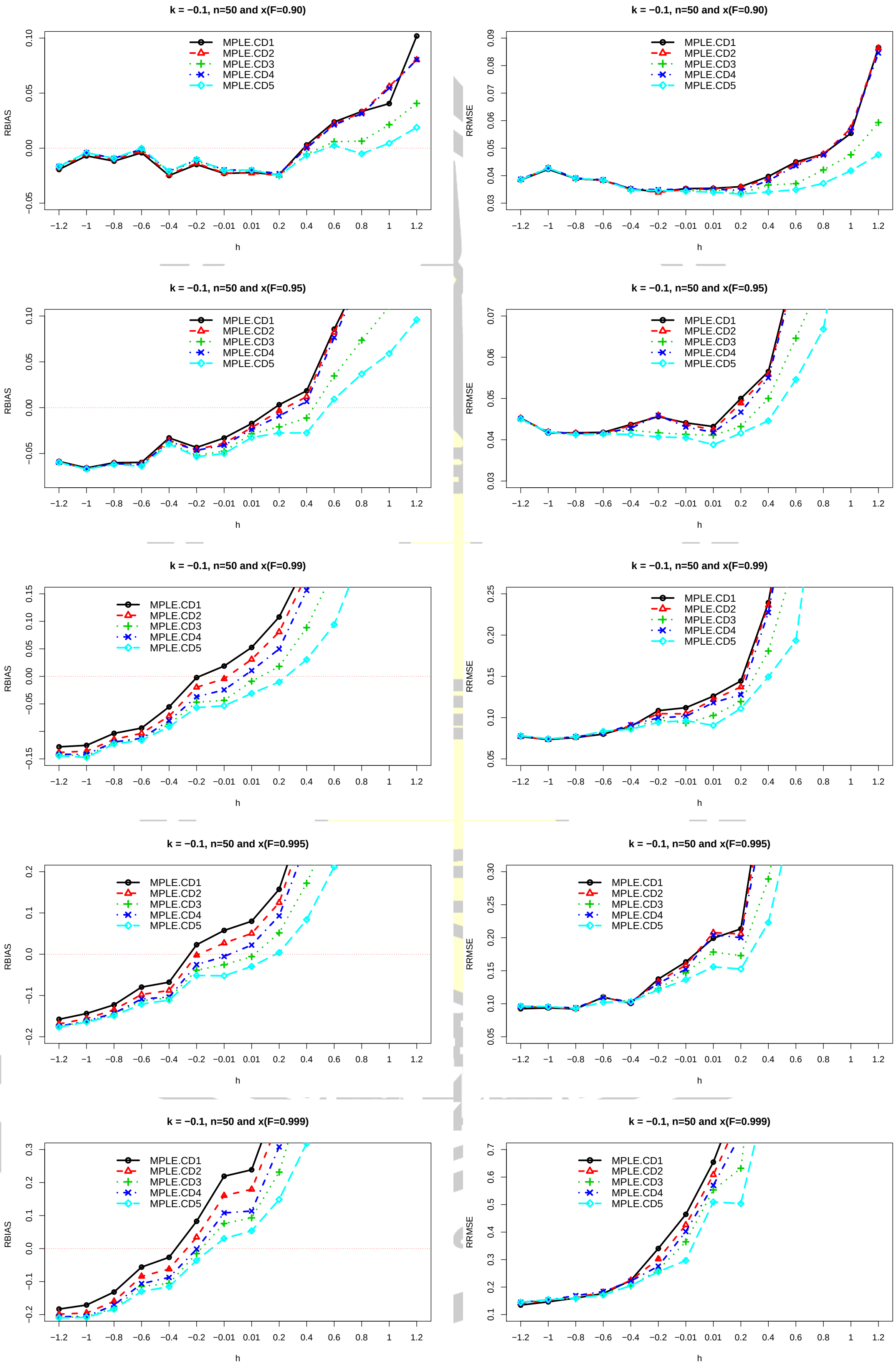

Figure A.27: RBIAS and RRMSE of the all quantile estimators of MPLE.CD1 to MPLE.CD5 for value of $k = -0.1$ and sample size $n = 50$.

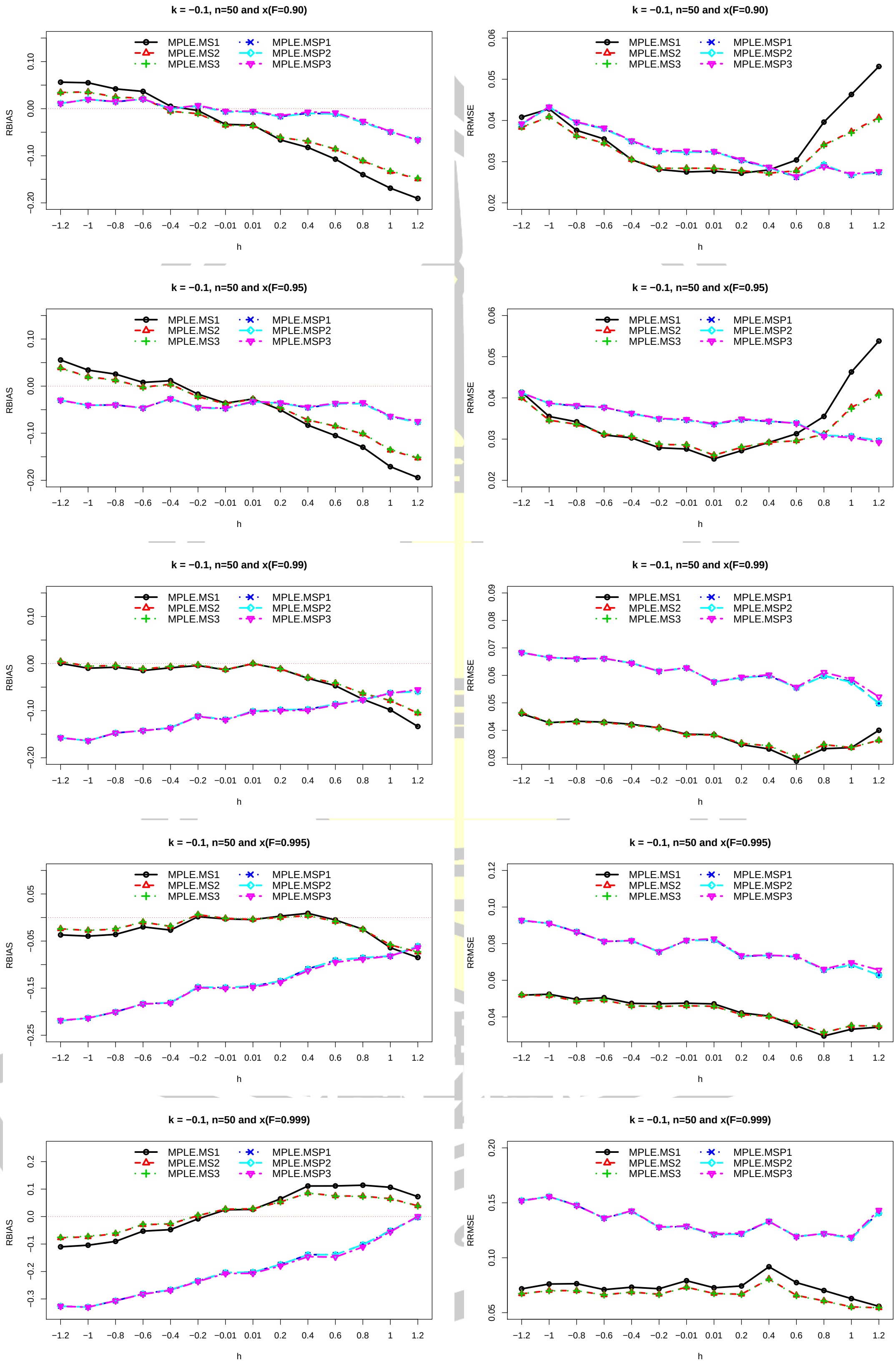

Figure A.28: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MPLE.MSP3 for value of $k = -0.1$ and sample size $n = 50$.

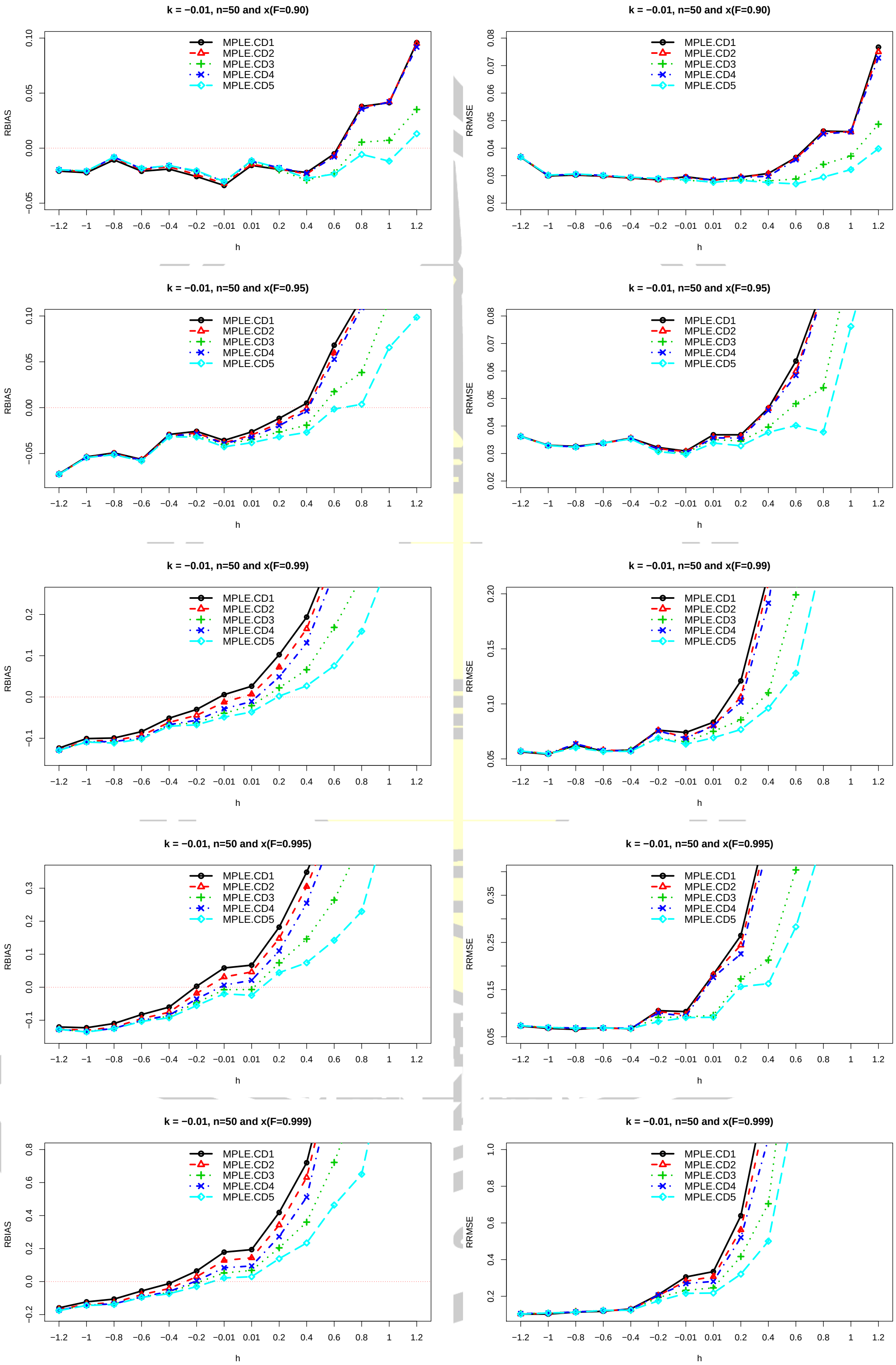

Figure A.29: RBIAS and RRMSE of the all quantile estimators of MPLE.CD1 to MPLE.CD5 for value of $k = -0.01$ and sample size $n = 50$.

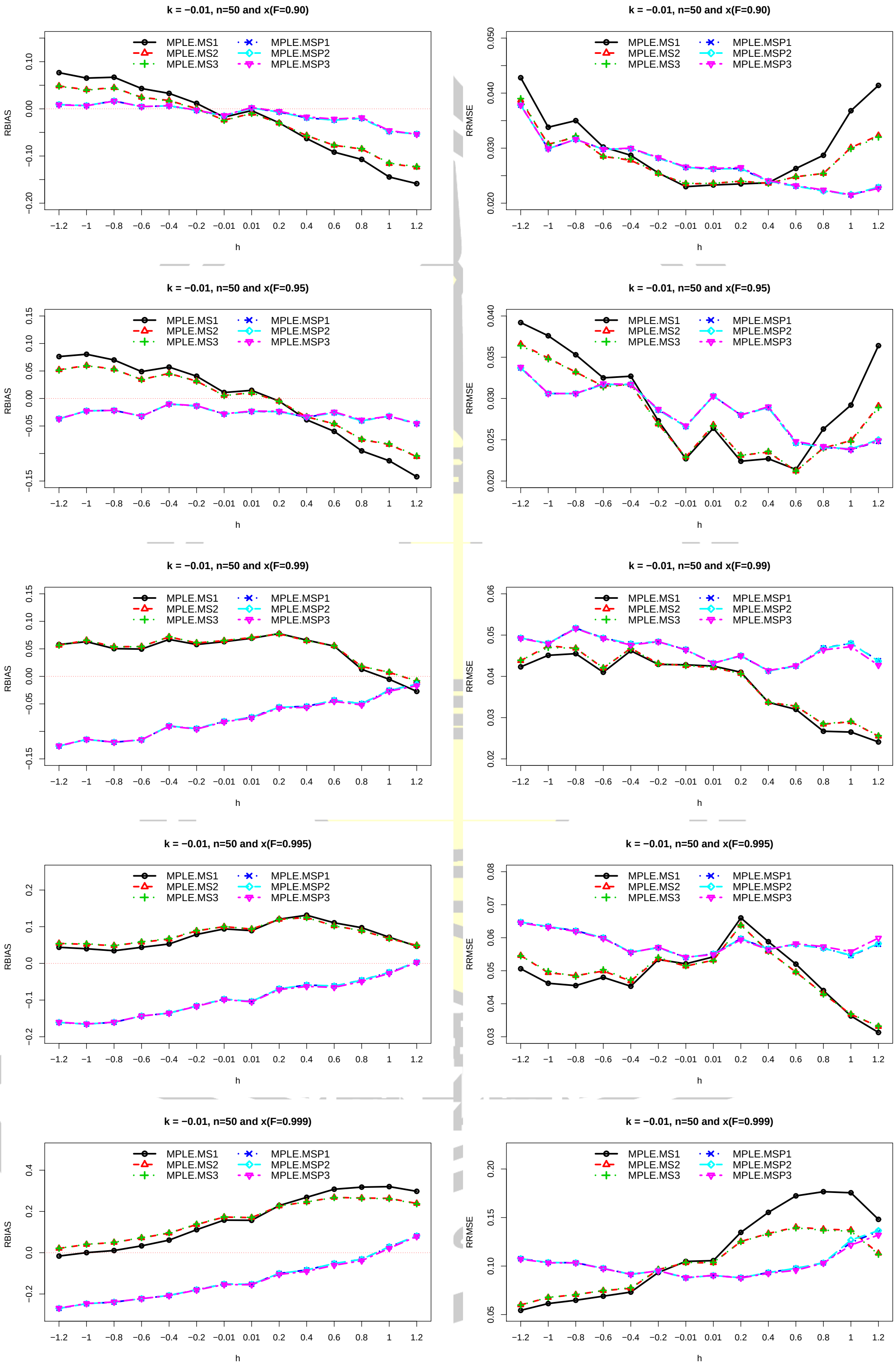

Figure A.30: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MPLE.MSP3 for value of $k = -0.01$ and sample size $n = 50$.

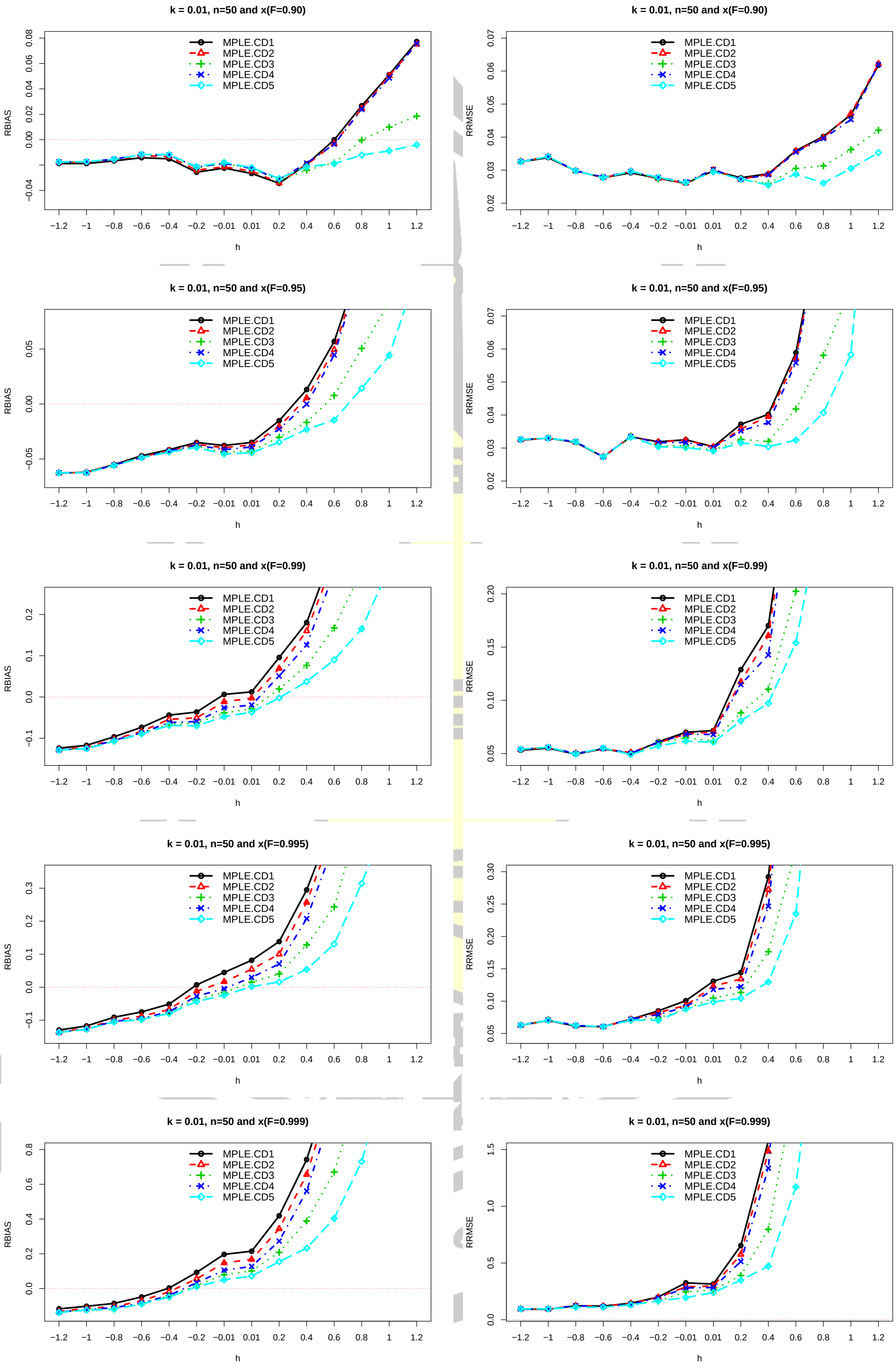

Figure A.31: RBIAS and RRMSE of the all quantile estimators of MPLE.CD1 to MPLE.CD5 for value of $k = 0.01$ and sample size $n = 50$.

