$\beta(\tau_1, \tau_2)$-CONTINUOUS MULTIFUNCTIONS ON BITOPOLOGICAL SPACES

KAEWTA LAPROM

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ฟังก์ชันหลายค่าต่อเนื่อง $\beta(\tau_1, \tau_2)$ บนปริภูมิเชิงไบทอพอโลยี

วิทยานิพนธ์

ของ

แก้วตา หล้าพรม

เสนอต่อมหาวิทยาลัยมหาสารคาม เพื่อเป็นส่วนหนึ่งของการศึกษาตามหลักสูตร
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มีนาคม 2563

The examining committee has unanimously approved this thesis, submitted by Miss Kaewta Laprom, as a partial fulfillment of the requirements for the Master of Science in Mathematics Education at Mahasarakham University.

Examining Committee

....	Chairman
(Asst. Prof. Supunnee Sompong, Ph.D.)	(Faculty graduate committee)
....	Committee
(Asst. Prof. Chawalit Boonpok, Ph.D.)	(Advisor)
....	Committee
(Asst. Prof. Chokchai Viriyapong, Ph.D.)	(Co-advisor)
....	Committee
(Asst. Prof. Jeeranunt Khampakdee, Ph.D.)	(Faculty graduate committee)
....	Committee
(Chalongchai Klanarong, Ph.D.)	(Faculty graduate committee)

Mahasarakham University has granted approval to accept this thesis as a partial fulfillment of the requirements for the Master of Science in Mathematics Education.

....	....
(Prof. Pairot Pramual , Ph.D.)	(Assoc. Prof. Krit Chaimoon, Ph.D.)
Dean of the Faculty of Science	Dean of Graduate School

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	Asst. Prof. Chokchai Viriyapong, Ph.D.
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ABSTRACT

The purpose of this paper is to introduce the concepts of $\beta(\tau_1, \tau_2)$-continuous multifunctions, almost $\beta(\tau_1, \tau_2)$-continuous multifunctions, and weakly $\beta(\tau_1, \tau_2)$-continuous multifunctions. Moreover, some characterizations of these multifunctions are investigated. Besides, the relationships between upper and lower $\beta(\tau_1, \tau_2)$-continuous multifunctions and the other types of continuity are also discussed.

Keywords : $\tau_1\tau_2$-β-open, upper $\beta(\tau_1, \tau_2)$-continuous multifunction, lower $\beta(\tau_1, \tau_2)$-continuous multifunction

ชื่อเรื่อง ฟังก์ชันหลายค่าต่อเนื่อง $\beta(\tau_1, \tau_2)$ บนปริภูมิเชิงไบทอพอโลยี
ผู้วิจัย นางสาวแก้วตา หล้าพรม
ปริญญา วิทยาศาสตรมหาบัณฑิต **สาขา** คณิตศาสตรศึกษา
อาจารย์ที่ปรึกษา ผู้ช่วยศาสตราจารย์ ดร. ชวลิต บุญปก
ผู้ช่วยศาสตราจารย์ ดร.โชคชัย วิริยะพงษ์
มหาวิทยาลัย มหาวิทยาลัยมหาสารคาม **ปีที่พิมพ์** 2563

บทคัดย่อ

วัตถุประสงค์ของการวิจัยครั้งนี้เพื่อนำเสนอแนวคิดของฟังก์ชันหลายค่าต่อเนื่อง $\beta(\tau_1, \tau_2)$ ฟังก์ชันหลายค่าเกือบต่อเนื่อง $\beta(\tau_1, \tau_2)$ และฟังก์ชันหลายค่าต่อเนื่องอย่างอ่อน $\beta(\tau_1, \tau_2)$ อีกทั้งแสดงให้เห็นลักษณะของฟังก์ชันหลายค่าเหล่านี้ นอกจากนี้ยังกล่าวถึงความสัมพันธ์ระหว่างฟังก์ชันหลายค่าต่อเนื่อง $\beta(\tau_1, \tau_2)$ กับความต่อเนื่องประเภทอื่นๆ

คำสำคัญ : เซตเปิด $\tau_1\tau_2$-β, ฟังก์ชันหลายค่าต่อเนื่องบน $\beta(\tau_1, \tau_2)$,
ฟังก์ชันหลายค่าต่อเนื่องล่าง $\beta(\tau_1, \tau_2)$

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Background

General topology is an important mathematical branch which is applied for many fields of applied sciences. Continuity is a basic concept for the study in topological spaces. Generalization of this concept by using weaker forms of open sets such as semi-open sets [1], preopen sets [2] and β-open sets [3] is one of the main research topics of general topology. In 1983, Monsef et al. [4] introduced the classes of β-open sets called semi-preopen sets by Andrijević in [3]; moreover, Monsef et al. [4] introduced almost β-continuous functions in topological spaces. From 1992 to 1993, the authors[5] obtained several characterizations of β-continuity and showed that almost quasi-continuity [6] investigated by Borsik and Dobos was equivalent to β-continuity. Therefore, in 1997, Nasef and Noiri [7] investigated fundamental characterizations of almost β-continuous functions. A year later, Popa and Noiri [8] investigated further characterizations of almost β-continuous functions. In 1992, Khedr et al.[9] generalized the notions of β-open sets and investigated β-continuous functions in bitopological spaces. Furthermore, in [10], [11], and [12] from 1996 to 2000, the authors extended these functions to multifunctions by introducing and characterizing the notions of β-continuous multifunctions, almost β-continuous multifunctions, and weakly β-continuous multifunctions in topological spaces.

Therefore, we are interested in defining upper and lower $\beta(\tau_1, \tau_2)$-continuous multifunctions and investigating some characterizations of these multifunctions. Furthermore, almost $\beta(\tau_1, \tau_2)$-continuous multifunctions and weakly $\beta(\tau_1, \tau_2)$-continuous multifunctions were investigated.

1.2 Objective of the research

The purposes of the research are:

1.2.1 To define and investigate the characterizations of upper and lower $\beta(\tau_1, \tau_2)$-continuous multifunctions.

1.2.2 To define and investigate the characterizations of upper and lower almost $\beta(\tau_1, \tau_2)$-continuous multifunctions.