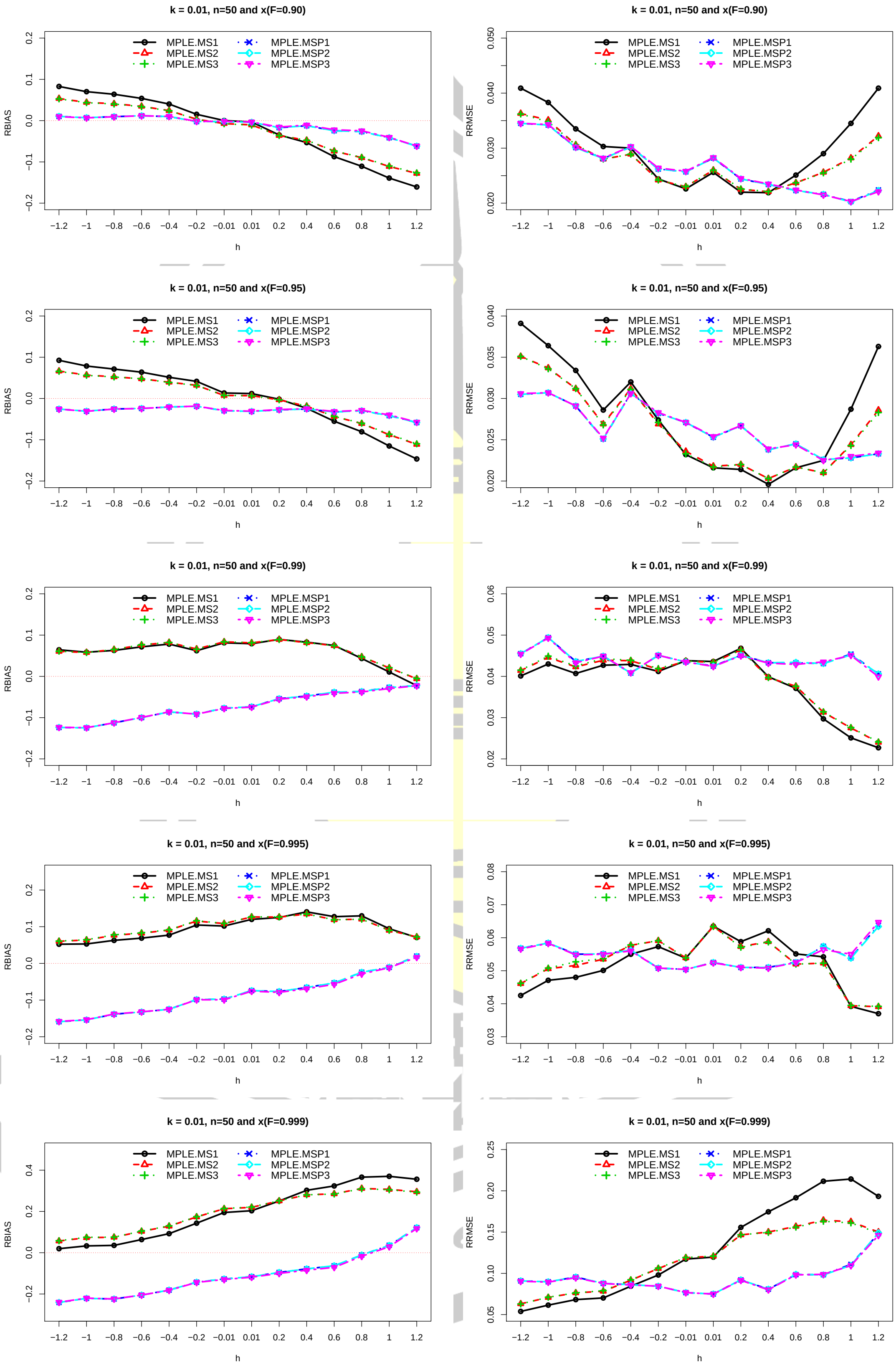

Figure A.32: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MPLE.MSP3 for value of $k = 0.01$ and sample size $n = 50$.

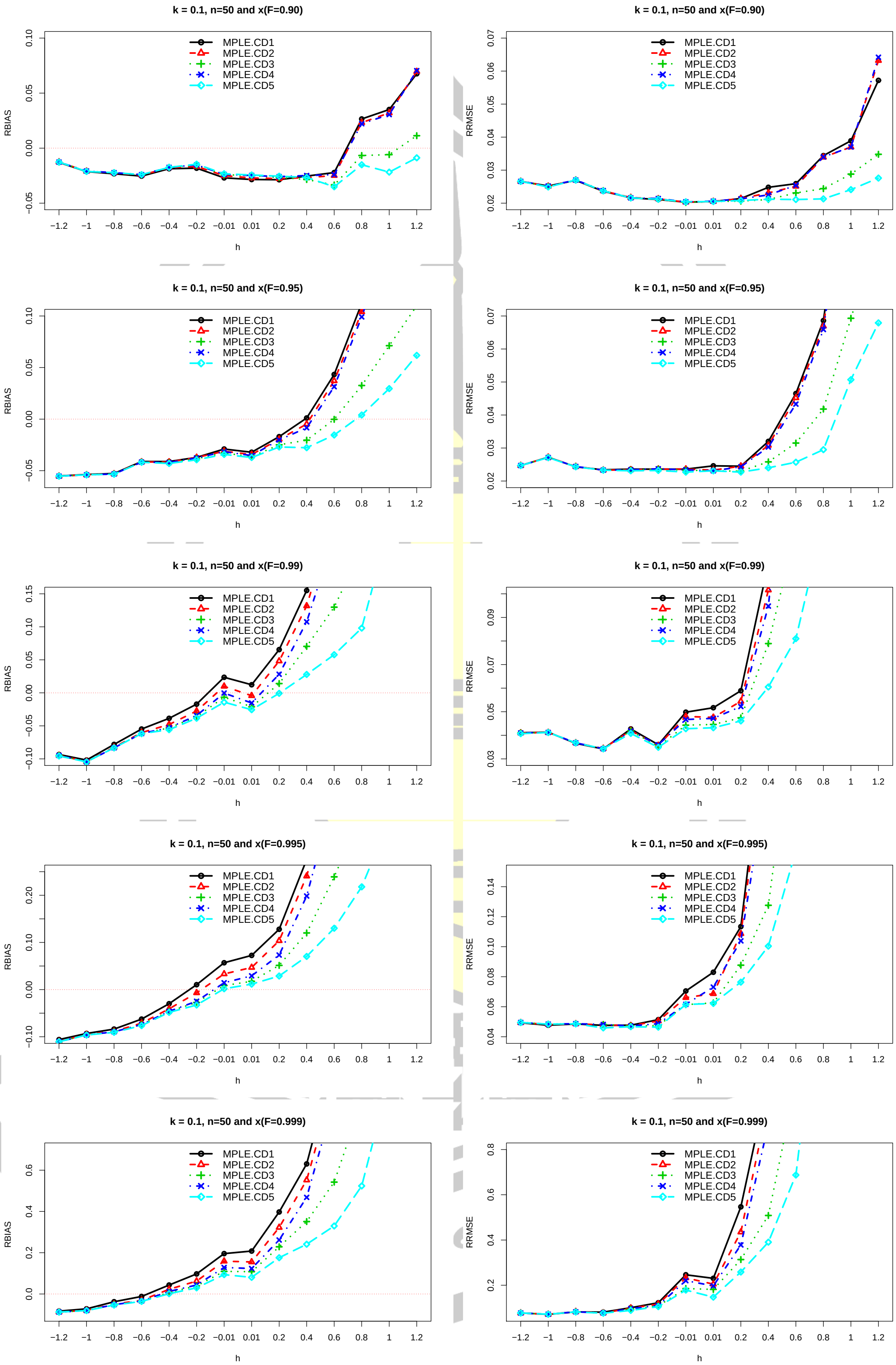

Figure A.33: RBIAS and RRMSE of the all quantile estimators of MPLE.CD1 to MPLE.CD5 for value of $k = 0.1$ and sample size $n = 50$.

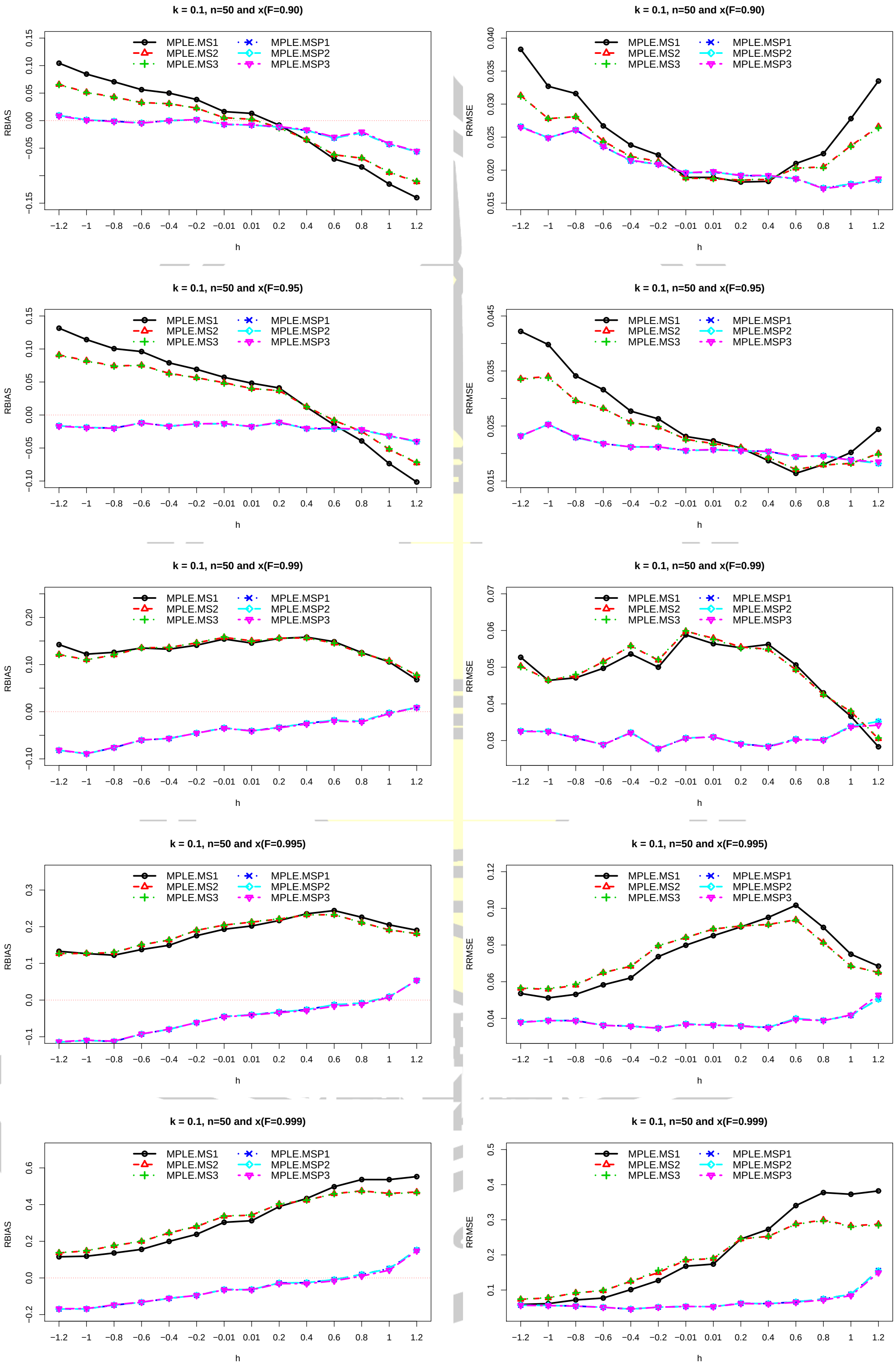

Figure A.34: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MPLE.MSP3 for value of $k = 0.1$ and sample size $n = 50$.

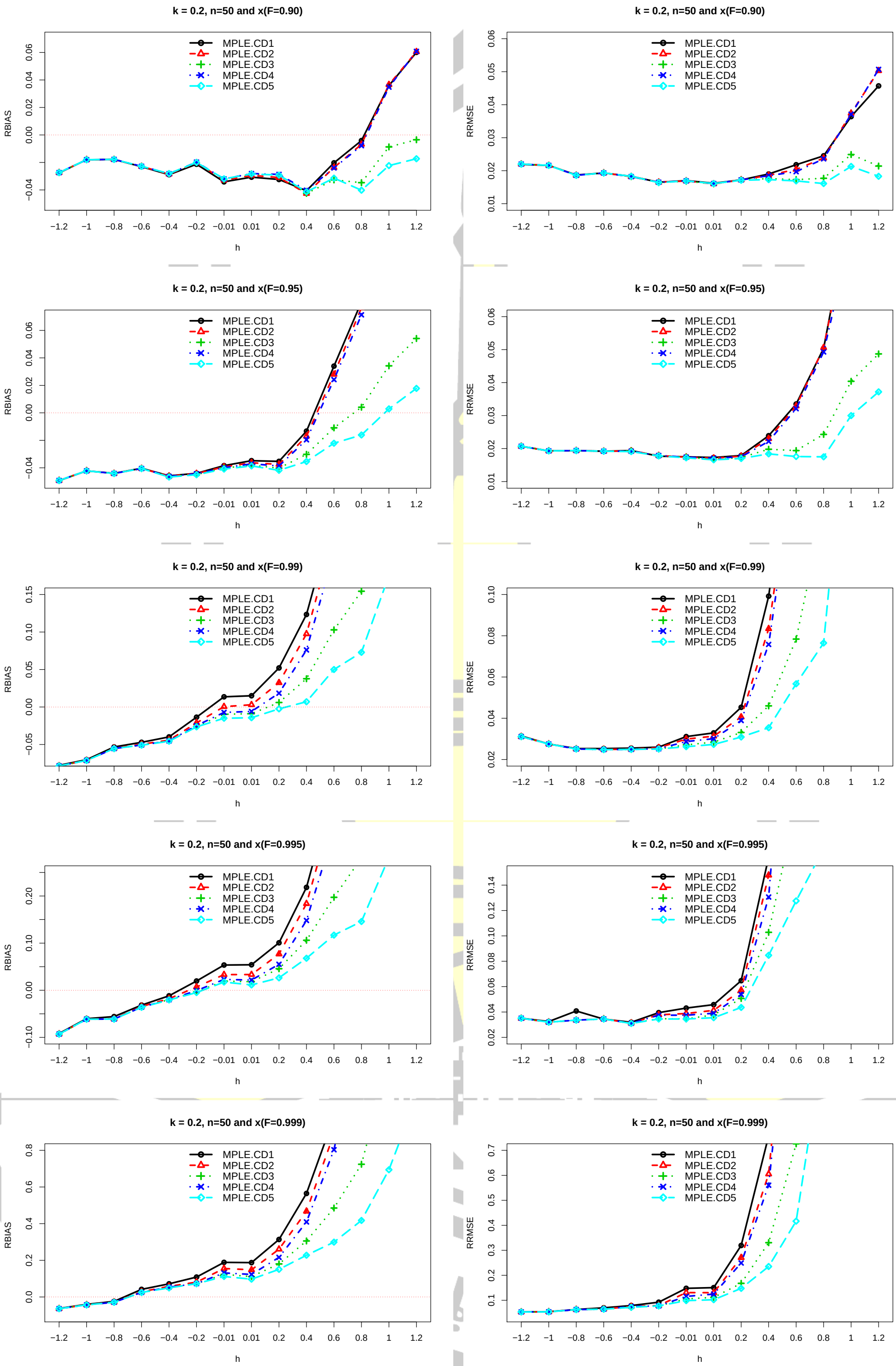

Figure A.35: RBIAS and RRMSE of the all quantile estimators of MPLE.CD1 to MPLE.CD5 for value of $k = 0.2$ and sample size $n = 50$.

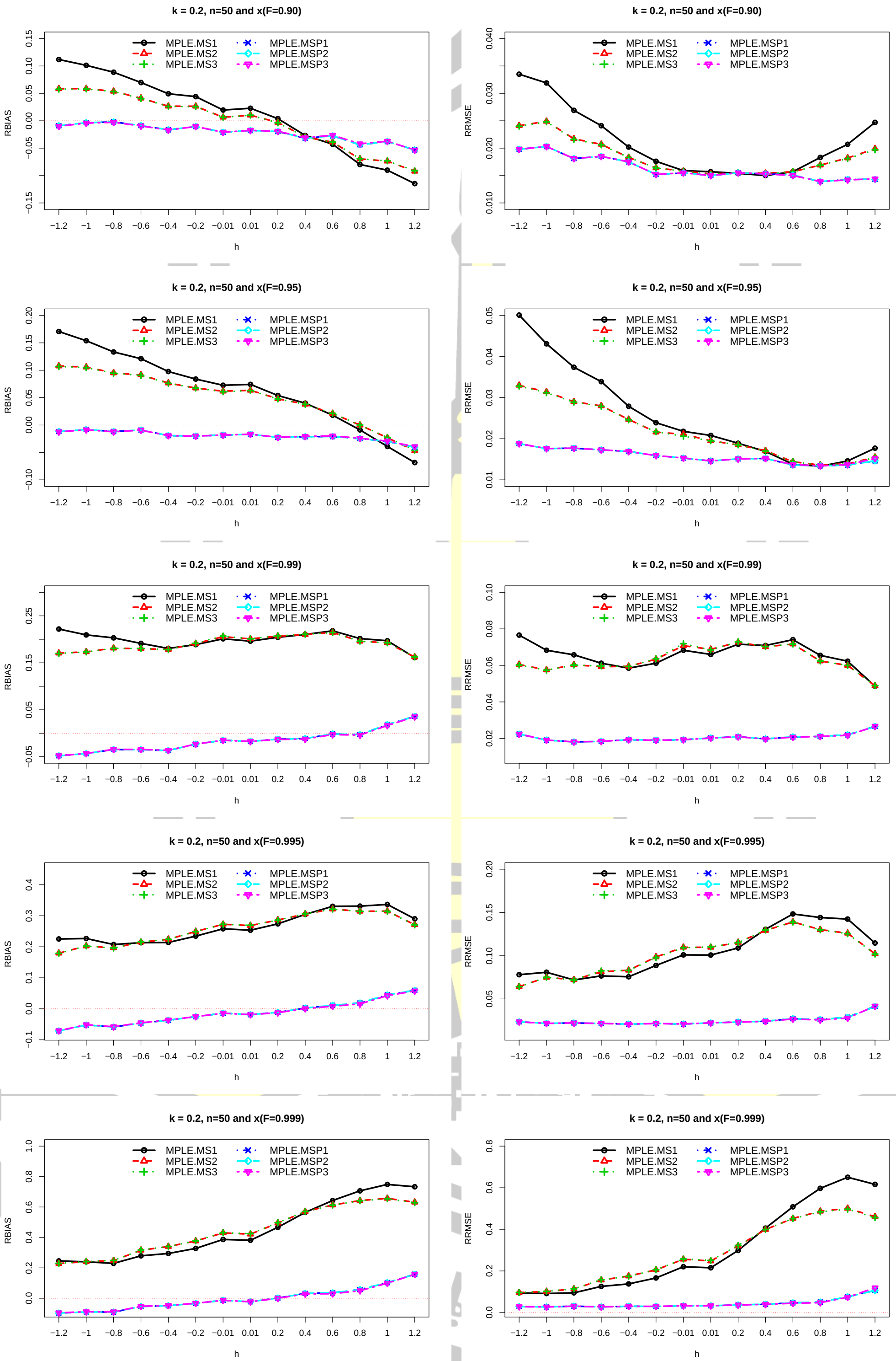

Figure A.36: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MPLE.MSP3 for value of $k = 0.2$ and sample size $n = 50$.

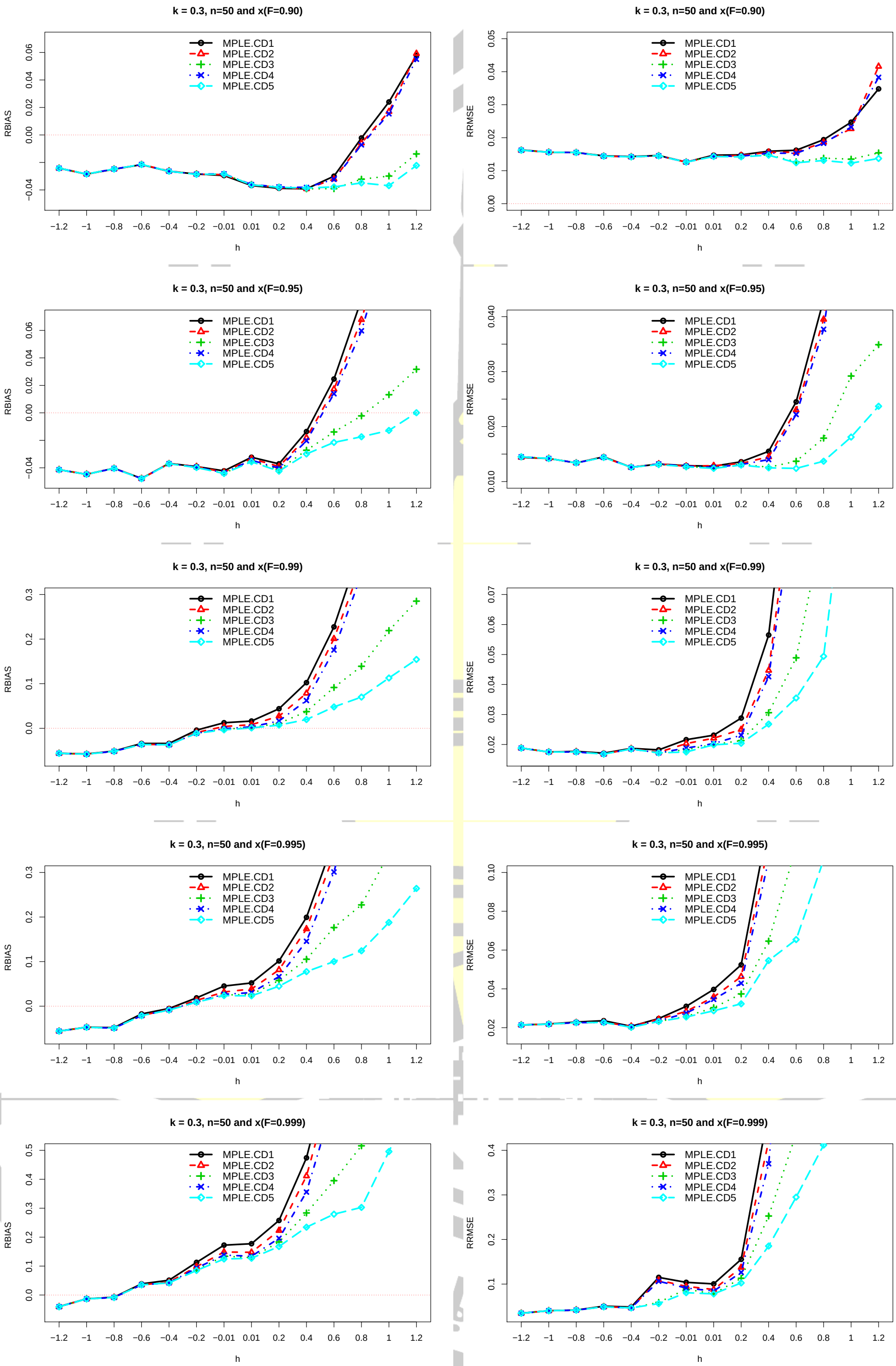

Figure A.37: RBIAS and RRMSE of the all quantile estimators of MPLE.CD1 to MPLE.CD5 for value of $k = 0.3$ and sample size $n = 50$.

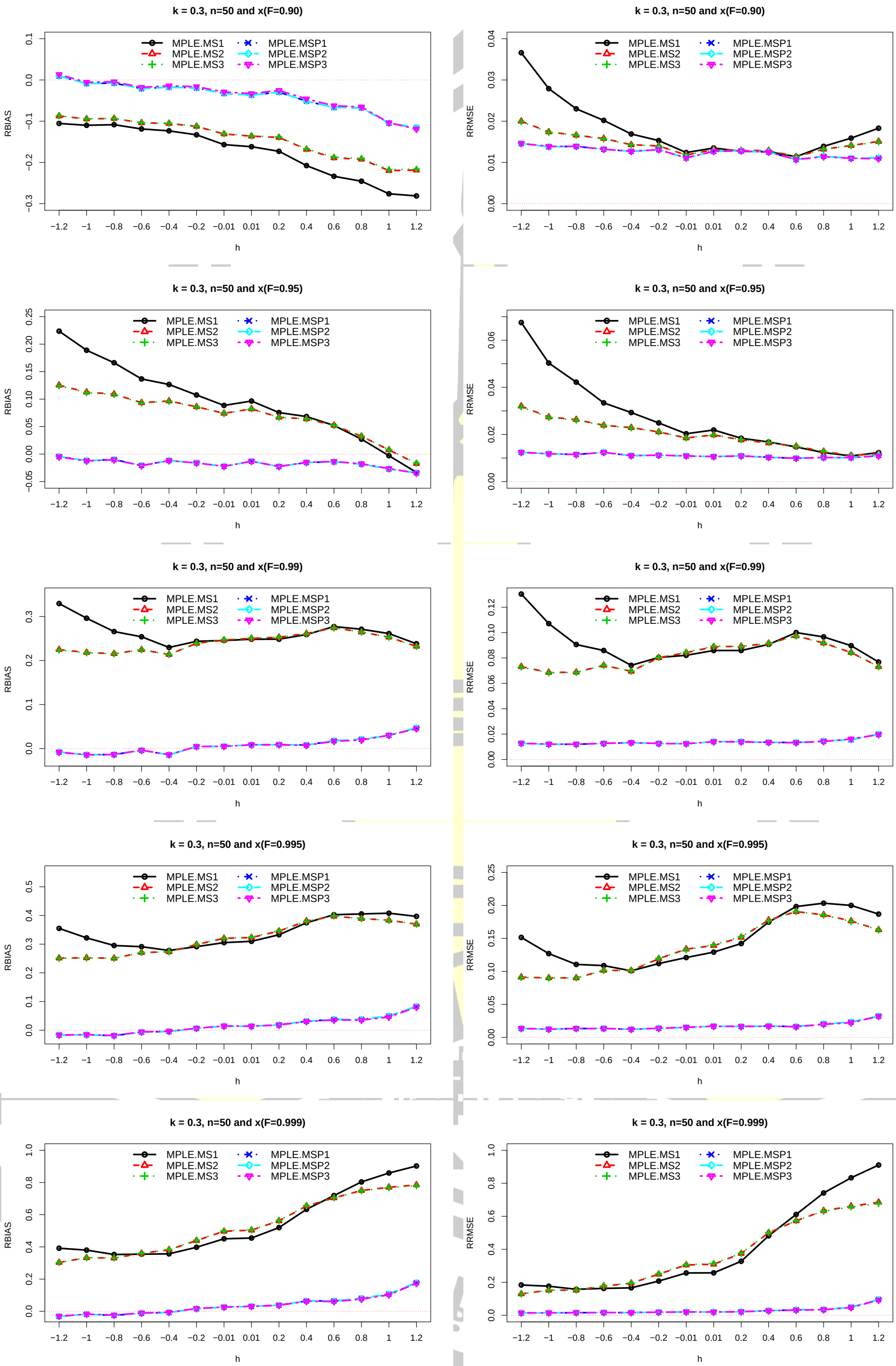

Figure A.38: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MPLE.MSP3 for value of $k = 0.3$ and sample size $n = 50$.

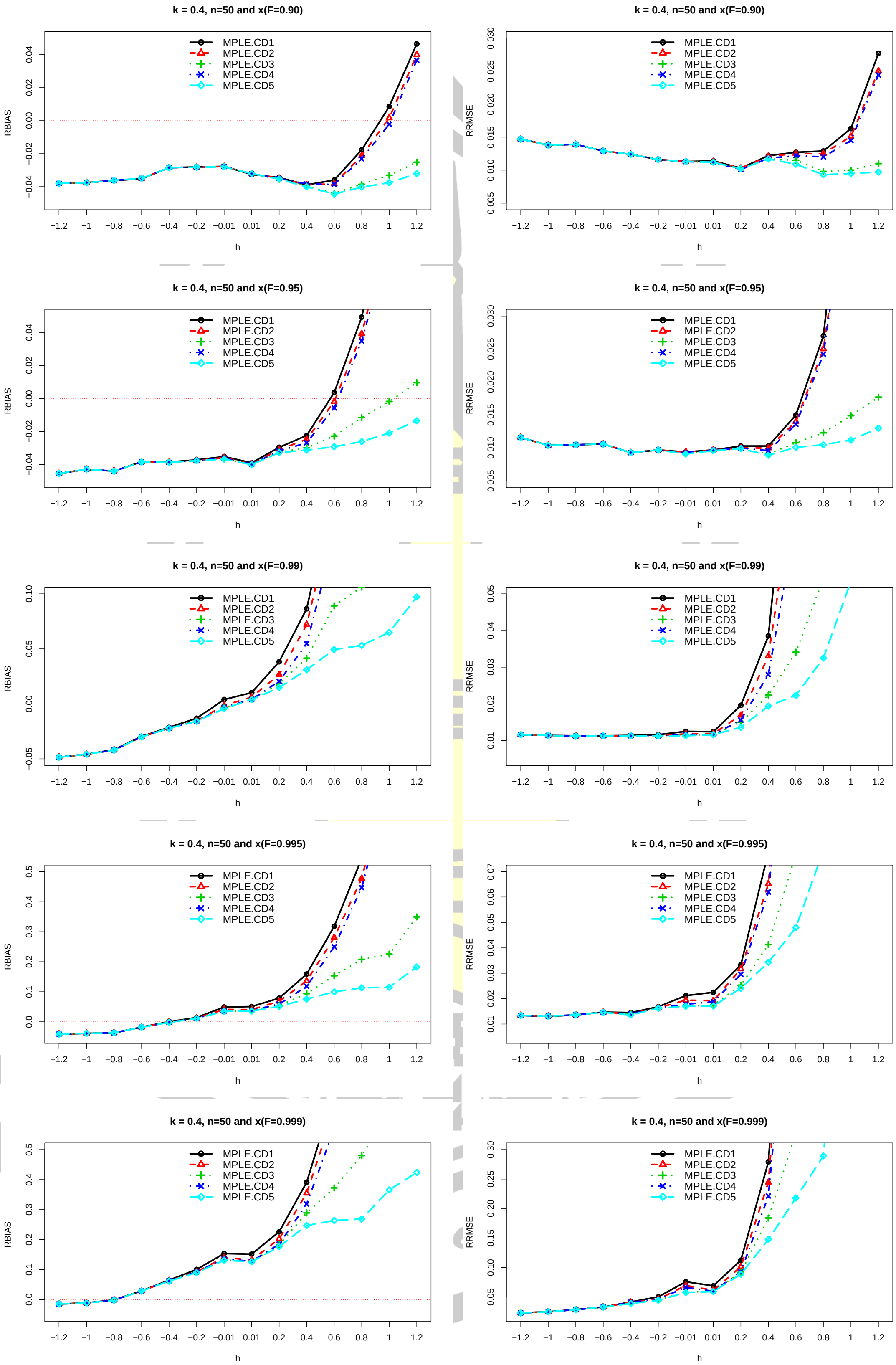

Figure A.39: RBIAS and RRMSE of the all quantile estimators of MPLE.CD1 to MPLE.CD5 for value of $k = 0.4$ and sample size $n = 50$.

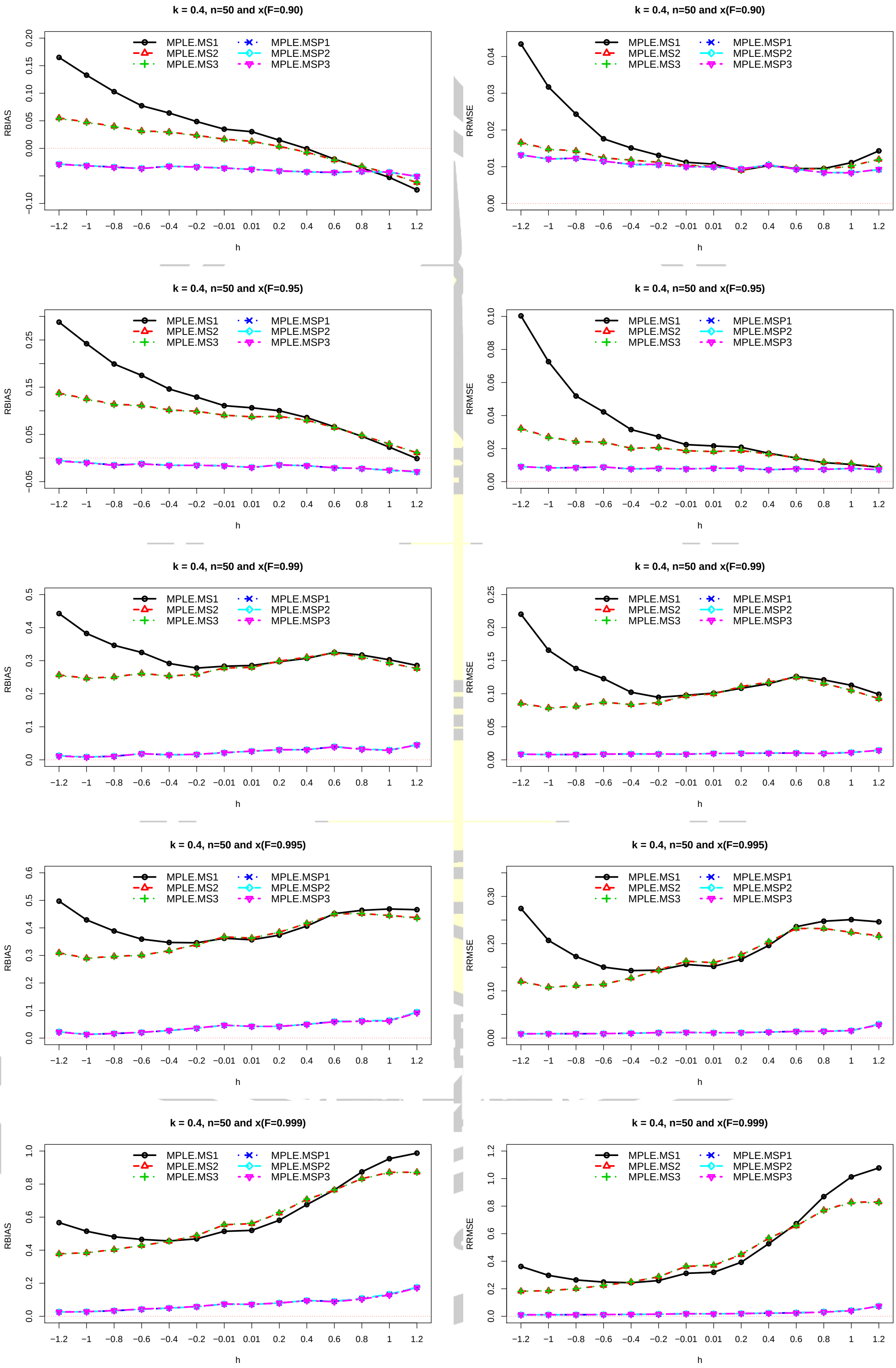

Figure A.40: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MPLE.MSP3 for value of $k = 0.4$ and sample size $n = 50$.

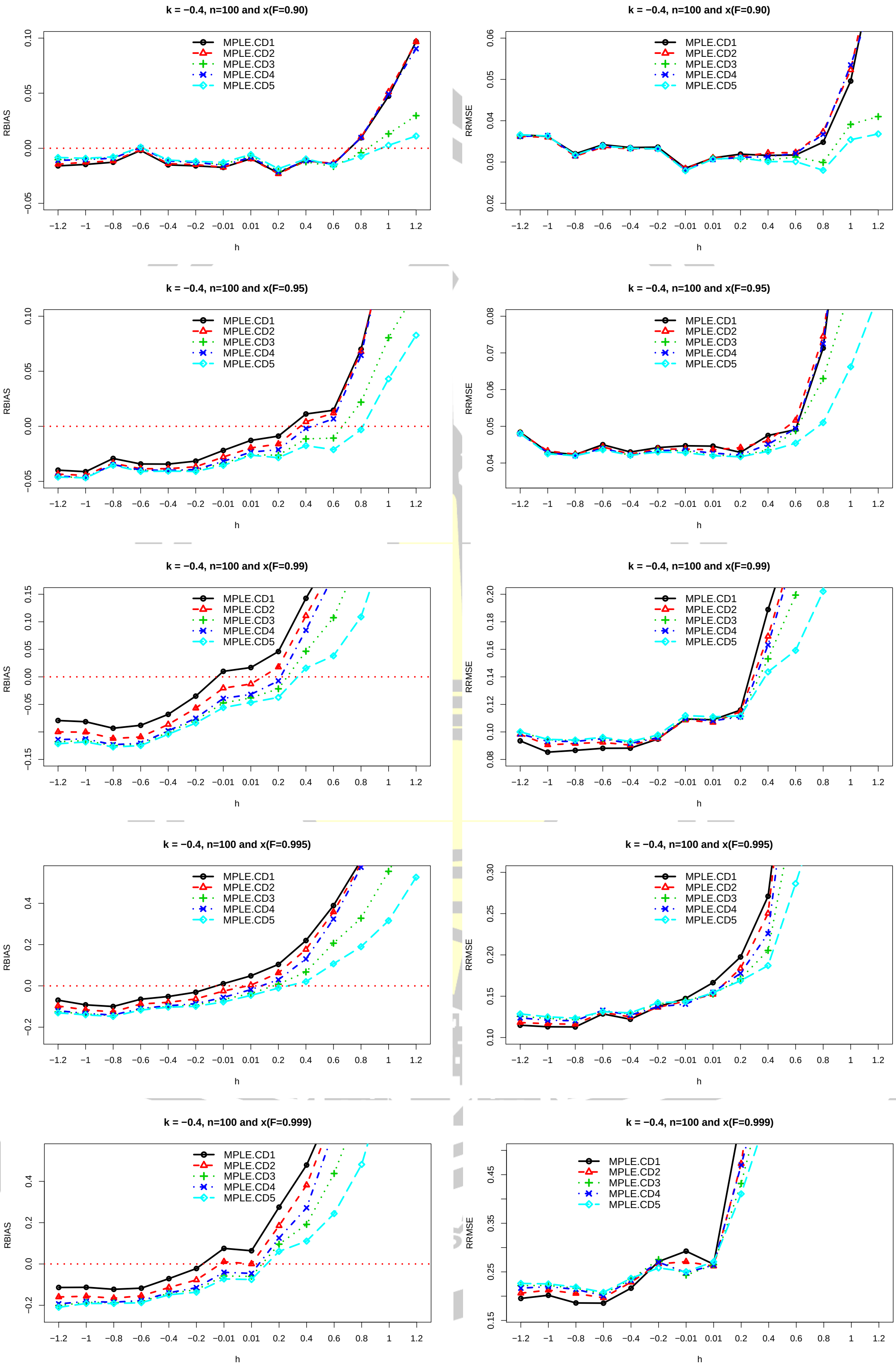

Figure A.41: RBIAS and RRMSE of the all quantile estimators of MPLE.CD1 to MPLE.CD5 for value of $k = -0.4$ and sample size $n = 100$.