1.2.3 To define and investigate the characterizations of upper and lower weakly $\beta(\tau_1, \tau_2)$-continuous multifunctions.

1.3 Research methodology

The research procedure of this thesis consists of the following steps:

1.3.1 Criticism and possible extension of the literature review.

1.3.2 To define and investigate the characterizations of upper and lower $\beta(\tau_1, \tau_2)$-continuous multifunctions.

1.3.3 To define and investigate the characterizations of upper and lower almost $\beta(\tau_1, \tau_2)$-continuous multifunctions.

1.3.4 To define and investigate the characterizations of upper and lower weakly $\beta(\tau_1, \tau_2)$-continuous multifunctions.

1.3.5 To make the conclusions and do a complete report to offer Mahasarakham University.

1.4 Scope of the study

The scopes of the study are: defining and investigating the characterizations of upper and lower $\beta(\tau_1, \tau_2)$-continuous multifunctions, the characterizations of upper and lower almost $\beta(\tau_1, \tau_2)$-continuous multifunctions, and the characterizations of upper and lower weakly $\beta(\tau_1, \tau_2)$-continuous multifunctions.

CHAPTER 2

PRELIMINARIES

In this chapter, we will give some definitions, notations, dealing with some preliminaries and some useful results that will be duplicated in later chapter.

2.1 Bitopological spaces

Definition 2.1.1. [13] Let $X \neq \emptyset$ and τ be a collection of subsets of X. Then, τ is called a topology on X if and only if τ satisfies the properties:

1. $\emptyset, X \in \tau$.

2. If $G_1, G_2 \in \tau$ then $G_1 \cap G_2 \in \tau$.

3. If $G_i \in \tau$ for all $i \in J$ then $\cup_{i \in J} G_i \in \tau$.

The pair (X, τ) is called a topological space. Sometimes this research, spaces (X, τ) (or simply X) always mean topological spaces.

Definition 2.1.2. [13] Let X be a topological space, and $A \subseteq X$. The *interior* of A is the set given by Int$(A) = \cup\{U \subseteq X : U \subseteq A$ and U is open$\}$.

Theorem 2.1.3. [13] If X is a space and $A \subseteq X$, then the following are true:

1. Int(A) is an open set.

2. Int$(A) \subseteq A$.

3. If $A \subseteq B \subseteq X$, then Int$(A) \subseteq$ Int(B).

4. If U is an open set with $U \subseteq A$, then $U \subseteq$ Int(A), that is Int(A) is the largest open set contained in A.

5. For every $A \subseteq X$ and $B \subseteq X$, Int$(A \cap B)$ =Int$(A) \cap$ Int(B).

6. For every $A \subseteq X$, A is open if and only if $A =$ Int(A).

Definition 2.1.4. [13] If A is a subset A of a topological space X, then the *closure* of A is the set given by Cl$(A) = \cap\{F \subseteq X : F$ is closed and $A \subseteq F\}$.

Theorem 2.1.5. [13] If X is a space and $A \subseteq X$, then the following are true:

1. Cl(A) is an closed set.
2. $A \subseteq$ Cl(A).
3. If $A \subseteq B \subseteq X$, then Cl$(A) \subseteq$ Cl(B).
4. If F is an closed set with $A \subseteq F$, then Cl$(A) \subseteq F$, that is Cl(A) is the smallest closed set containing in A.
5. For every $A \subseteq X$ and $B \subseteq X$, Cl$(A \cup B) =$ Cl$(A) \cup$ Cl(B).
6. For every $A \subseteq X$, A is closed if and only if $A =$ Cl(A).

Proposition 2.1.6. [3] For a subset A of a topological space (X, τ), the following properties hold:

(1) Cl$(A) \cap G \subseteq$ Cl$(A \cap G)$ for every open set G.

(2) Int$(A \cup F) \subseteq$ Int$(A) \cup F$ for every closed set F.

Definition 2.1.7. [15] A collection $\mathcal{A}$ of subsets of a space X is said to *cover* X, or to be a *covering* of X, if the union of the element of $\mathcal{A}$ is equal to X. It is called an *open covering* of X if its elements are open subsets of X.

In this investigation, we study on bitopological spaces. According to Kelly [14], a bitopological space (X, τ_1, τ_2) is a set X with two topologies, τ_1 and τ_2 on the space. Let A be a subsets of bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$.

Definition 2.1.8. [16] Let (X, τ_1, τ_2) be a bitopological space and let $A \subseteq X$. Then, A is called (i, j)-*semi-open* if $A \subseteq j$Cl$(i$Int$(A))$, where $i \neq j = 1, 2$. The complement of (i, j)-semi-open set is (i, j)-*semi-closed*.

Definition 2.1.9. [17] Let (X, τ_1, τ_2) be a bitopological space and let $A \subseteq X$. Then, A is called (i, j)-*regular-open* if $A = i\text{Int}(j\text{Cl}(A))$, where $i \neq j = 1, 2$. The complement of (i, j)-regular-open set is (i, j)-*regular-closed*.

Definition 2.1.10. [18] Let (X, τ_1, τ_2) be a bitopological space and let $A \subseteq X$. Then, A is called (i, j)-*preopen* if $A \subseteq i\text{Int}(j\text{Cl}(A))$. The complement of (i, j)-preopen set is (i, j)-*preclosed*.

Definition 2.1.11. [9] Let (X, τ_1, τ_2) be a bitopological space and let $A \subseteq X$. Then, A is called (i, j)-β-*open* if $A \subseteq j\text{Cl}(i\text{Int}(j\text{Cl}(A)))$. The complement of (i, j)-β-open set is (i, j)-β-*closed*.

Note: In our research, we let $\tau_1\tau_2$-semi-open, $\tau_1\tau_2$-preopen, $\tau_1\tau_2$-β-open and $\tau_1\tau_2$-regular-open represent $(2, 1)$-*semi-open*, $(1, 2)$-*preopen*, $(2, 1)$-β-*open* and $(1, 2)$-*regular-open* respectively; moreover, we do complement of those in the similar way.

Example 2.1.12. Let $X = \{a, b, c\}$ with topologies $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{a, b\}, X\}$.