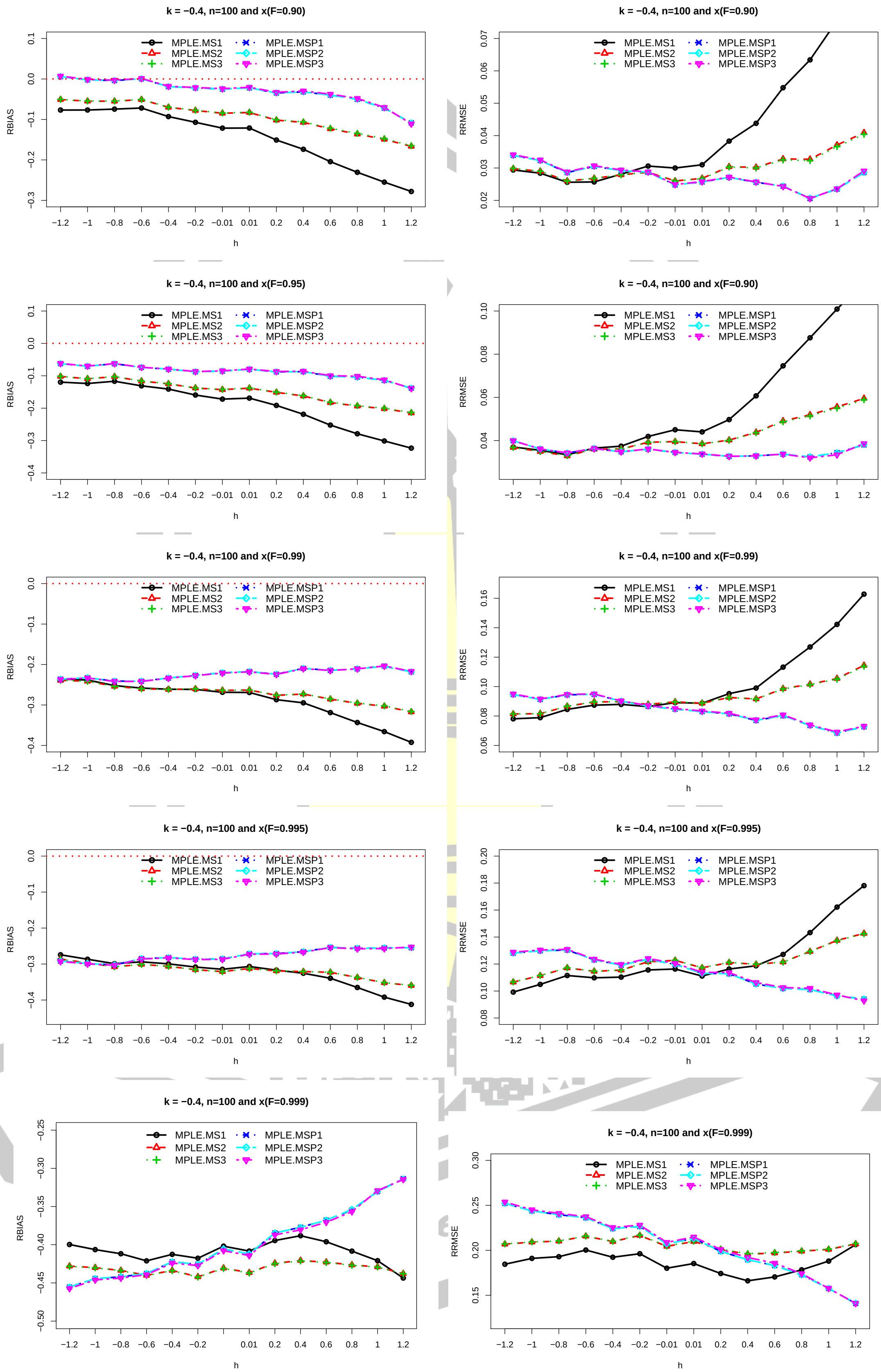

Figure A.42: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MPLE.MSP3 for value of $k = -0.4$ and sample size $n = 100$.

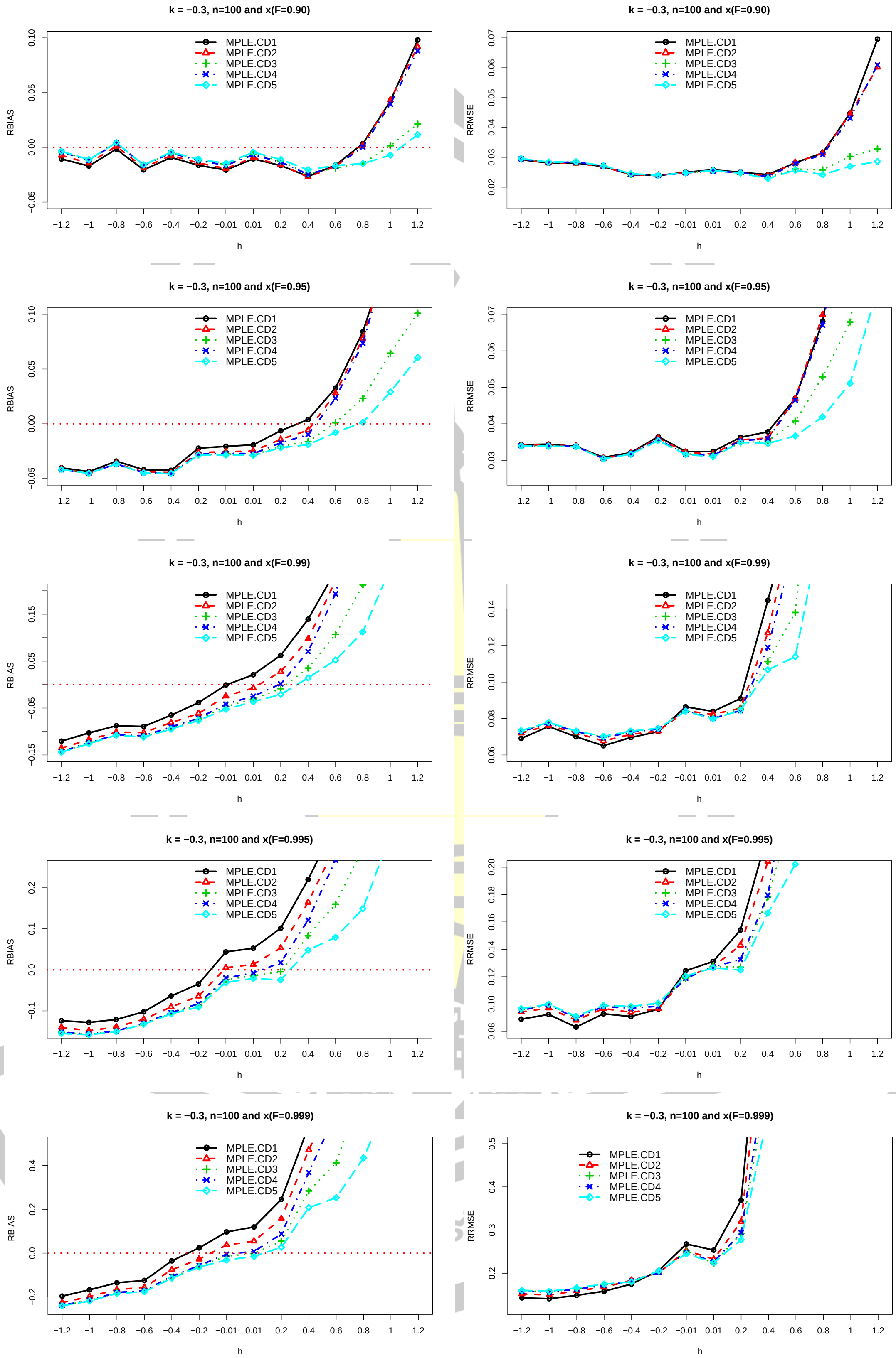

Figure A.43: RBIAS and RRMSE of the all quantile estimators of MPLE.CD1 to MPLE.CD5 for value of $k = -0.3$ and sample size $n = 100$.

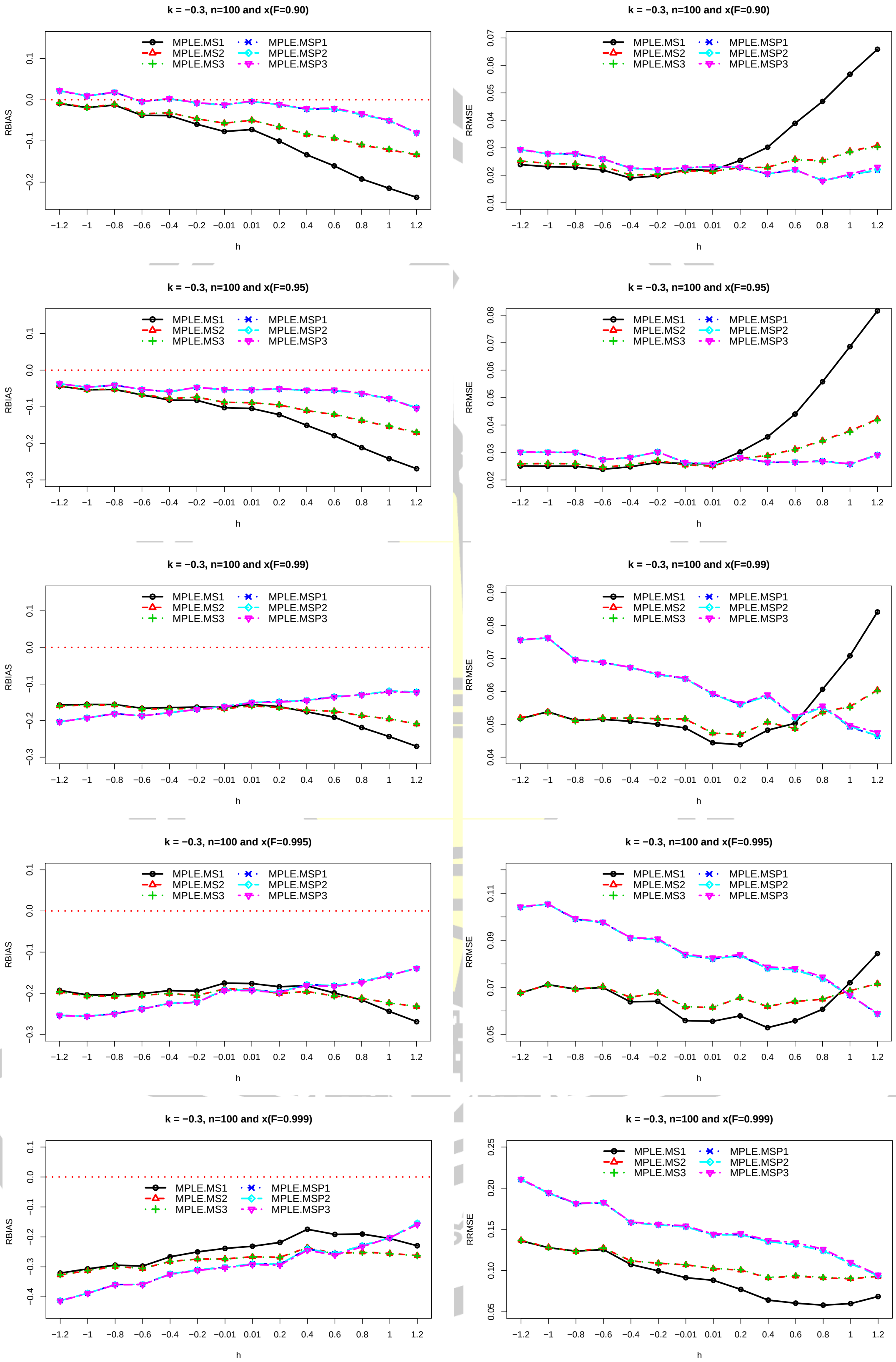

Figure A.44: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MPLE.MSP3 for value of $k = -0.3$ and sample size $n = 100$.

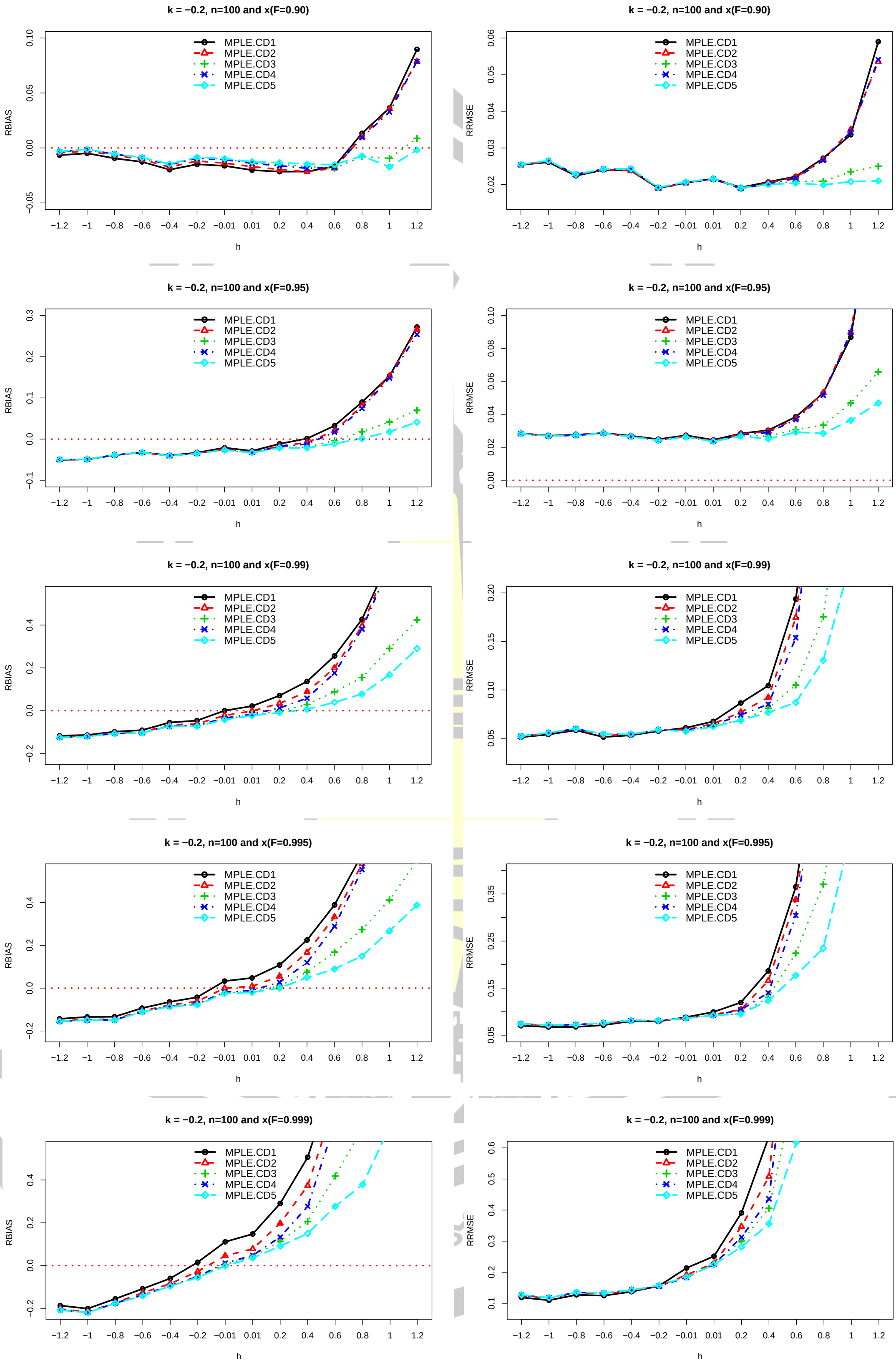

Figure A.45: RBIAS and RRMSE of the all quantile estimators of MPLE.CD1 to MPLE.CD5 for value of $k = -0.2$ and sample size $n = 100$.

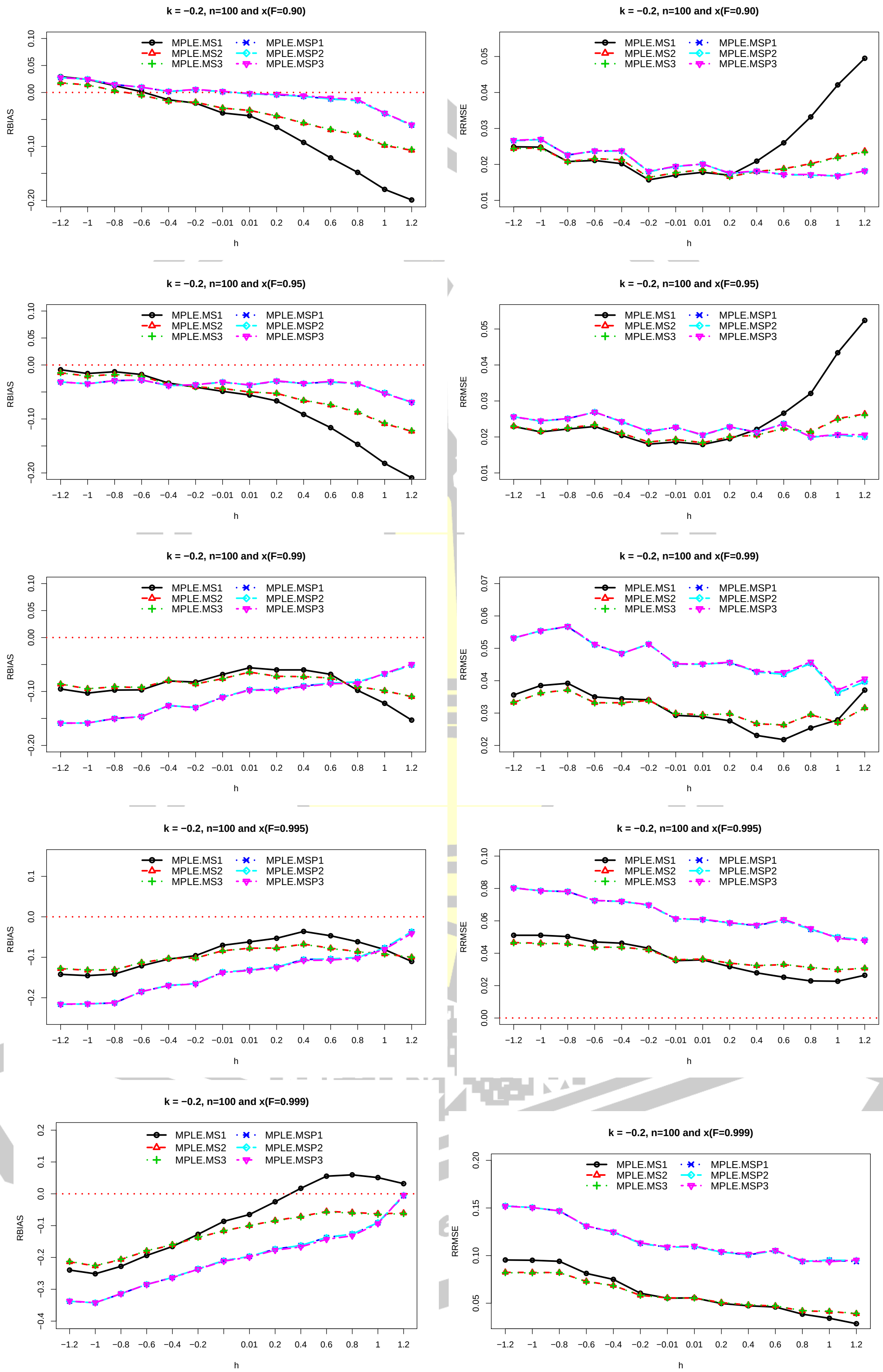

Figure A.46: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MPLE.MSP3 for value of $k = -0.2$ and sample size $n = 100$.

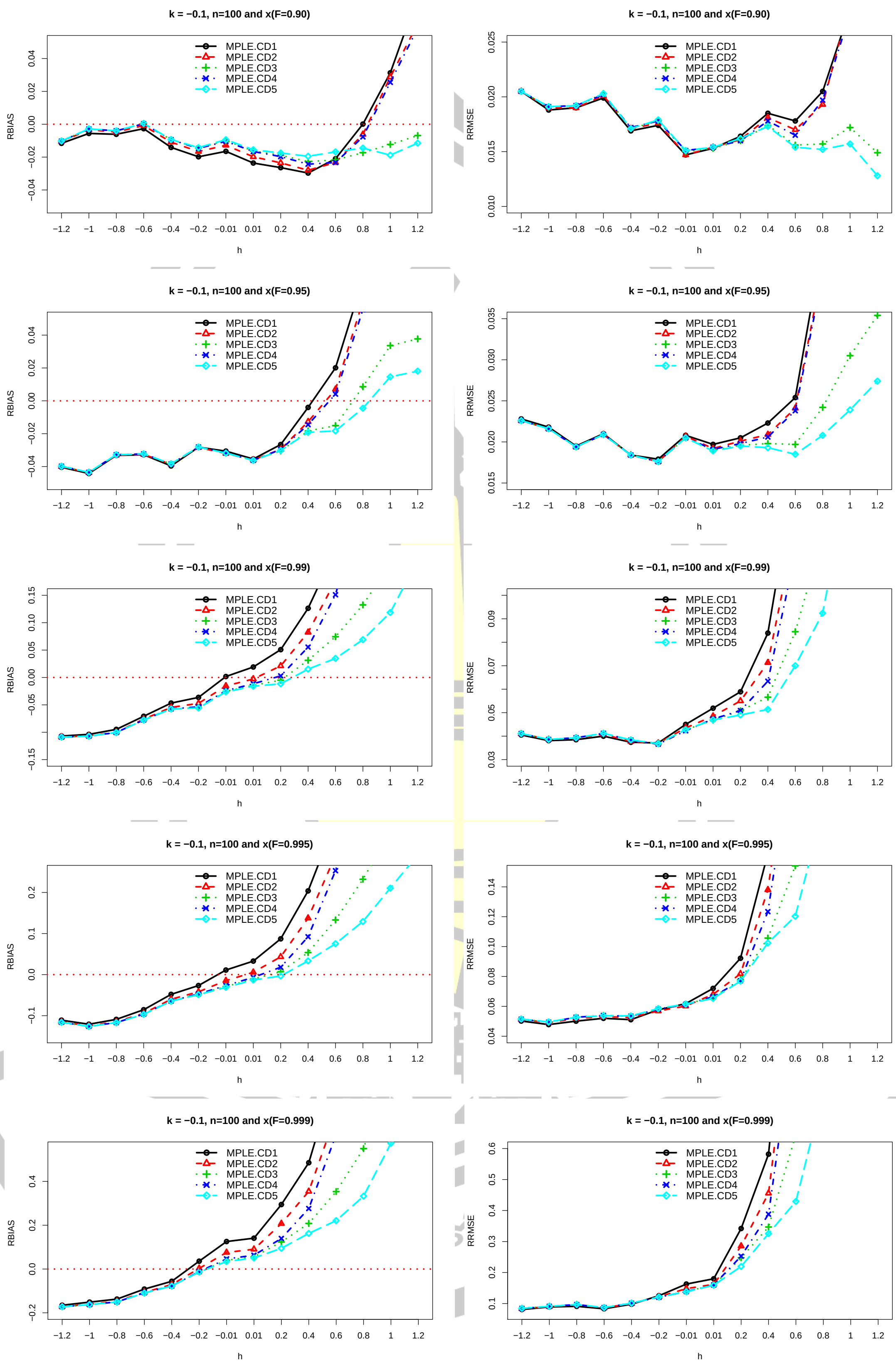

Figure A.47: RBIAS and RRMSE of the all quantile estimators of MPLE.CD1 to MPLE.CD5 for value of $k = -0.1$ and sample size $n = 100$.

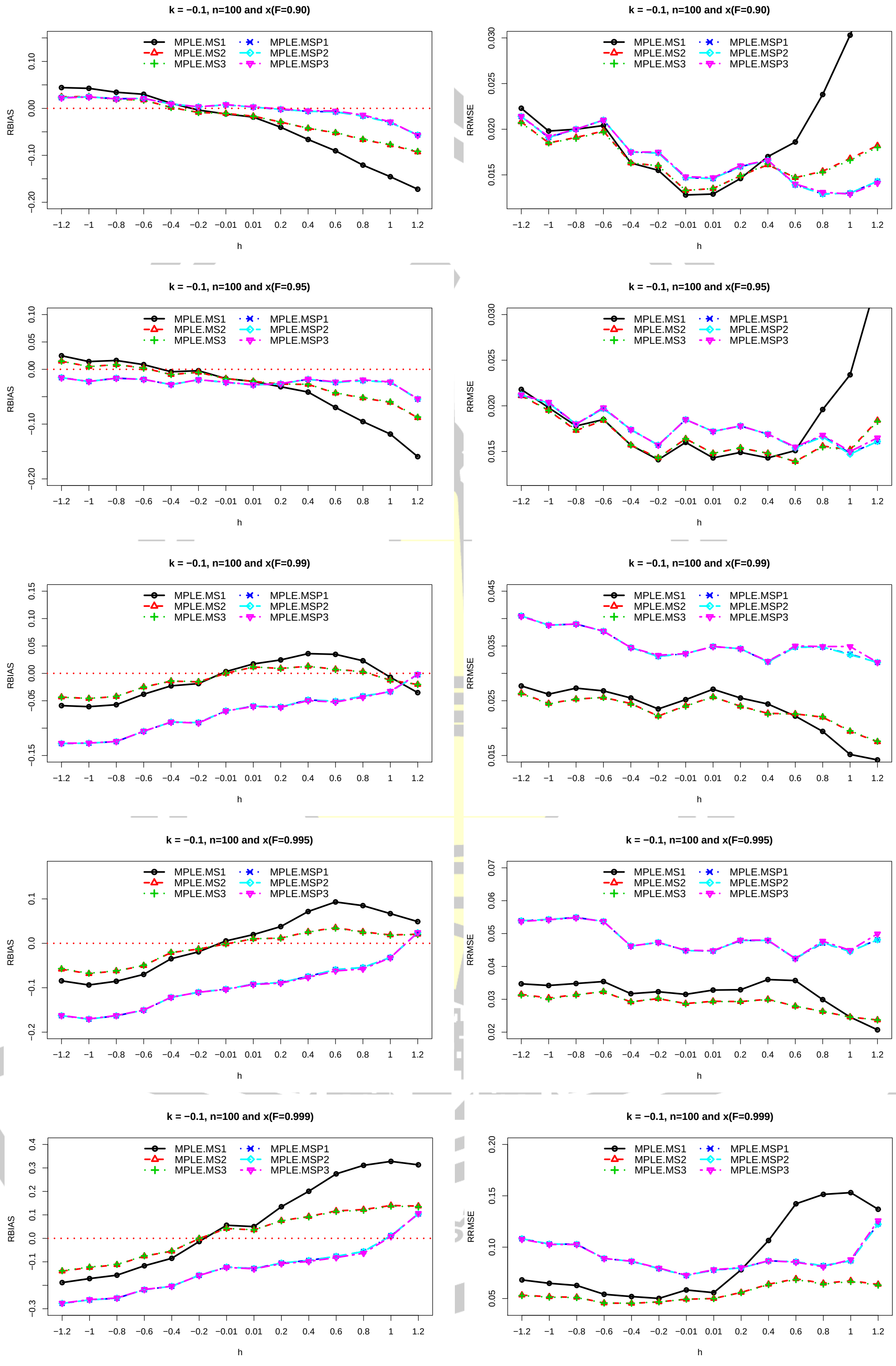

Figure A.48: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MPLE.MSP3 for value of $k = -0.1$ and sample size $n = 100$.

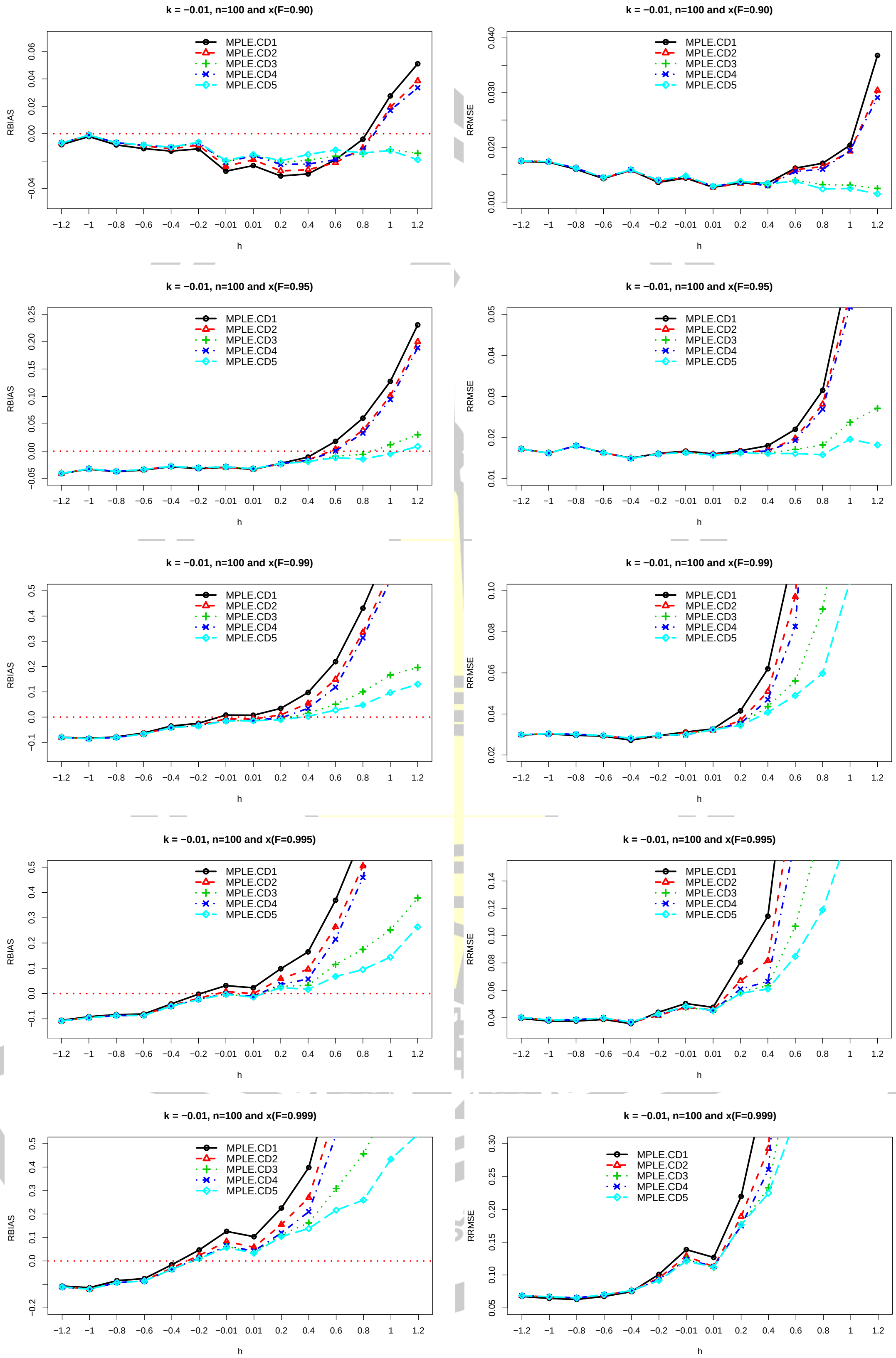

Figure A.49: RBIAS and RRMSE of the all quantile estimators of MPLE.CD1 to MPLE.CD5 for value of $k = -0.01$ and sample size $n = 100$.

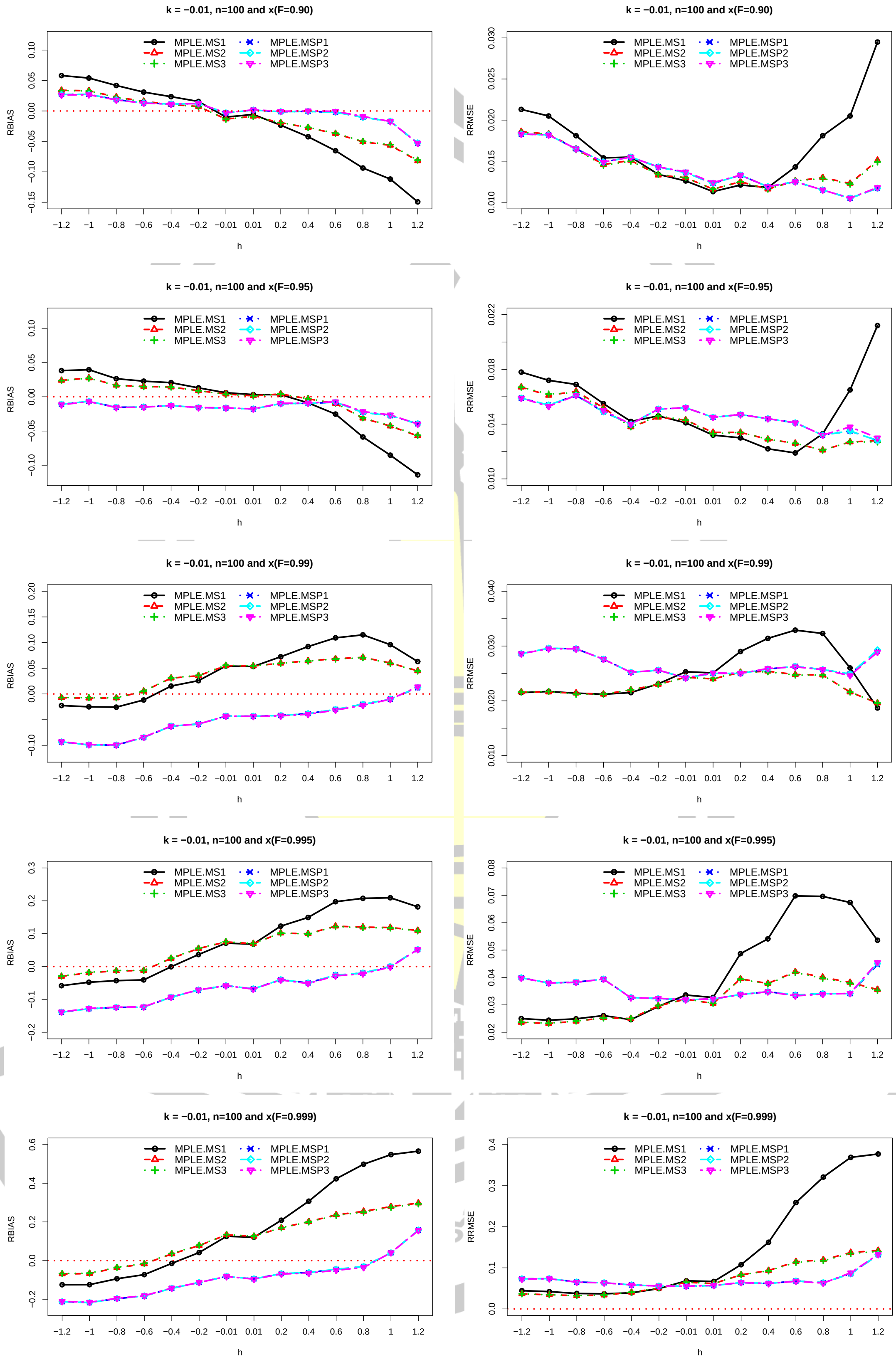

Figure A.50: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MPLE.MSP3 for value of $k = -0.01$ and sample size $n = 100$.

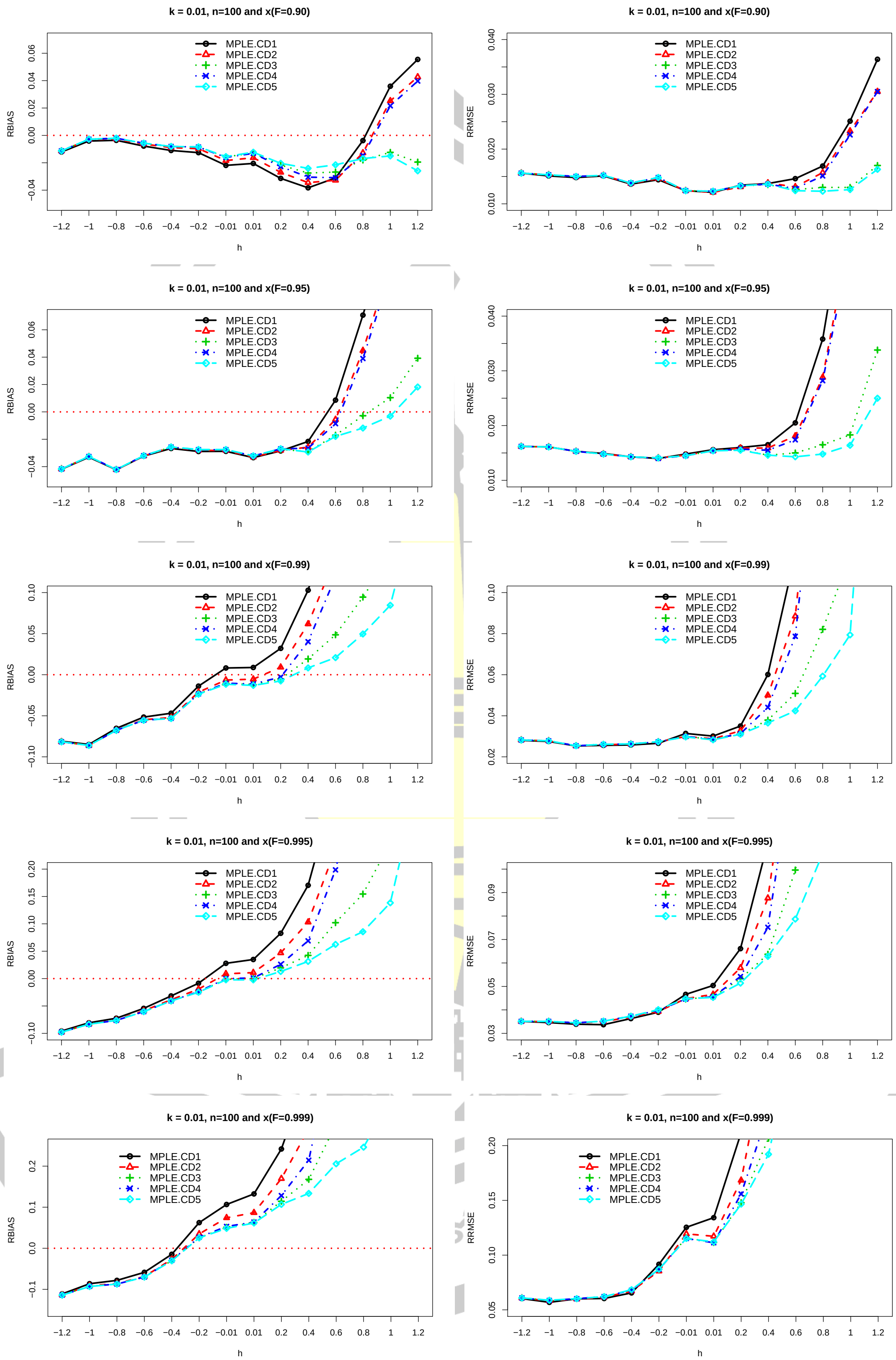

Figure A.51: RBIAS and RRMSE of the all quantile estimators of MPLE.CD1 to MPLE.CD5 for value of $k = 0.01$ and sample size $n = 100$.

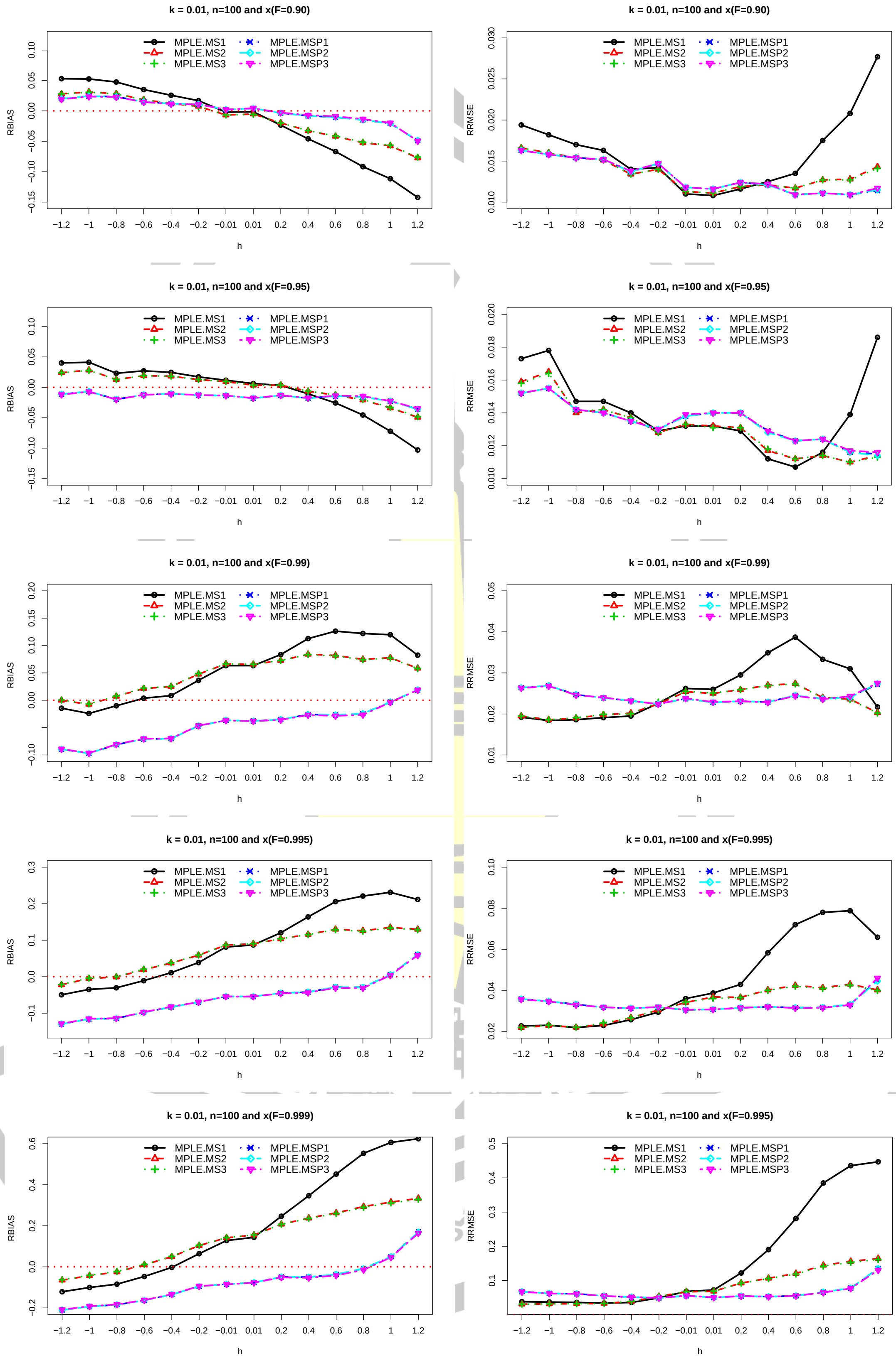

Figure A.52: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MPLE.MSP3 for value of $k = 0.01$ and sample size $n = 100$.