1. It is easy to verify that there are five $\tau_1\tau_2$-semi-open such as $\emptyset, \{a\}, \{a, b\}, \{a, c\}$, and X; furthermore, five $\tau_1\tau_2$-semi-closed sets on X are $\emptyset, \{b, c\}, \{c\}, \{b\}$, and X.

2. It is easy to verify that there are three $\tau_1\tau_2$-regular-open such as $\emptyset, \{b\}$ and X; furthermore, three $\tau_1\tau_2$-regular-closed sets on X are $\emptyset, \{a, c\}$, and X.

3. It is easy to verify that the $\tau_1\tau_2$-preopen on X are $\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}$ and X; furthermore, six $\tau_1\tau_2$-preclosed sets on X are $\emptyset, \{b, c\}, \{a, c\}, \{c\}, \{b\}$, and X.

4. It is easy to check that $\{a\}$ is a $\tau_1\tau_2$-β-open set but $\{c\}$ is not a $\tau_1\tau_2$-β-open set. Therefore, we know the complement of $\{a\}$ is $\tau_1\tau_2$-β-closed set.

Definition 2.1.13. [19] Let (X, τ_1, τ_2) be a bitopological space and A be a subset of X. Then, $\tau_1\tau_2$-*semi-closure* of A denoted by $\tau_1\tau_2$-$sCl(A)$ is defined as

$$\tau_1\tau_2\text{-sCl}(A) = \cap\{F \subseteq X : F \text{ is } \tau_1\tau_2\text{-semi-closed in } X \text{ and } A \subseteq F\}.$$

Definition 2.1.14. [9] Let (X, τ_1, τ_2) be a bitopological space and A be a subset of X. Then, $\tau_1\tau_2$-*preclosure* of A denoted by $\tau_1\tau_2$-pCl(A) is defined as

$$\tau_1\tau_2\text{-pCl}(A) = \cap\{F \subseteq X : F \text{ is } \tau_1\tau_2\text{-preclosed in } X \text{ and } A \subseteq F\}.$$

Definition 2.1.15. [9] Let (X, τ_1, τ_2) be a bitopological space and A be a subset of X. Then, $\tau_1\tau_2$-β-closure of A denoted by $\tau_1\tau_2$-βCl(A) is defined as

$$\tau_1\tau_2\text{-}\beta\text{Cl}(A) = \cap\{F \subseteq X : F \text{ is } \tau_1\tau_2\text{-}\beta\text{-closed in } X \text{ and } A \subseteq F\}.$$

Example 2.1.16. Let $X = \{a, b, c\}$ with topologies $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{a, b\}, X\}$. Let $A = \{c\}$. Then, (X, τ_1, τ_2) is a bitopological space. In example 2.1.12, $\tau_1\tau_2$-semi-closed in X are $\emptyset, \{b, c\}, \{c\}, \{b\}$, and X. Clearly, $A \subseteq \{b, c\}$ and $A \subseteq \{c\}$, then $\tau_1\tau_2$-sCl$(A) = \{b, c\} \cap \{c\} = \{c\}$. Moreover, in example 2.1.12, $\tau_1\tau_2$-preclosed in X are $\emptyset, \{b, c\}, \{a, c\}, \{b\}, \{c\}$, and X. Clearly, $A \subseteq \{b, c\}$, $A \subseteq \{c\}$ and $A \subseteq \{a, c\}$. Hence, $\tau_1\tau_2$-pCl$(A) = \{b, c\} \cap \{c\} \cap \{a, c\} = \{c\}$. Moreover, it is easy to check that $\tau_1\tau_2$-βCl$(A) = \{c\}$.

Definition 2.1.17. [19] Let (X, τ_1, τ_2) be a bitopological space and A be a subset of X. Then, $\tau_1\tau_2$-*semi-interior* of A denoted by $\tau_1\tau_2$-sInt(A) is defined as

$$\tau_1\tau_2\text{-sInt}(A) = \cup\{G \subseteq X : G \text{ is } \tau_1\tau_2\text{-semi-open in } X \text{ and } G \subseteq A\}.$$

Definition 2.1.18. [20] Let (X, τ_1, τ_2) be a bitopological space and A be a subset of X. Then, $\tau_1\tau_2$-*preinterior* of A denoted by $\tau_1\tau_2$-pInt(A) is defined as

$$\tau_1\tau_2\text{-pInt}(A) = \cup\{G \subseteq X : G \text{ is } \tau_1\tau_2\text{-preopen in } X \text{ and } G \subseteq A\}.$$

Definition 2.1.19. [9] Let (X, τ_1, τ_2) be a bitopological space and A be a subset of X. Then, $\tau_1\tau_2$-β-*interior* of A denoted by $\tau_1\tau_2$-βInt(A) is defined as

$$\tau_1\tau_2\text{-}\beta\text{Int}(A) = \cup\{G \subseteq X : G \text{ is } \tau_1\tau_2\text{-}\beta\text{-open in } X \text{ and } G \subseteq A\}.$$

Example 2.1.20. Let $X = \{a, b, c\}$ with topologies $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{a, b\}, X\}$. Let $A = \{a, c\}$. Then, (X, τ_1, τ_2) is a bitopological space.

In example 2.1.12, $\tau_1\tau_2$-semi-open in X are $\emptyset, \{a\}, \{a, b\}, \{a, c\}$, and X. Clearly, $\emptyset \subseteq A$, $\{a\} \subseteq A$ and $\{a, c\} \subseteq A$, then $\tau_1\tau_2$-sInt$(A) = \emptyset \cup \{a\} \cup \{a, c\} = \{a, c\}$.

Moreover, in example 2.1.12 is that $\tau_1\tau_2$-preopen in X are $\emptyset, \{a\}, \{b\}, \{a,c\}, \{a,b\}$, and X. Clearly, $\emptyset \subseteq A$, $\{a\} \subseteq A$ and $\{a,c\} \subseteq A$. Hence, $\tau_1\tau_2$-pInt$(A) = \emptyset \cup \{a\} \cup \{a,c\} = \{a,c\}$. Moreover, it is easy to check that $\tau_1\tau_2$-βInt$(A) = \{a,c\}$.