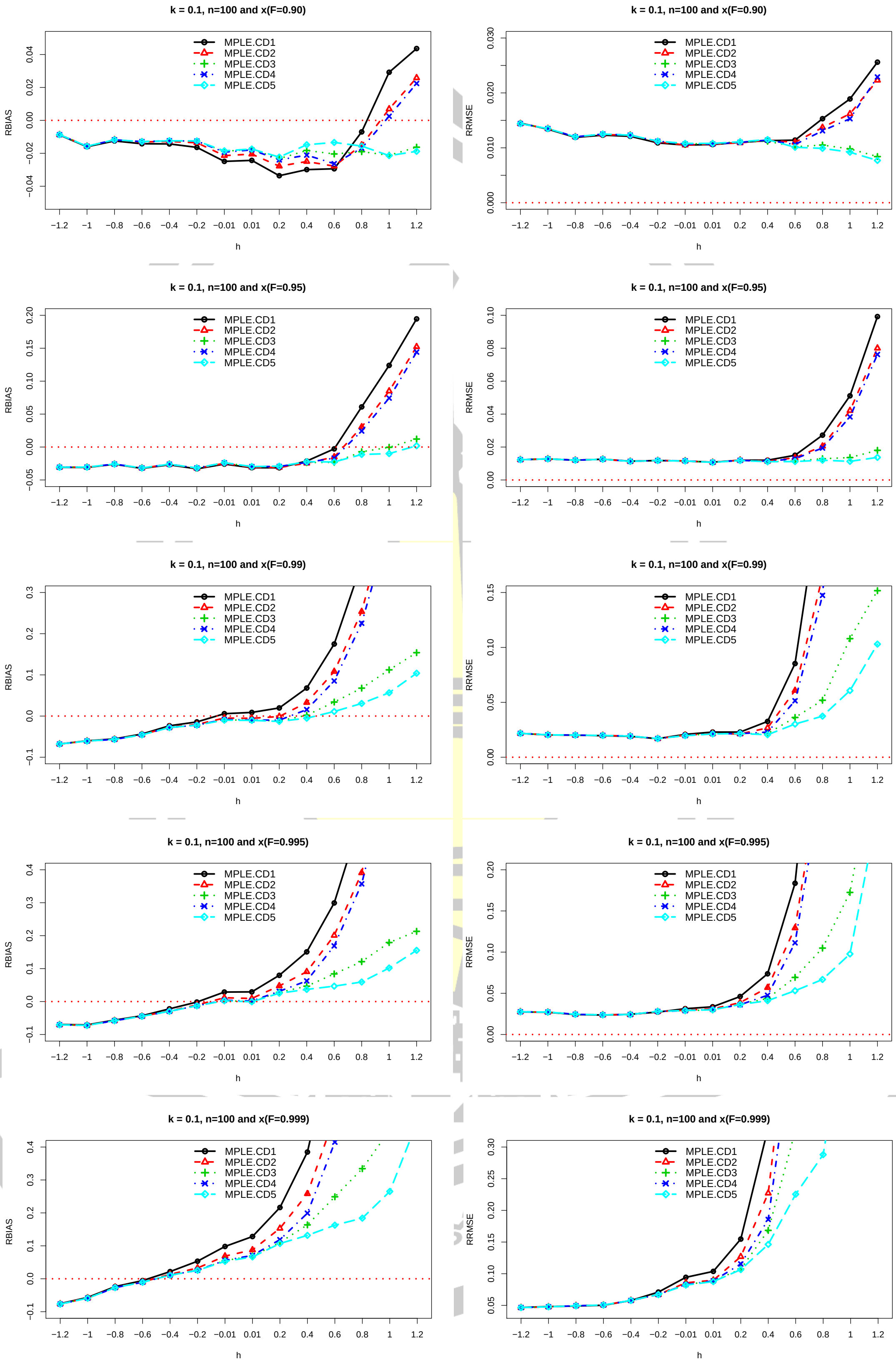

Figure A.53: RBIAS and RRMSE of the all quantile estimators of MPLE.CD1 to MPLE.CD5 for value of $k = 0.1$ and sample size $n = 100$.

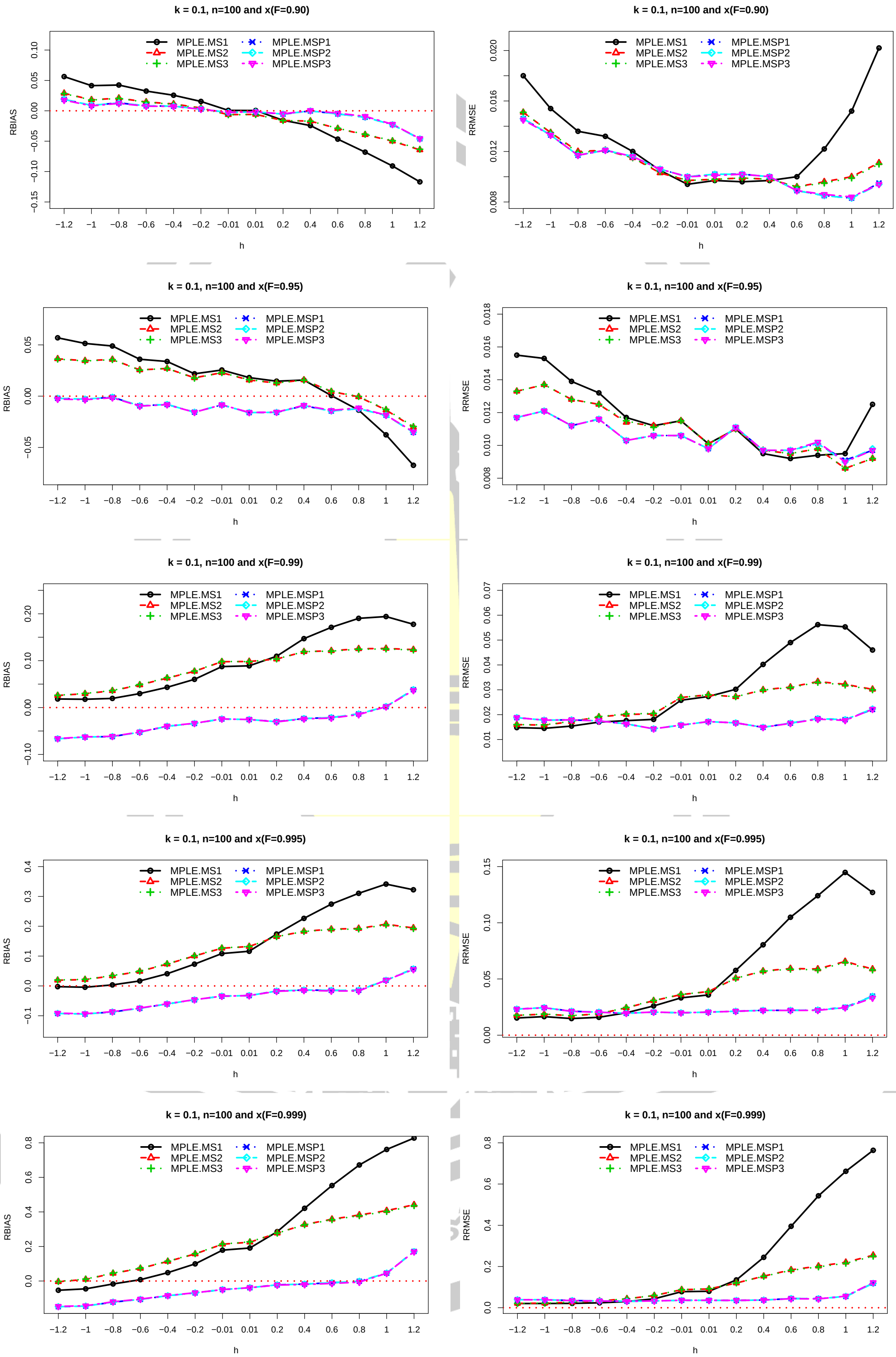

Figure A.54: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MPLE.MSP3 for value of $k = 0.1$ and sample size $n = 100$.

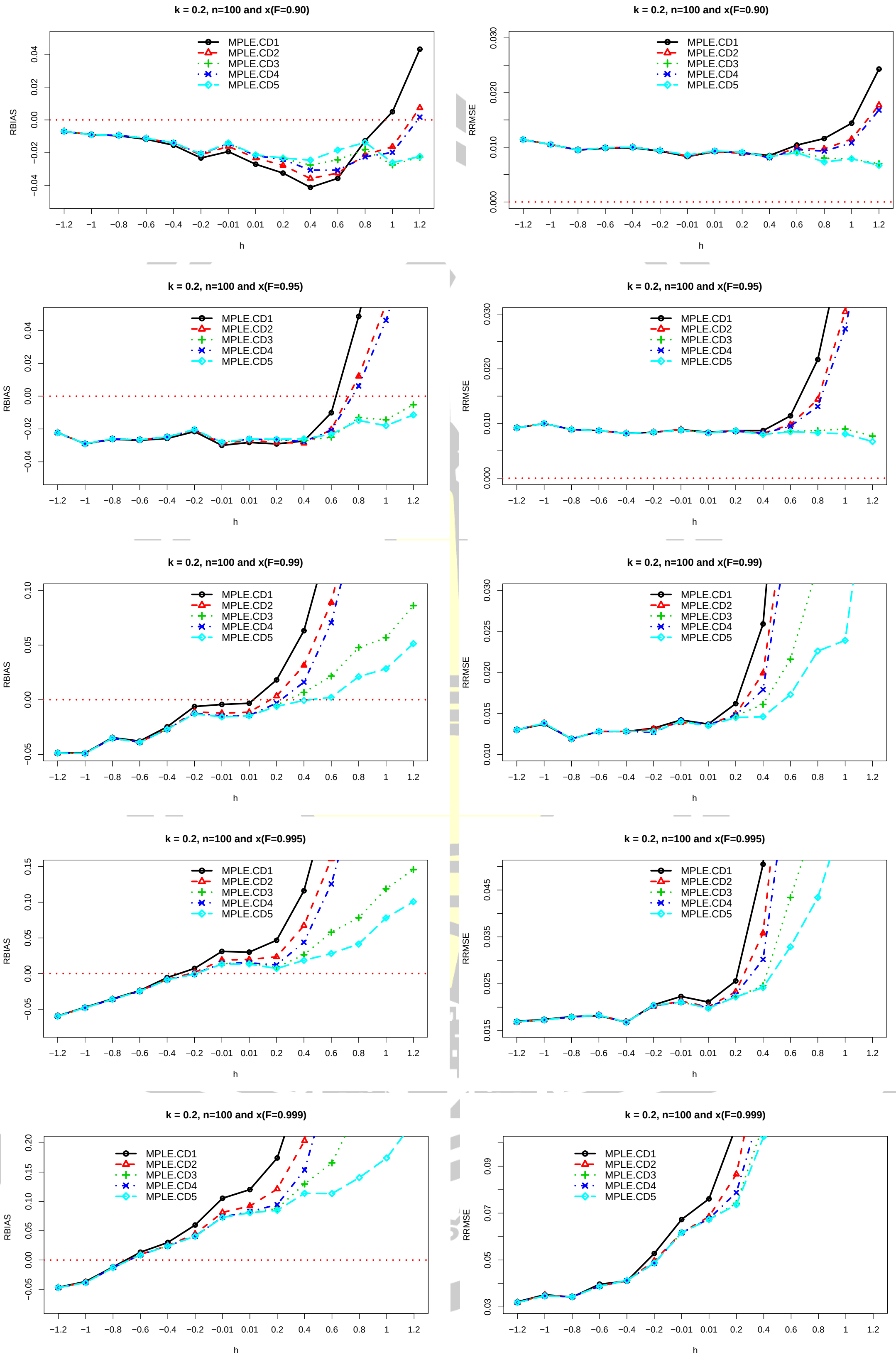

Figure A.55: RBIAS and RRMSE of the all quantile estimators of MPLE.CD1 to MPLE.CD5 for value of $k = 0.2$ and sample size $n = 100$.

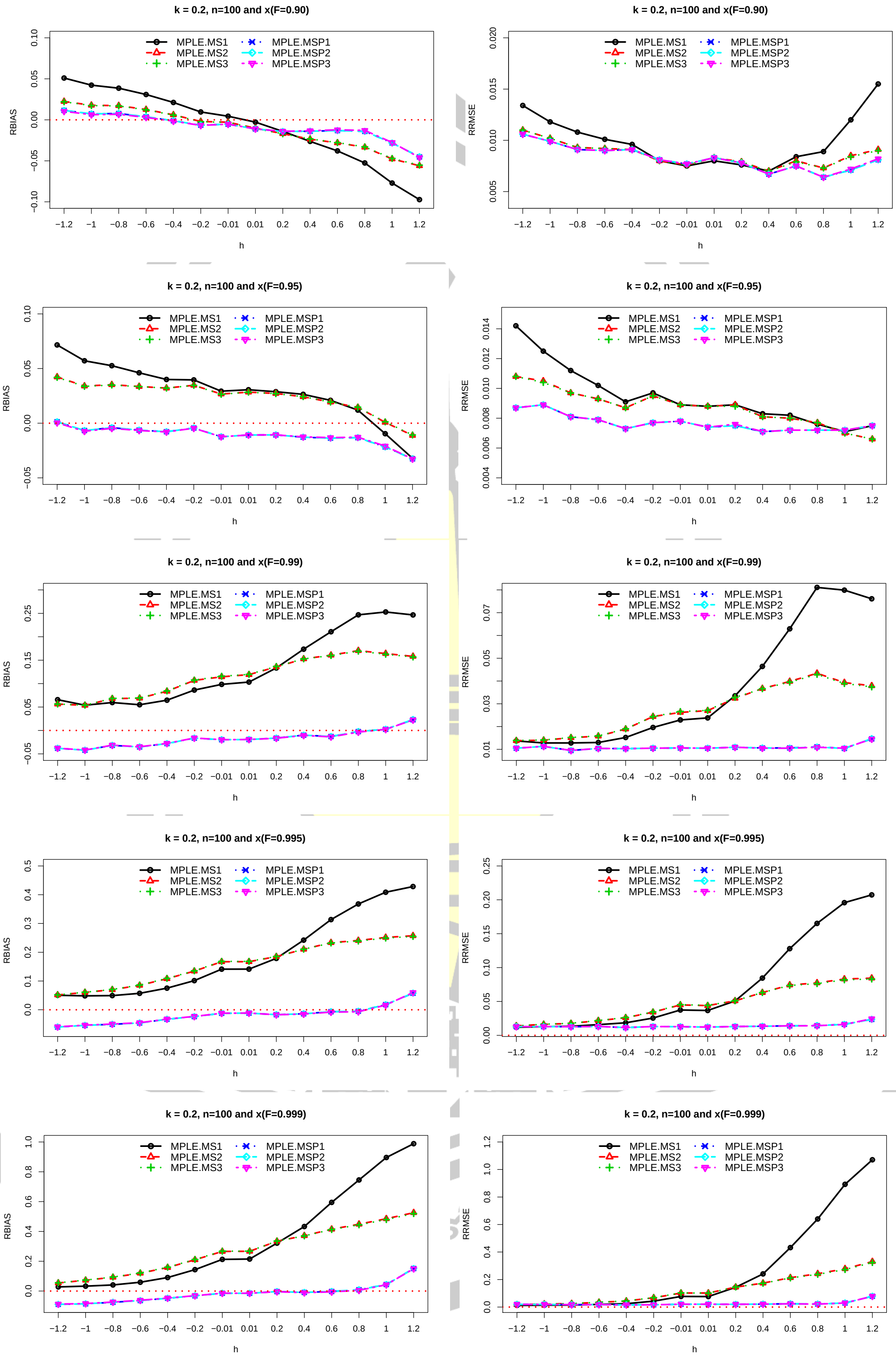

Figure A.56: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MPLE.MSP3 for value of $k = 0.2$ and sample size $n = 100$.

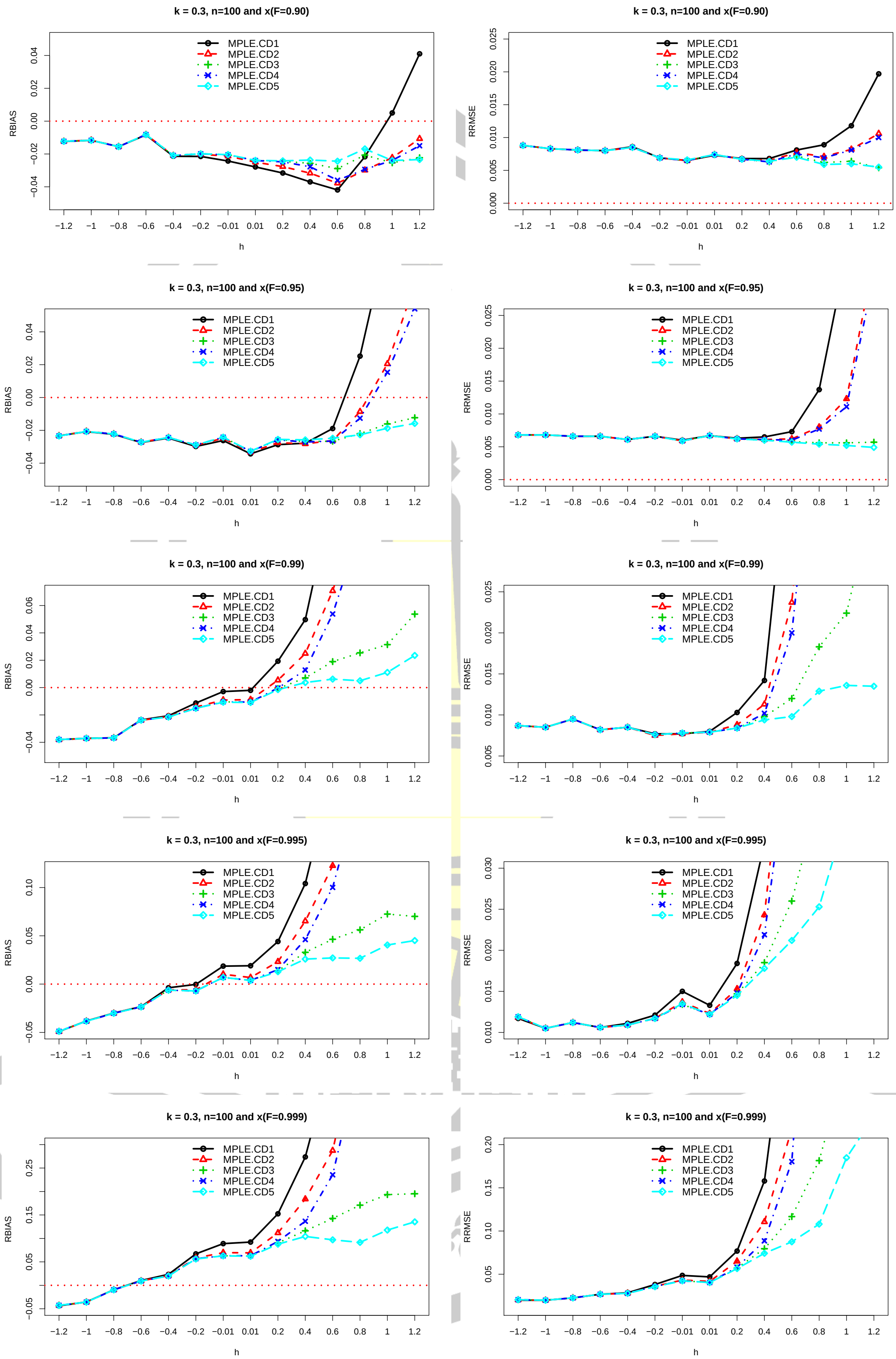

Figure A.57: RBIAS and RRMSE of the all quantile estimators of MPLE.CD1 to MPLE.CD5 for value of $k = 0.3$ and sample size $n = 100$.