Definition 2.1.21. [21] Let (X, τ_1, τ_2) be a bitopological space and let $A \subseteq X$. Then, A is called $\tau_1\tau_2$-*open* if $A = \tau_1$-Int$(\tau_2$-Int$(A))$. The complement of $\tau_1\tau_2$-open set is $\tau_1\tau_2$-*closed*.

Example 2.1.22. Let $X = \{a,b,c\}$ with topologies $\tau_1 = \{\emptyset, \{b\}, \{c\}, \{b,c\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b,c\}, X\}$. It is easy to check that $\{b,c\}$ is a $\tau_1\tau_2$-open set but $\{c\}$ is not a $\tau_1\tau_2$-open set. Therefore, we know the complement of $\{b,c\}$ is $\tau_1\tau_2$-closed set.

Definition 2.1.23. [21] Let (X, τ_1, τ_2) be a bitopological space and A be a subset of X. Then, $\tau_1\tau_2$-*closure* of A denoted by $\tau_1\tau_2$-Cl(A) is defined as

$$\tau_1\tau_2\text{-Cl}(A) = \cap\{F \subseteq X : F \text{ is } \tau_1\tau_2\text{-closed in } X \text{ and } A \subseteq F\}.$$

Example 2.1.24. Let $X = \{a,b,c\}$ with topologies $\tau_1 = \{\emptyset, \{b\}, \{c\}, \{b,c\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b,c\}, X\}$. Let $A = \{a\}$. Then, (X, τ_1, τ_2) is a bitopological space. It is easy to verify that $\tau_1\tau_2$-closed in X are $\emptyset, \{a\}$, and X. Clearly, $A \subseteq \{a\}$ and $A \subseteq X$, then $\tau_1\tau_2$-Cl$(A) = \{a\} \cap X = \{a\}$.

Definition 2.1.25. [21] Let (X, τ_1, τ_2) be a bitopological space and A be a subset of X. Then, $\tau_1\tau_2$-*interior* of A denoted by $\tau_1\tau_2$-Int(A) is defined as

$$\tau_1\tau_2\text{-Int}(A) = \cup\{G \subseteq X : G \text{ is } \tau_1\tau_2\text{-open in } X \text{ and } G \subseteq A\}.$$

Example 2.1.26. Let $X = \{a,b,c\}$ with topologies $\tau_1 = \{\emptyset, \{b\}, \{c\}, \{b,c\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b,c\}, X\}$. Let $A = \{b,c\}$. Then, (X, τ_1, τ_2) is a bitopological space. It is easy to verify that $\tau_1\tau_2$-*open* in X are $\emptyset, \{b,c\}$, and X. Clearly, $\emptyset \subseteq A$ and $\{b,c\} \subseteq A$, then $\tau_1\tau_2$-Int$(A) = \emptyset \cup \{b,c\} = \{b,c\}$.

Definition 2.1.27. [21] A set N of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*neighbourhood* of $x \in X$ if there exists a $\tau_1\tau_2$-open set V of (X, τ_1, τ_2) such that $x \in V \subseteq N$.

Example 2.1.28. Let $X = \{a, b, c\}$ with topologies $\tau_1 = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$. Let $N = \{a, b\}$.
Then, (X, τ_1, τ_2) is a bitopological space. It is easy to verify that $\tau_1\tau_2$-open in X are $\emptyset, \{b\}, \{b, c\}$, and X. Clearly, $b \in \{b\} \subseteq N$, then N is $\tau_1\tau_2$-neighbourhood of $b \in X$.

Definition 2.1.29. A set N of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*preneighbourhood* of $x \in X$ if there exists a $\tau_1\tau_2$-preopen set V of (X, τ_1, τ_2) such that $x \in V \subseteq N$.

Example 2.1.30. Let $X = \{a, b, c\}$ with topologies $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $N = \{a, c\}$, then (X, τ_1, τ_2) is the bitopological space. In example 2.1.12, the $\tau_1\tau_2$-preopen on X are $\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}$ and X. Clearly, $a \in \{a\} \subseteq N$, then N is $\tau_1\tau_2$-preneighbourhood of $a \in X$.

Proposition 2.1.31. [21] Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:

(1) $A \subseteq \tau_1\tau_2$-Cl(A) and $\tau_1\tau_2$-Cl($\tau_1\tau_2$-Cl(A)) $= \tau_1\tau_2$-Cl(A).

(2) If $A \subseteq B$, then $\tau_1\tau_2$-Cl(A) $\subseteq \tau_1\tau_2$-Cl(B).

(3) $\tau_1\tau_2$-Cl(A) is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2$-Cl(A).

(5) $\tau_1\tau_2$-Cl($X - A$) $= X - \tau_1\tau_2$-Int(A).

Proposition 2.1.32. Let (X, τ_1, τ_2) be a bitopological space. If A is $\tau_1\tau_2$-semi-open and B is $\tau_1\tau_2$-open in X, then $A \cap B$ is $\tau_1\tau_2$-semi-open.

Proof. Suppose that A is $\tau_1\tau_2$-semi-open and B is $\tau_1\tau_2$-open in X.
Then, $A \subseteq \tau_1$-Cl(τ_2-Int(A)) and $B = \tau_1$-Int(B) $= \tau_2$-Int(B). Therefore, we obtain $A \cap B \subseteq \tau_1$-Cl($\tau_2$-Int($A$)) $\cap B$. By Proposition 2.1.6(1), we have
$A \cap B \subseteq \tau_1$-Cl($\tau_2$-Int($A$)) $\cap B \subseteq \tau_1$-Cl(τ_2-Int(A) $\cap B$) $= \tau_1$-Cl(τ_2-Int($A \cap B$)).
Hence, $A \cap B$ is $\tau_1\tau_2$-semi-open. □

Theorem 2.1.33. [9] Let (X, τ_1, τ_2) be a bitopological. Let $\{A_\gamma | \gamma \in \bigtriangledown\}$ be a family of subsets of X. The following properties are hold:

(1) If A_γ is $\tau_1\tau_2$-semi-open for each $\gamma \in \bigtriangledown$, then $\cup_{\gamma\in\bigtriangledown} A_\gamma$ is $\tau_1\tau_2$-semi-open.

(2) If A_γ is $\tau_1\tau_2$-semi-closed for each $\gamma \in \bigtriangledown$, then $\cap_{\gamma\in\bigtriangledown} A_\gamma$ is $\tau_1\tau_2$-semi-closed.

Proof. 1. Suppose that A_γ is $\tau_1\tau_2$-semi-open for each $\gamma \in \bigtriangledown$. Then, we have $A_\gamma \subseteq \tau_1\text{-Cl}(\tau_2\text{-Int}(A_\gamma)) \subseteq \tau_1\text{-Cl}(\tau_2\text{-Int}(\cup_{\gamma\in\bigtriangledown} A_\gamma))$, and hence $\cup_{\gamma\in\bigtriangledown} A_\gamma \subseteq \tau_1\text{-Cl}(\tau_2\text{-Int}(\cup_{\gamma\in\bigtriangledown} A_\gamma))$. This shows that $\cup_{\gamma\in\bigtriangledown} A_\gamma$ is $\tau_1\tau_2$-semi-open.

2. Suppose that A_γ is $\tau_1\tau_2$-semi-closed for each $\gamma \in \bigtriangledown$. Then, we have $X - A_\gamma$ is $\tau_1\tau_2$-semi-open and $X - \cap_{\gamma\in\bigtriangledown} A_\gamma = \cup_{\gamma\in\bigtriangledown}(X - A_\gamma)$. Therefore, by (1), $X - \cap_{\gamma\in\bigtriangledown} A_\gamma$ is $\tau_1\tau_2$-semi-open, and hence $\cap_{\gamma\in\bigtriangledown} A_\gamma$ is $\tau_1\tau_2$-semi-closed.

□

Proposition 2.1.34. For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:

(1) $\tau_1\tau_2$-sInt(A) is $\tau_1\tau_2$-semi-open.

(2) $\tau_1\tau_2$-sCl(A) is $\tau_1\tau_2$-semi-closed.

(3) A is $\tau_1\tau_2$-semi-open if and only if $A = \tau_1\tau_2$-sInt(A).

(4) A is $\tau_1\tau_2$-semi-closed if and only if $A = \tau_1\tau_2$-sCl(A).

Proof. (1) and (2) follows from Proposition 2.1.33 (3) and (4) follows from (1) and (2). □

Proposition 2.1.35. For a subset A of a bitopological space (X, τ_1, τ_2),$x \in \tau_1\tau_2$-sCl(A) if and only if $U \cap A \neq \emptyset$ for every $\tau_1\tau_2$-semi-open set U containing x.

Proof. Let $x \in \tau_1\tau_2$-sCl(A). We shall show that $U \cap A \neq \emptyset$ for every $\tau_1\tau_2$-semi-open set U containing x. Suppose that $U \cap A = \emptyset$ for some $\tau_1\tau_2$-semi-open set U containing x. Then, $A \subseteq X - U$ and $X - U$ is $\tau_1\tau_2$-semi-closed. Since $x \in \tau_1\tau_2$-sCl(A), we have $x \in \tau_1\tau_2$-sCl$(X - U) = X - U$; hence $x \notin U$, which is a contradiction that $x \in U$. Therefore, $U \cap A \neq \emptyset$.

Conversely, we assume that $U \cap A \neq \emptyset$ for every $\tau_1\tau_2$-semi-open set U containing x. We shall show that $x \in \tau_1\tau_2$-sCl(A). Suppose that $x \notin \tau_1\tau_2$-sCl(A). Then, there exists a $\tau_1\tau_2$-semi-closed set F such that $A \subseteq F$ and $x \notin F$. Therefore, we obtain

$X - F$ is a $\tau_1\tau_2$-semi-open set containing x such that $(X - F) \cap A = \emptyset$. This a contradiction to $U \cap A \neq \emptyset$; hence $x \in \tau_1\tau_2$-sCl(A). □

Proposition 2.1.36. For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:

(1) $X - \tau_1\tau_2$-sCl$(A) = \tau_1\tau_2$-sInt$(X - A)$.

(2) $X - \tau_1\tau_2$-sInt$(A) = \tau_1\tau_2$-sCl$(X - A)$.

Proof. (1) Let $x \in X - \tau_1\tau_2$-sCl(A). Then, $x \notin \tau_1\tau_2$-sCl(A). By Proposition 2.1.35, there exists a $\tau_1\tau_2$-semi-open set V containing x such that $V \cap A = \emptyset$. Then, $V \subseteq X - A$, and hence $x \in \tau_1\tau_2$-sInt$(X - A)$. This shows that $X - \tau_1\tau_2$-sCl$(A) \subseteq \tau_1\tau_2$-sInt$(X - A)$.

Let $x \in \tau_1\tau_2$-sInt$(X - A)$. Then, there exists a $\tau_1\tau_2$-semi-open set V containing x such that $V \subseteq X - A$, and hence $V \cap A = \emptyset$. By Proposition 2.1.35, we have $x \notin \tau_1\tau_2$-sCl(A); hence $x \in X - \tau_1\tau_2$-sCl(A). Therefore,

$$\tau_1\tau_2\text{-sInt}(X - A) \subseteq X - \tau_1\tau_2\text{-sCl}(A).$$

Consequently, we obtain $X - \tau_1\tau_2$-sCl$(A) = \tau_1\tau_2$-sInt$(X - A)$.

(2) This follow from (1). □

Proposition 2.1.37. For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:

(1) $\tau_1\tau_2$-sCl$(A) = \tau_1$-Int$(\tau_2$-Cl$(A)) \cup A$.

(2) If A is τ_1-open in X, then $\tau_1\tau_2$-sCl$(A) = \tau_1$-Int$(\tau_2$-Cl$(A))$.