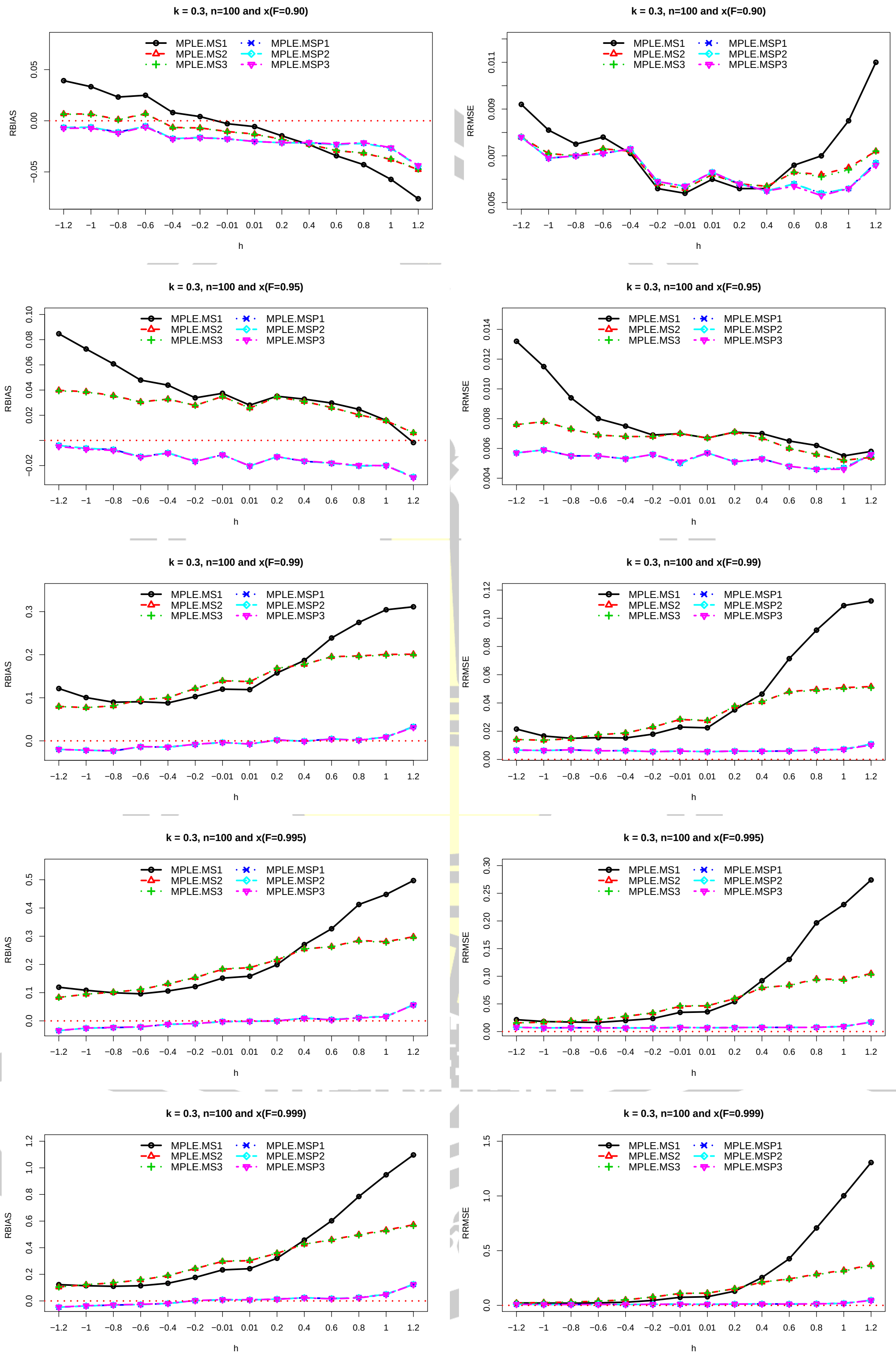

Figure A.58: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MPLE.MSP3 for value of $k = 0.3$ and sample size $n = 100$.

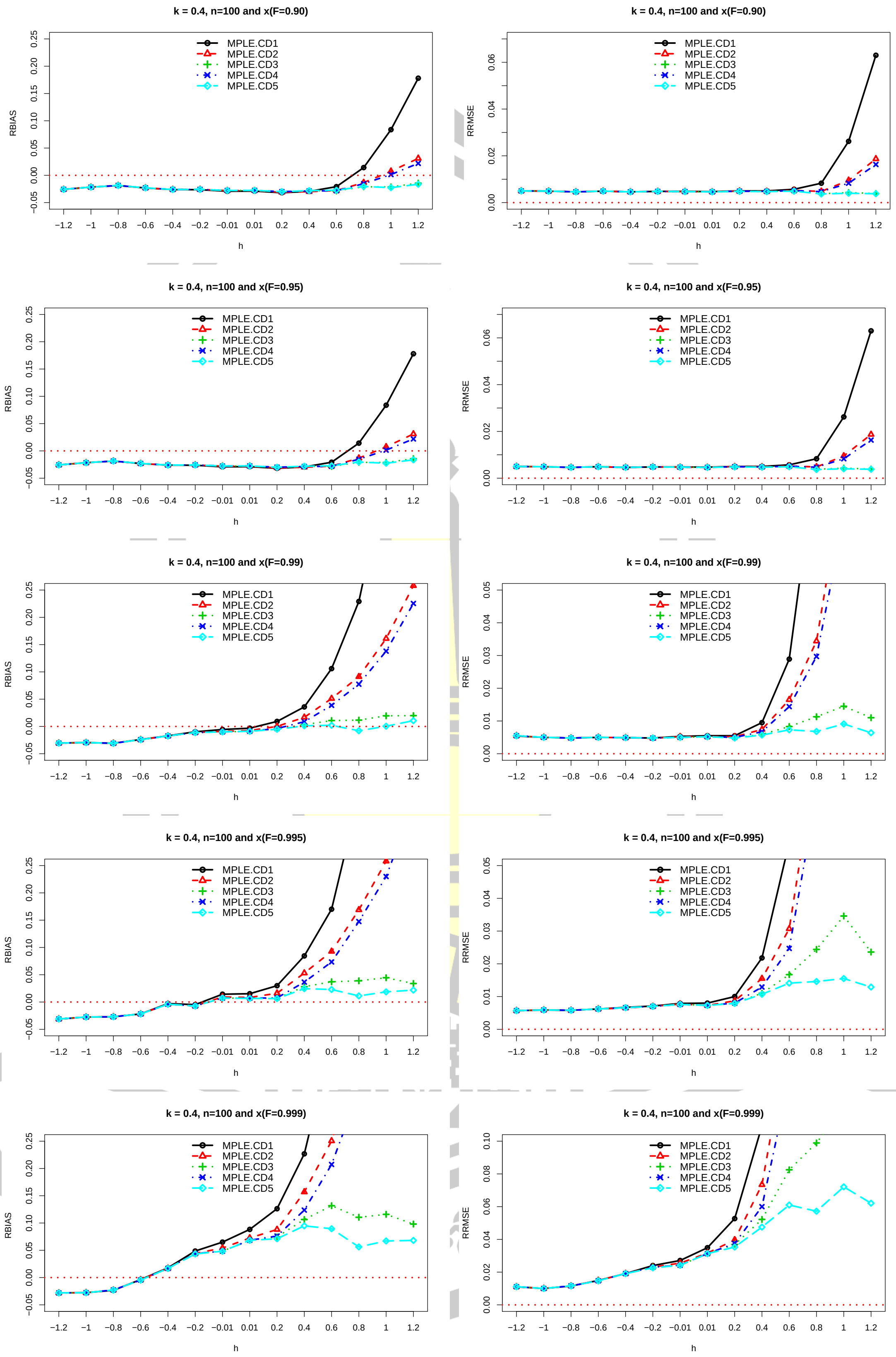

Figure A.59: RBIAS and RRMSE of the all quantile estimators of MPLE.CD1 to MPLE.CD5 for value of $k = 0.4$ and sample size $n = 100$.

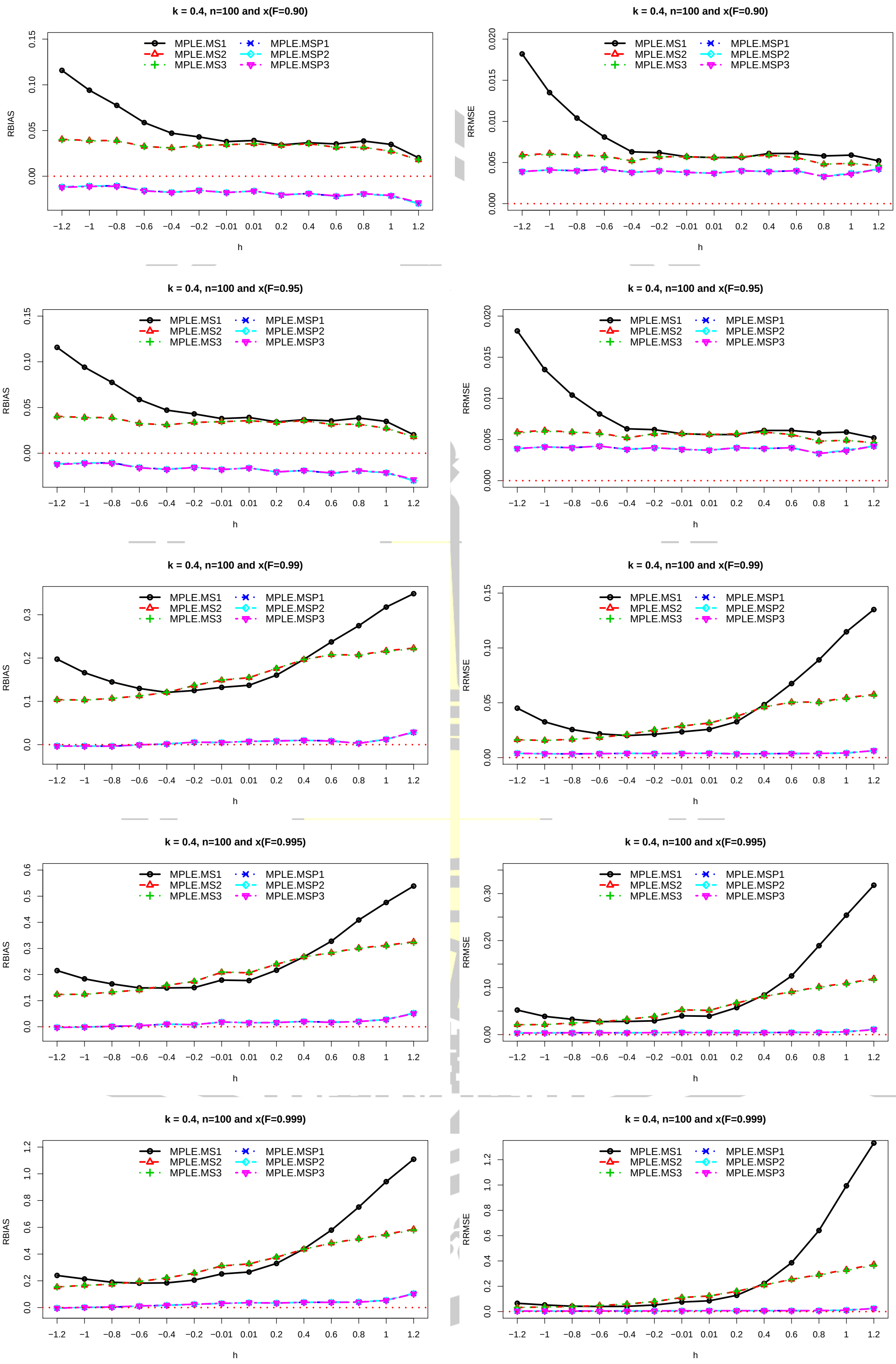

Figure A.60: RBIAS and RRMSE of the all quantile estimators of MPLE.MS1 to MPLE.MSP3 for value of $k = 0.4$ and sample size $n = 100$.

APPENDIXES B

```
###################################
##### Main Program  ###############

rm(list=ls())
library(ismev)
library(plyr)
library(lmomco)
library(FAdist)
library(ADGofTest)
library(alabama)
library(lmom)
setwd("D:/12_pmle_mle_K4D_11032019")
source("cons_mle_Park-3.r")
source("kap4mle_Park-3.r")      ;source("loglik_mle-park-3.r")
source("loglik_pmle_cd1_may.r") ;source("kap4pmle_cd1_Park-3.r")
source("loglik_pmle_cd2_may.r") ;source("kap4pmle_cd2_Park-3.r")
source("loglik_pmle_cd3_may.r") ;source("kap4pmle_cd3_Park-3.r")
source("loglik_pmle_cd4_may.r") ;source("kap4pmle_cd4_Park-3.r")
source("loglik_pmle_cd5_may.r") ;source("kap4pmle_cd5_Park-3.r")
source("loglik_pmle_ms1_may.r") ;source("kap4pmle_ms1_Park-3.r")
source("loglik_pmle_ms2_may.r") ;source("kap4pmle_ms2_Park-3.r")
source("loglik_pmle_ms4_may.r") ;source("kap4pmle_ms4_Park-3.r")
source("loglik_pmle_msp1_may.r");source("kap4pmle_msp1_Park-3.r")
source("loglik_pmle_msp2_may.r");source("kap4pmle_msp2_Park-3.r")
source("loglik_pmle_msp4_may.r");source("kap4pmle_msp4_Park-3.r")


############################################################################
ptm <- proc.time()
ki <- c(-0.4)
hi <- c(0.01)
prob <- c(0.90,0.95,0.99,0.995,0.999)
x.dat <- vector()
n = 30
for(k in ki)
{
for(h in hi)
{
summary_out1 = NULL
```

```
for(pr in prob)
{
#Initial Parameter
k0  = k-0.01     #shape1
h0  = h-0.01     #shape2
xi0 = 1          #scale
mu0 = 0          #location

replication <- 3
no <- 1
mm <- 1
data_out1 = NULL

while(no <= replication){

x.dat <- rkappa4(n,shape1=k,shape2=h,scale=1,location=0)
maxxi = max(x.dat)
minxi = min(x.dat)


###################################################
### estimate by MLE ########### method = "L-BFGS-B"

sink("nul")
fit2.mle = kap4mle2(x.dat,para=c(k0,h0,xi0,mu0))
mle_k  = fit2.mle$parmle[1];    mle_h  = fit2.mle$parmle[2]
mle_xi = fit2.mle$parmle[3];    mle_mu = fit2.mle$parmle[4]
para.mle = c(mle_k,mle_h,mle_xi,mle_mu)
sink()

###################################################
### estimate by MPLE.CD ######## method = "L-BFGS-B"

sink("nul")
fit2.mple.cd1 = kap4pmle.cd3.1(x.dat,para=c(k0,h0,xi0,mu0))
mple.cd1_k  = fit2.mple.cd1$par[1];mple.cd1_h  = fit2.mple.cd1$par[2]
mple.cd1_xi = fit2.mple.cd1$par[3];mple.cd1_mu = fit2.mple.cd1$par[4]
para.mple.cd1 = c(mple.cd1_k,mple.cd1_h,mple.cd1_xi,mple.cd1_mu)
sink()
```

```
sink("nul")
fit2.mple.cd2 = kap4pmle.cd3.2(x.dat,para=c(k0,h0,xi0,mu0))
mple.cd2_k  = fit2.mple.cd2$par[1];mple.cd2_h  = fit2.mple.cd2$par[2]
mple.cd2_xi = fit2.mple.cd2$par[3];mple.cd2_mu = fit2.mple.cd2$par[4]
para.mple.cd2 = c(mple.cd2_k,mple.cd2_h,mple.cd2_xi,mple.cd2_mu)
sink()

sink("nul")
fit2.mple.cd3 = kap4pmle.cd3.3(x.dat,para=c(k0,h0,xi0,mu0))
mple.cd3_k  = fit2.mple.cd3$par[1];mple.cd3_h  = fit2.mple.cd3$par[2]
mple.cd3_xi = fit2.mple.cd3$par[3];mple.cd3_mu = fit2.mple.cd3$par[4]
para.mple.cd3 = c(mple.cd3_k,mple.cd3_h,mple.cd3_xi,mple.cd3_mu)
sink()

sink("nul")
fit2.mple.cd4 = kap4pmle.cd3.4(x.dat,para=c(k0,h0,xi0,mu0))
mple.cd4_k  = fit2.mple.cd4$par[1];mple.cd4_h  = fit2.mple.cd4$par[2]
mple.cd4_xi = fit2.mple.cd4$par[3];mple.cd4_mu = fit2.mple.cd4$par[4]
para.mple.cd4 = c(mple.cd4_k,mple.cd4_h,mple.cd4_xi,mple.cd4_mu)
sink()

sink("nul")
fit2.mple.cd5 = kap4pmle.cd3.5(x.dat,para=c(k0,h0,xi0,mu0))
mple.cd5_k  = fit2.mple.cd5$par[1];mple.cd5_h  = fit2.mple.cd5$par[2]
mple.cd5_xi = fit2.mple.cd5$par[3];mple.cd5_mu = fit2.mple.cd5$par[4]
para.mple.cd5 = c(mple.cd5_k,mple.cd5_h,mple.cd5_xi,mple.cd5_mu)
sink()

####################################################
### estimate by MPLE.MS ####### method = "L-BFGS-B"

sink("nul")
fit2.mple.ms1 = kap4pmle.ms.3.1(x.dat,para=c(k0,h0,xi0,mu0))
mple.ms1_k  = fit2.mple.ms1$par[1];mple.ms1_h  = fit2.mple.ms1$par[2]
mple.ms1_xi = fit2.mple.ms1$par[3];mple.ms1_mu = fit2.mple.ms1$par[4]
para.mple.ms1 = c(mple.ms1_k,mple.ms1_h,mple.ms1_xi,mple.ms1_mu)
sink()

sink("nul")
fit2.mple.ms2 = kap4pmle.ms.3.2(x.dat,para=c(k0,h0,xi0,mu0))
mple.ms2_k  = fit2.mple.ms2$par[1];mple.ms2_h  = fit2.mple.ms2$par[2]
```

```
mple.ms2_xi = fit2.mple.ms2$par[3];mple.ms2_mu = fit2.mple.ms2$par[4]
para.mple.ms2 = c(mple.ms2_k,mple.ms2_h,mple.ms2_xi,mple.ms2_mu)
sink()

sink("nul")
fit2.mple.ms3 = kap4pmle.ms.3.4(x.dat,para=c(k0,h0,xi0,mu0))
mple.ms3_k  = fit2.mple.ms3$par[1];mple.ms3_h  = fit2.mple.ms3$par[2]
mple.ms3_xi = fit2.mple.ms3$par[3];mple.ms3_mu = fit2.mple.ms3$par[4]
para.mple.ms3 = c(mple.ms3_k,mple.ms3_h,mple.ms3_xi,mple.ms3_mu)
sink()

####################################################
### estimate by MPLE.MSP ####### method = "L-BFGS-B"

sink("nul")
fit2.mple.msp1 = kap4pmle.msp.3.1(x.dat,para=c(k0,h0,xi0,mu0))
mple.msp1_k  = fit2.mple.msp1$par[1]
mple.msp1_h  = fit2.mple.msp1$par[2]
mple.msp1_xi = fit2.mple.msp1$par[3]
mple.msp1_mu = fit2.mple.msp1$par[4]
para.mple.msp1 = c(mple.msp1_k,mple.msp1_h,mple.msp1_xi,mple.msp1_mu)
sink()

sink("nul")
fit2.mple.msp2 = kap4pmle.msp.3.2(x.dat,para=c(k0,h0,xi0,mu0))
mple.msp2_k  = fit2.mple.msp2$par[1]
mple.msp2_h  = fit2.mple.msp2$par[2]
mple.msp2_xi = fit2.mple.msp2$par[3]
mple.msp2_mu = fit2.mple.msp2$par[4]
para.mple.msp2 = c(mple.msp2_k,mple.msp2_h,mple.msp2_xi,mple.msp2_mu)
sink()

sink("nul")
fit2.mple.msp3 = kap4pmle.msp.3.4(x.dat,para=c(k0,h0,xi0,mu0))
mple.msp3_k  = fit2.mple.msp3$par[1]
mple.msp3_h  = fit2.mple.msp3$par[2]
mple.msp3_xi = fit2.mple.msp3$par[3]
mple.msp3_mu = fit2.mple.msp3$par[4]
para.mple.msp3 = c(mple.msp3_k,mple.msp3_h,mple.msp3_xi,mple.msp3_mu)
sink()
```

```
##################################################
### estimate by LM #######
lm <- lmoms(x.dat)
fit.lm <- parkap(lm)
lm_k  = fit.lm$para[3]
lm_h  = fit.lm$para[4]
lm_xi = fit.lm$para[2]
lm_mu = fit.lm$para[1]
est.lm = c(lm_k,lm_h,lm_xi,lm_mu)
y_ = matrix(NA,nrow=replication,ncol=length(est.lm))
y_[no,] = est.lm
tt <- 1
for (jj in 1 : length(est.lm)){
if( (y_[no,jj] == 0) || is.na(y_[no,jj]) || is.nan(y_[no,jj]) ){
tt <- 2
}else if( (jj==4) && (tt==1) ){
cat("\n\n k ",k," h ",h,"x(F)=",pr," Rep:: ",no,"\n")

######## rbias and rmse ########################
parametera = rbind(para.mle,est.lm,para.mple.cd1,para.mple.cd2,
para.mple.cd3,para.mple.cd4,para.mple.cd5,para.mple.ms1,para.mple.ms2,
para.mple.ms3,para.mple.msp1,para.mple.msp2,para.mple.msp3)

qt <- qkappa4(p=pr,shape1=k,shape2=h,scale=xi0,location=mu0)
q <- NULL; biase <- NULL; rrmse <- NULL
for(r in 1:nrow(parametera)){
q[r] <- qkappa4(p=pr,shape1=parametera[r,1],shape2=parametera[r,2],
scale=parametera[r,3],location=parametera[r,4])
biase[r] <- (q[r] - qt)/qt
rrmse[r] <- ((q[r] - qt)/qt )^2
#cat("i=",r,q[r],qt,biase[r],rrmse[r],"\n")
}

#################################################
####### write to files ########################

data_out  = c(no,biase, rrmse)
data_out1 = rbind(data_out1,data_out)
dir_output <- paste("D:/12_pmle_mle_K4D_11032019/ ",sep="")
file_name_output <- paste(dir_output,"k",k,"h",h,"x(F)",pr,"n",n,".csv")
```

```
data_out1 <- round(data_out1, digits = 4)
write.csv(data_out1, file_name_output)

no = no + 1

} ## end loop if else

mm = mm + 1

} ## end loop for

}## end while loop

######## mean rbias and rrmse ##########
dir_output <- paste("D:/12_pmle_mle_K4D_11032019/ ",sep="")
file_name_output <- paste(dir_output,"k",k,"h",h,"x(F)",pr,"n",n,".csv")
output <- read.csv(file_name_output, header=T)
summary_out = c(pr,colMeans(output[,c(-1,-2)]))
summary_out1 <- rbind(summary_out1,summary_out)
summary_out1 <- round(summary_out1, digits = 4)
colnames(summary_out1) <- c("pr",'rbias.mle','rbias.lm',
'rbias.mple.cd1','rbias.mple.cd2','rbias.mple.cd3',
'rbias.mple.cd4','rbias.mple.cd5',
'rbias.mple.ms1','rbias.mple.ms2','rbias.mple.ms3',
'rbias.mple.msp1','rbias.mple.msp2','rbias.mple.msp3',
'rrmse.mle','rrmse.lm','rrmse.mple.cd1','rrmse.mple.cd2',
'rrmse.mple.cd3','rrmse.mple.cd4','rrmse.mple.cd5',
'rrmse.mple.ms1','rrmse.mple.ms2','rrmse.mple.ms3',
'rrmse.mple.msp1','rrmse.mple.msp2','rrmse.mple.msp3')
dir <- paste("D:/12_pmle_mle_K4D_11032019/ ",sep="")
file_name <- paste(dir,"rbias and rrmse k",k,"h",h,"n",n,".csv")
write.csv(summary_out1, file_name)


} ## end for pr
} ## end for hi
} ## end for ki

cat("Time :: ",time.used = proc.time() - ptm,"\n")
```

```
#### Const.k4d #########
cons2.mle <- function(para)

{
# Programmed by Jeong-Soo Park, 10 Dec. 2018
# Dept of Stat. Chonnam National University, Korea

p <- rep(NA,5)
maxxi <- max(x.dat)
minxi <- min(x.dat)

eps.con <- c(1.e-10)
Uu <- para[4] + (para[3]/para[1])
if (para[2] > 0) {
w0 <-exp( (-para[1])*log(para[2]) )
ww <- para[3]- (para[3]*w0)
Ee <- para[4]+ ww/para[1]
}

p[1]= 10
if (para[1] > 0) { p[1] <- Uu -maxxi -eps.con}
else if (para[1] < 0 && para[2] < 0 )  {p[1] <- minxi-Uu -eps.con}

p[2] = 10
if (para[2] > 0) {
p[2] <- minxi-Ee -eps.con
}

p[3] <- para[3]-eps.con
p[4] <- abs(para[1])-eps.con
p[5] <- abs(para[2])-eps.con

p[is.na(p)] <- -10
p[p < 0] <- -10

return(p)
}
```

```
######## MLE program for the 4-parameter Kappa distribution #######

kap4mle2 = function(data,para)
{
library(ismev)
library(plyr)
library(FAdist)
library(alabama)
library(lmom)
# Programmed by Jeong-Soo Park, 10 Dec. 2018
# Dept of Stat. Chonnam National University, Korea

maxit <-20
newp <-matrix(NA, nrow=4,ncol=maxit*2)
x.dat <-data
maxxi <-max(x.dat)           # The data set name should be "x.dat"
minxi <-min(x.dat)

fminstar <- 1.0e+20
c<-cons2.mle(para)
if( min(c) > 0 ) {
fit2.mle = constrOptim.nl(par = para, fn=loglik2.mle, hin=cons2.mle)
fminstar <-fit2.mle$value
parstar <-fit2.mle$par
iterstar <- 1
# cat("iter, minfn, par= ", "1", fit2.mle$value, fit2.mle$par,"\n")
}

sd0 <-sd(x.dat)
mean0 <-mean(x.dat)
newp[1,2] <- 0.001                               # k =para[1]
newp[2,2] <- -.001                               # h =para[2]
newp[3,2] <- sd(x.dat)*0.7797                    # alpha or sigma =para[3]
newp[4,2] <- mean(x.dat)-0.45005*sd(x.dat)       # xi or mu =para[4]

para<-newp[,2]
c2 <-cons2.mle(para)
if( min(c2) > 0 ) {
fit2.mle = constrOptim.nl(par = para, fn=loglik2.mle, hin=cons2.mle)
# cat("iter, minfn, par= ", "2", fit2.mle$value, fit2.mle$par,"\n")
```

```
if(fit2.mle$value < fminstar) {
fminstar <- fit2.mle$value
parstar <- fit2.mle$par
iterstar <- 2
}
}
newp[,3]<- c(.01, -.3, 61, 177)
newp[,4] <- c(-.1,-.2,120,115)
newp[,5] <- c(.3,-1.3,90,30)
newp[,6] <- c(-.05,.1,40,60)
newp[,7] <- c(.01,.02,100,80)
newp[,8] <- c(-.09,1.9,58,96)
newp[,9] <- c(-.3,.3,60,60)
newp[,10] <- c(.001,.001,60,60)
newp[,11] <- c(-.5,-1.,60,60)
newp[3,11] <- 0.7*sd0
newp[4,11] <- mean0-0.4*sd0
newp[,12] <- c(-.4,.5,60,60)
newp[3,12] <- 0.7*sd0
newp[4,12] <- mean0-0.4*sd0
newp[,13] <- c(.4,.5,60,60)
newp[3,13] <- 0.7*sd0
newp[4,13] <- mean0-0.4*sd0
newp[,14] <- c(.4,-.5,60,60)
newp[3,14] <- 0.7*sd0
newp[4,14] <- mean0
newp[,15] <- c(-.5,-1.5,60,60)
newp[3,15] <- 2*sd0
newp[4,15] <- mean0
newp[,16] <- c(-.5,-1.,60,60)
newp[3,16] <- sd0*2
newp[4,16] <- mean0
newp[,17] <- c(-.2,-2.,60,60)
newp[3,17] <- sd0
newp[4,17] <- mean0*2
newp[,18] <- c(-.01,-.01,60,60)
newp[3,18] <- sd0
newp[4,18] <- mean0*2
newp[,19] <- c(-1,-.5,60,60)
newp[3,19] <- sd0
newp[4,19] <- mean0-0.5*sd0
```

```
newp[,20] <- c(-.5,-.5,60,60)
newp[3,20] <- sd0/2
newp[4,20] <- sd0/2


for (iter in 3:maxit) {
para <- newp[,iter]
c3 <- cons2.mle(para)
# cat("iter, C3,para=",iter, c3, para,"\n")

if(min(c3) > 0 ) {
fit2.mle = constrOptim.nl(par = para, fn=loglik2.mle, hin=cons2.mle)
# cat("iter, minfn, par= ", iter, fit2.mle$value, fit2.mle$par,"\n")

if(fit2.mle$value < fminstar) {
fminstar <- fit2.mle$value
parstar <- fit2.mle$par
iterstar <- iter
}
}
}

###### estimate by LM of GEVD for initial values #######
lm <- samlmu(x.dat)
fit.lm <- pelgev(lm)

iter<-maxit+1
newp[1,iter] <- fit.lm[3]   # k
newp[2,iter] <- 0.01        # h
newp[3,iter] <- fit.lm[2]   # scale
newp[4,iter] <- fit.lm[1]   # location
# est.lm = c(lm_k,lm_h,lm_xi,lm_mu)

para<-newp[,iter]
c4<-cons2.mle(para)
# cat("iter, C=",iter, c4,"\n")

if(min(c4) > 0 ) {
fit2.mle = constrOptim.nl(par = para, fn=loglik2.mle, hin=cons2.mle)
# cat("iter, minfn, par= ", iter, fit2.mle$value, fit2.mle$par,"\n")
```

```
if(fit2.mle$value < fminstar) {
fminstar <- fit2.mle$value
parstar <- fit2.mle$par
iterstar <- iter
}
}

options(digits=10)
# cat("iter, minfn, par= ", iterstar, fminstar,parstar, "\n")
return(list(fcnmin=fminstar, parmle=parstar, miniter=iterstar))

}
####negative log-Likelihood function evaluation for MLE in K4D #########
#auglag in alabama

loglik2.mle <- function(theta)
{
# Programmed by Jeong-Soo Park, 10 Dec. 2018
# Dept of Stat. Chonnam National University, Korea

eps.like <- c(1.e-40)
max.like <- 1.e+15
c1<- rep(NA,5)

c1<- cons2.mle(theta)
if( min(c1) < 0 ) {
max.like<- max.like +(runif(1)*40 -10)
#     cat( "mllnew=", mllnew,"\n" )
#     mllnew <- mllnew + (runif(1) -.5)
#     return(mllnew)
return(max.like)
}

k <- theta[1]
h <- theta[2]
al <- theta[3]
xi <- theta[4]

nobs <- length(x.dat)
G <- rep(NA, nobs)
W <- rep(NA, nobs)
```

```
work <- rep(NA, nobs)
G2 <- rep(NA, nobs)
W2 <- rep(NA, nobs)

work <- x.dat-xi
G <- 1-(k*work)/al
G2 <- G^((k-1)/k)
W  <-  1- h*exp( log(G)/k )
W2 <- exp( log(W)*(h-1)/h )

W2 <- W2[!is.na(W2)]
G2 <- G2[!is.na(G2)]

W2 <- W2[!is.infinite(W2)]
G2 <- G2[!is.infinite(G2)]

W2 <- W2[!(W2 < 0)]
G2 <- G2[!(G2 < 0)]

W2[W2 < eps.like]<- eps.like
G2[G2 < eps.like]<- eps.like

nobs_new<- min( length(W2), length(G2) )
if( nobs_new < nobs ) {
mllnew <- max.like
return(mllnew)
}

mllnew<- nobs_new*log(al) + sum(log(G2)) + sum(log(W2))
if(mllnew < 0) { mllnew<- max.like }

return(mllnew)
}
```

```
######## MPLE.CD1 program for the 4-parameter Kappa distribution #######

kap4pmle.cd3.1 = function(data,para)
{

maxit <- 20
newp <- matrix(NA, nrow=4,ncol=maxit*2)
x.dat <- data
maxxi <- max(x.dat)           # The data set name should be "x.dat"
minxi <- min(x.dat)
fminstar <- 1.0e+20
c<-cons2.mle(para)
if( min(c) > 0 ) {
fit2.mle = constrOptim.nl(par = para, fn=loglik2.pmle.cd, hin=cons2.mle)
fminstar <- fit2.mle$value
parstar <- fit2.mle$par
iterstar <- 1
}

sd0 <- sd(x.dat)
mean0 <- mean(x.dat)
newp[1,2] <- 0.001                              # k =para[1]
newp[2,2] <- -.001                              # h =para[2]
newp[3,2] <- sd(x.dat)*0.7797                   # alpha or sigma =para[3]
newp[4,2] <- mean(x.dat)-0.45005*sd(x.dat)      # xi or mu =para[4]

para<-newp[,2]
c2<-cons2.mle(para)
if( min(c2) > 0 ) {
fit2.mle = constrOptim.nl(par = para, fn=loglik2.pmle.cd, hin=cons2.mle)
if(fit2.mle$value < fminstar) {
fminstar <- fit2.mle$value
parstar <- fit2.mle$par
iterstar <- 2
}
}

newp[,3] <- c(.01, -.3, 61, 177)
newp[,4] <- c(-.1,-.2,120,115)
newp[,5] <- c(.3,-1.3,90,30)
newp[,6] <- c(-.05,.1,40,60)
```

```
newp[,7] <- c(.01,.02,100,80)
newp[,8] <- c(-.09,1.9,58,96)
newp[,9] <- c(-.3,.3,60,60)
newp[,10] <- c(.001,.001,60,60)
newp[,11] <- c(-.5,-1.,60,60)
newp[3,11] <- 0.7*sd0
newp[4,11] <- mean0-0.4*sd0
newp[,12] <- c(-.4,.5,60,60)
newp[3,12] <- 0.7*sd0
newp[4,12] <- mean0-0.4*sd0
newp[,13] <- c(.4,.5,60,60)
newp[3,13] <- 0.7*sd0
newp[4,13] <- mean0-0.4*sd0
newp[,14] <- c(.4,-.5,60,60)
newp[3,14] <- 0.7*sd0
newp[4,14] <- mean0
newp[,15] <- c(-.5,-1.5,60,60)
newp[3,15] <- 2*sd0
newp[4,15] <- mean0
newp[,16] <- c(-.5,-1.,60,60)
newp[3,16] <- sd0*2
newp[4,16] <- mean0
newp[,17] <- c(-.2,-2.,60,60)
newp[3,17] <- sd0
newp[4,17] <- mean0*2
newp[,18] <- c(-.01,-.01,60,60)
newp[3,18] <- sd0
newp[4,18] <- mean0*2
newp[,19] <- c(-1,-.5,60,60)
newp[3,19] <- sd0
newp[4,19] <- mean0-0.5*sd0
newp[,20] <- c(-.5,-.5,60,60)
newp[3,20] <- sd0/2
newp[4,20] <- sd0/2


for (iter in 3:maxit) {
para<-newp[,iter]
c3<-cons2.mle(para)
if(min(c3) > 0 ) {
fit2.mle = constrOptim.nl(par = para, fn=loglik2.pmle.cd, hin=cons2.mle)
```

```
if(fit2.mle$value < fminstar) {
fminstar <-fit2.mle$value
parstar<-fit2.mle$par
iterstar<-iter
}
}
}

###### estimate by LM of GEVD for initial values #######
lm <- samlmu(x.dat)
fit.lm <- pelgev(lm)
iter<-maxit+1
newp[1,iter] <- fit.lm[3]   # k
newp[2,iter] <- 0.01        # h
newp[3,iter] <- fit.lm[2]   # scale
newp[4,iter] <- fit.lm[1]   # location
para <- newp[,iter]
c4<-cons2.mle(para)
if(min(c4) > 0 ) {
fit2.mle = constrOptim.nl(par = para, fn=loglik2.pmle.cd, hin=cons2.mle)
if(fit2.mle$value < fminstar) {
fminstar <-fit2.mle$value
parstar<-fit2.mle$par
iterstar<-iter
}
}
options(digits=10)
return(list(fcnmin=fminstar, parmle=parstar, miniter=iterstar))
}

loglik2.pmle.cd <- function(theta)
{
eps.like <- c(1.e-40)
max.like <- 1.e+15
c1 <- rep(NA,5)
c1 <- cons2.mle(theta)
if( min(c1) < 0 ) {
max.like <- max.like +(runif(1)*40 -10)
return(max.like)
}
```

```
k <-theta[1]
h <-theta[2]
al <-theta[3]
xi <-theta[4]

nobs<-length(x.dat)
G <- rep(NA, nobs)
W <- rep(NA, nobs)
work <- rep(NA, nobs)
G2 <- rep(NA, nobs)
W2 <- rep(NA, nobs)

work <- x.dat-xi
G <- 1-(k*work)/al
G2 <- G^((k-1)/k)
W <- 1- h*exp( log(G)/k )
W2 <- exp( log(W)*(h-1)/h )

W2 <- W2[!is.na(W2)]
G2 <- G2[!is.na(G2)]

W2 <- W2[!is.infinite(W2)]
G2 <- G2[!is.infinite(G2)]

W2 <- W2[!(W2 < 0)]{\large {\large {\tiny }}}
G2 <- G2[!(G2 < 0)]

W2[W2 < eps.like]<- eps.like
G2[G2 < eps.like]<- eps.like

nobs_new<- min( length(W2), length(G2) )
if( nobs_new < nobs ) {
mllnew<- max.like
return(mllnew)
}
a = 1; b =1
pk <- exp (-a* ( (1/(1-k)) - 1)^b)
ph <- exp (-a* ( (1/(1-h)) - 1)^b )
if(k < 0){pk=1}
if(h < 0){ph=1}
if(k >= 1){pk=eps.like} #might be change for k >=1
```

```
if(h >= 1){ph=eps.like} #might be change for k >=1.2

mllnew <- nobs_new*log(al) + sum(log(G2)) + sum(log(W2))
- log(pk) - log(ph)
if(mllnew < 0) { mllnew<- max.like }
return(mllnew)
}
```

BIOGRAPHY

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NAME	Nipada Papukdee
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