Proof. (1) Since $\tau_1\tau_2$-sCl(A) is $\tau_1\tau_2$-semi-closed, we have

$$\tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-sCl}(A))) \subseteq \tau_1\tau_2\text{-sCl}(A).$$

Thus, τ_1-Int$(\tau_2$-Cl$(A)) \subseteq \tau_1\tau_2$-sCl$(A)$. Hence, τ_1-Int$(\tau_2$-Cl$(A)) \cup A \subseteq \tau_1\tau_2$-sCl$(A)$. To establish the opposite inclusion, we observe that

$$\begin{aligned}\tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\text{-Int}(\tau_2\text{-Cl}(A)) \cup A)) &\subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(A) \cup A)\\ &= \tau_1\text{-Int}(\tau_2\text{-Cl}(A)).\end{aligned}$$

Therefore,

$$\tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\text{-Int}(\tau_2\text{-Cl}(A)) \cup A)) \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(A)) \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(A)) \cup A.$$

Hence, τ_1-Int$(\tau_2$-Cl$(A)) \cup A$ is $\tau_1\tau_2$-semi-closed. Then,

$$\tau_1\tau_2\text{-sCl}(A) \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(A)) \cup A.$$

Consequently, we obtain $\tau_1\tau_2$-sCl$(A) = \tau_1$-Int$(\tau_2$-Cl$(A)) \cup A$.

(2) Let A be a τ_1-open set, then $A = \tau_1$-Int$(A) \subseteq \tau_1$-Int$(\tau_2$-Cl$(A))$. Therefore, by (1), we have $\tau_1\tau_2$-sCl$(A) = \tau_1$-Int$(\tau_2$-Cl$(A))$. □

Proposition 2.1.38. Let (X, τ_1, τ_2) be a bitopological space and $\{A_\gamma \mid \gamma \in \Gamma\}$ a family of subsets of X. The following properties hold:

(1) If A_γ is $\tau_1\tau_2$-β-open for each $\gamma \in \Gamma$, then $\bigcup_{\gamma\in\Gamma} A_\gamma$ is $\tau_1\tau_2$-β-open.

(2) If A_γ is $\tau_1\tau_2$-β-closed for each $\gamma \in \Gamma$, then $\bigcap_{\gamma\in\Gamma} A_\gamma$ is $\tau_1\tau_2$-β-closed.

Proof. 1. The proof follows from Theorem 3.2 of [9].

2. The proof follows from Lemma 2.1 of [22].

□

Proposition 2.1.39. [22] For a subset A of a bitopoligical space (X, τ_1, τ_2), the following properties hold:

(1) $\tau_1\tau_2$-βInt(A) is $\tau_1\tau_2$-β-open.

(2) $\tau_1\tau_2$-βCl(A) is $\tau_1\tau_2$-β-closed.

(3) A is $\tau_1\tau_2$-β-open if and only if $A = \tau_1\tau_2$-βInt(A).

(4) A is $\tau_1\tau_2$-β-closed if and only if $A = \tau_1\tau_2$-βCl(A).

Proposition 2.1.40. [22] For a subset A of a bitopological space (X, τ_1, τ_2), $x \in \tau_1\tau_2$-βCl(A) if and only if $U \cap A \neq \emptyset$ for every $\tau_1\tau_2$-β-open set U containing x.

Proposition 2.1.41. [22] For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:

(1) $X-$ $\tau_1\tau_2$-βCl(A) = $\tau_1\tau_2$-βInt$(X - A)$.

(2) $X-$ $\tau_1\tau_2$-βInt(A) = $\tau_1\tau_2$-βCl$(X - A)$.

2.2 Multifunctions

A multifunction $F : X \to Y$ is a point-to-set correspondence from X to Y.

We always assume that $F(x) \neq \emptyset$ for all $x \in X$. We shall denote the upper and lower inverse [23] of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is $F^+(B) = \{x \in X | F(x) \subseteq B\}$ and $F^-(B) = \{x \in X | F(x) \cap B \neq \emptyset\}$.

Example 2.2.1. Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c, d, e\}$. A multifunction $F : X \to Y$ is defined as follows : $F(1) = \{c\}, F(2) = \{b, d\}$ and $F(3) = \{a, e\}$. Then, $F^+(\{a, b, c, d\}) = \{1, 2\}$ and $F^-(\{a, b, c, d\}) = \{1, 2, 3\}$. Moreover, it is easy to check that $F^-(\{a\}) = \{3\}$.

Definition 2.2.2. [23] A multifunction $F : X \to Y$ and $A \subseteq X$, then *direct image* of A under the multifunction F is the set $F(A) = \cup_{x \in A} F(x)$.

Example 2.2.3. Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c, d, e\}$. A multifunction $F : X \to Y$ is defined as follows : $F(1) = \{c\}, F(2) = \{b, d\}$ and $F(3) = \{a, e\}$.
Therefore, if $A = \{1, 2\} \subseteq X$, then $F(A) = F(1) \cup F(2) = \{c\} \cup \{b, d\} = \{b, c, d\}$.

Lemma 2.2.4. Let $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a multifunction and $U \subseteq X, V \subseteq Y$. Then, the following properties hold:

(1) If $F(U) \subseteq V$, then $U \subseteq F^+(V)$.

(2) If $F(z) \cap V \neq \emptyset$ for every $z \in U$, then $U \subseteq F^-(V)$.

(3) If $U \subseteq V$ then $F^+(U) \subseteq F^+(V)$.

(4) If $U \subseteq V$ then $F^-(U) \subseteq F^-(V)$.

(5) $X - F^-(V) = F^+(Y - V)$.

Proof. (1) Let $x \in U$. Since $\cup_{x \in U} F(x) = F(U) \subseteq V$, $F(x) \subseteq F(U) \subseteq V$. Thus, $F(x) \subseteq V$; hence $x \in F^+(V)$. This shows that $U \subseteq F^+(V)$.

(2) Suppose that $F(z) \cap V \neq \emptyset$ for every $z \in U$. Let $x \in U$. Therefore, $F(x) \cap V \neq \emptyset$. Hence, $x \in F^-(V)$. This shows that $U \subseteq F^-(V)$.

(3) Let $x \in F^+(U)$. Therefore, $F(x) \subseteq U$. Since $U \subseteq V$, $F(x) \subseteq V$. Thus, $x \in F^+(V)$. This shows that $F^+(U) \subseteq F^+(V)$.

(4) Let $x \in F^-(U)$. Therefore, $F(x) \cap U \neq \emptyset$. Since $U \subseteq V$, $F(x) \cap V \neq \emptyset$. Thus, $x \in F^-(V)$. This shows that $F^-(U) \subseteq F^-(V)$.

(5) Let $x \in X - F^-(V)$. Then, $x \notin F^-(V)$. Therefore, $F(x) \cap V = \emptyset$. Thus, $F(x) \subseteq Y - V$. Hence, $x \in F^+(Y - V)$. This implies that $X - F^-(V) \subseteq F^+(Y - V)$. On the other hand, suppose that $x \in F^+(Y - V)$, so $F(x) \subseteq Y - V$. Therefore, $F(x) \cap V = \emptyset$. Then, we obtain $x \notin F^-(V)$. Thus, $x \in X - F^-(V)$; hence, $F^+(Y - V) \subseteq X - F^-(V)$. Consequently, $X - F^-(V) = F^+(Y - V)$. $\square$

CHAPTER 3

$\beta(\tau_1, \tau_2)$-CONTINUOUS MULTIFUNCTIONS

3.1 Characterizations of upper and lower $\beta(\tau_1, \tau_2)$-continuous multifunctions

In this section, we introduce the notions of upper and lower $\beta(\tau_1, \tau_2)$-continuous multifunctions, and investigate some characterizations of these multifunctions.

Definition 3.1.1. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be:

(1) *upper* $\beta(\tau_1, \tau_2)$*-continuous* at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists a $\tau_1\tau_2$-β-open set U containing x such that $F(U) \subseteq V$;

(2) *lower* $\beta(\tau_1, \tau_2)$*-continuous* at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau_1\tau_2$-β-open set U containing x such that $F(z) \cap V \neq \emptyset$ for every $z \in U$;

(3) *upper* (resp. *lower*) $\beta(\tau_1, \tau_2)$*-continuous* if F has this property at each point of X.

Example 3.1.2. Let $X = \{a, b, c\}$ with topologies $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{a, b\}, X\}$. Let $Y = \{1, 2, 3, 4, 5\}$ with topologies
$\sigma_1 = \{\emptyset, \{1\}, \{2, 3, 4, 5\}, Y\}$ and $\sigma_2 = \{\emptyset, \{2\}, \{3\}, \{2, 3\}, \{2, 3, 4, 5\}, Y\}$.
A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is defined as follow: $F(a) = \{2, 3\}$, $F(b) = \{1, 2\}$, $F(c) = \{1, 4, 5\}$. Then, F is upper and lower $\beta(\tau_1, \tau_2)$-continuous.

Theorem 3.1.3. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is upper $\beta(\tau_1, \tau_2)$-continuous at $x \in X$ if and only if $x \in \tau_1\tau_2$-$\beta\text{Int}(F^+(V))$ for every $\sigma_1\sigma_2$-open set V of Y containing $F(x)$.

Proof. Let V be a $\sigma_1\sigma_2$-open set containing $F(x)$. Consequently, there exists a $\tau_1\tau_2$-β-open set U containing x such that $F(U) \subseteq V$. Therefore, $x \in U \subseteq F^+(V)$. Since U is $\tau_1\tau_2$-β-open, we have $x \in \tau_1\tau_2$-$\beta\text{Int}(F^+(V))$.

Conversely, let V be a $\sigma_1\sigma_2$-open set containing $F(x)$. By the hypothesis, $x \in \tau_1\tau_2$-βInt$(F^+(V))$. There exists a $\tau_1\tau_2$-β-open set G containing x such that $G \subseteq F^+(V)$; hence, $F(G) \subseteq V$. This shows that F is upper $\beta(\tau_1, \tau_2)$-continuous at x. □

Theorem 3.1.4. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower $\beta(\tau_1, \tau_2)$-continuous at $x \in X$ if and only if $x \in \tau_1\tau_2$-βInt$(F^-(V))$ for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$.

Proof. The proof is similar to that of Theorem 3.1.3. □

Theorem 3.1.5. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) F is upper $\beta(\tau_1, \tau_2)$-continuous;

(2) $F^+(V)$ is $\tau_1\tau_2$-β-open in X for every $\sigma_1\sigma_2$-open set V of Y;

(3) $F^-(K)$ is $\tau_1\tau_2$-β-closed in X for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $\tau_1\tau_2$-$\beta Cl(F^-(B)) \subseteq F^-(\sigma_1\sigma_2$-Cl$(B))$ for every subset B of Y;

(5) τ_1-Int$(\tau_2$-Cl$(\tau_1$-Int$(F^-(B)))) \subseteq F^-(\sigma_1\sigma_2$-Cl$(B))$ for every subset B of Y.

Proof. (2) ⇒ (1): Let $x \in X$ and V be a $\sigma_1\sigma_2$-open set containing $F(x)$. By (2), $F^+(V)$ is a $\tau_1\tau_2$-β-open set containing x. Putting $U = F^+(V)$, we obtain U is a $\tau_1\tau_2$-β-open set containing x such that $F(U) \subseteq V$. This shows that F is upper $\beta(\tau_1, \tau_2)$-continuous.

(1) ⇒ (2): Let V be a $\sigma_1\sigma_2$-open set of Y and $x \in F^+(V)$. By 2.2.4, $F(x) \subseteq V$, then there exists a $\tau_1\tau_2$-β-open set U containing x such that $F(U) \subseteq V$. Consequently, we obtain $x \in U \subseteq \tau_1$-Cl$(\tau_2$-Int$(\tau_1$-Cl$(U))) \subseteq \tau_1$-Cl$(\tau_2$-Int$(\tau_1$-Cl$(F^+(V))))$. Thus, $F^+(V) \subseteq \tau_1$-Cl$(\tau_2$-Int$(\tau_1$-Cl$(F^+(V))))$. This shows $F^+(V)$ is $\tau_1\tau_2$-β-open in X.

(2) ⇒ (3): This follows from the fact that $F^+(Y - B) = X - F^-(B)$ for every subset B of Y.

(3) ⇒ (4): For each subset B of Y, $\sigma_1\sigma_2$-Cl(B) is $\sigma_1\sigma_2$-closed in Y. By (3), $F^-(\sigma_1\sigma_2$-Cl$(B))$ is $\tau_1\tau_2$-β-closed in X; therefore,$\tau_1\tau_2$-βCl$(F^-(B)) \subseteq F^-(\sigma_1\sigma_2$-Cl$(B))$.

(4) ⇒ (5): Let B be a subset of Y. By Proposition 2.1.39(2), we obtain

$$\tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\text{-Int}(F^-(B)))) \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\text{-Int}(\tau_1\tau_2\text{-}\beta Cl(F^-(B)))))$$

$$\subseteq \tau_1\tau_2\text{-}\beta Cl(F^-(B)).$$

Consequently, τ_1-Int(τ_2-Cl(τ_1-Int($F^-(B)$))) $\subseteq F^-(\sigma_1\sigma_2$-Cl$(B))$ by (4).

(5) $\Rightarrow$ (2): Let V be a $\sigma_1\sigma_2$-open set of Y, so $Y - V$ is $\sigma_1\sigma_2$-closed in Y. By (5) and Lemma 2.2.4,

$$\begin{aligned} X - \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\text{-Cl}(F^+(V)))) &= \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\text{-Int}(X - F^+(V)))) \\ &= \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\text{-Int}(F^-(Y - V)))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(Y - V)) \\ &= X - F^+(\sigma_1\sigma_2\text{-Int}(V)) \\ &\subseteq X - F^+(V). \end{aligned}$$

Therefore, we obtain $F^+(V) \subseteq \tau_1$-Cl(τ_2-Int(τ_1-Cl($F^+(V)$))), and hence $F^+(V)$ is $\tau_1\tau_2$-β-open in X. □

Theorem 3.1.6. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) F is lower $\beta(\tau_1, \tau_2)$-continuous;

(2) $F^-(V)$ is $\tau_1\tau_2$-β-open in X for every $\sigma_1\sigma_2$-open set V of Y;

(3) $F^+(K)$ is $\tau_1\tau_2$-β-closed in X for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $\tau_1\tau_2$-βCl$(F^+(B)) \subseteq F^+(\sigma_1\sigma_2$-Cl$(B))$ for every subset B of Y;

(5) τ_1-Int(τ_2-Cl(τ_1-Int($F^+(B)$))) $\subseteq F^+(\sigma_1\sigma_2$-Cl$(B))$ for every subset B of Y.

Proof. It is shown similarly to the proof of Theorem 3.1.5 that the statements (1), (2), (3), (4) and (5) are equivalent. □

Definition 3.1.7. [24] Let (X, τ_1, τ_2) be bitopological space. A covering $\mathcal{B}$ is called a refinement of covering $\mathcal{U}$ if every $\tau_1\tau_2$-open set of $\mathcal{B}$ is contained in some $\tau_1\tau_2$-open set of $\mathcal{U}$.

Definition 3.1.8. [21] A collection $\mathfrak{U}$ of subsets of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-$locally finite$ if every $x \in X$ has a $\tau_1\tau_2$-neighborhood which intersects only finitely many elements of $\mathfrak{U}$.

Definition 3.1.9. [21] A subset A of a bitopological space (X, τ_1, τ_2) is said to be:

(1) $\tau_1\tau_2$-*paracompact* if every cover of A by $\tau_1\tau_2$-open sets of X is refined by a cover of A which consists of $\tau_1\tau_2$-open sets of X and is $\tau_1\tau_2$-locally finite in X;

(2) $\tau_1\tau_2$-*regular* if for each $x \in A$ and each $\tau_1\tau_2$-open set U of X containing x, there exists a $\tau_1\tau_2$-open set V of X such that $x \in V \subseteq \tau_1\tau_2\text{-Cl}(V) \subseteq U$.

Lemma 3.1.10. [21] If A is a $\tau_1\tau_2$-regular $\tau_1\tau_2$-paracompact set of a bitopological space (X, τ_1, τ_2) and U is a $\tau_1\tau_2$-open neighbourhood of A, then there exists a $\tau_1\tau_2$-open set V of X such that $A \subseteq V \subseteq \tau_1\tau_2\text{-Cl}(V) \subseteq U$.

Definition 3.1.11. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called *punctually* (τ_1, τ_2)*-paracompact* (resp. *punctually* (τ_1, τ_2)*-regular*) if for each $x \in X$, $F(x)$ is $\tau_1\tau_2$-paracompact (resp. $\tau_1\tau_2$-regular).

For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, by

$$\beta\text{Cl}F_{\circledast} : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2),$$

we denote the multifunction defined as follows: $\beta\text{Cl}F_{\circledast}(x) = \sigma_1\sigma_2\text{-}\beta\text{Cl}(F(x))$ for each $x \in X$.

Example 3.1.12. Let $X = \{1, 2, 3\}$ with topologies

$$\tau_1 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X\} \text{ and } \tau_2 = \{\emptyset, \{1\}, \{1, 2\}, X\}.$$

Let $Y = \{a, b, c\}$ with topologies $\sigma_1 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, Y\}$, $\sigma_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is defined as follow: $F(1) = \{a, b\}$, $F(2) = \{a\}$, $F(3) = \{b\}$. Then, F is punctually (τ_1, τ_2)-paracompact.

Example 3.1.13. Let $X = \{1, 2, 3\}$ with topologies

$$\tau_1 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X\}$$

and $\tau_2 = \{\emptyset, \{1\}, \{1, 2\}, X\}$. Let $Y = \{a, b, c\}$ with topologies $\sigma_1 = \{\emptyset, \{b, c\}, Y\}$ and $\sigma_2 = \{\emptyset, \{a, c\}, Y\}$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is defined as follow: $F(1) = F(2) = F(3) = \{a, b\}$. Then, F is punctually (τ_1, τ_2)-regular.

Lemma 3.1.14. Let (X, τ_1, τ_2) be a bitopological space. Then, $\tau_1\tau_2$-βCl$(A) \subseteq \tau_1\tau_2$-Cl$(A)$ for every subset A of X.

Proof. Let $x \in X - \tau_1\tau_2$-Cl(A). By Proposition 2.1.31, $x \in \tau_1\tau_2$-Int$(X - A)$ and there exists a $\tau_1\tau_2$-open set V such that $x \in V \subseteq X - A$. Since every $\tau_1\tau_2$-open set is $\tau_1\tau_2$-β-open, we have $x \in \tau_1\tau_2$-$\beta Int(X - A)$. By Proposition 2.1.41(1), $x \in X - \tau_1\tau_2$-βCl(A), so $X - \tau_1\tau_2$-Cl$(A) \subseteq X-$ $\tau_1\tau_2$-βCl(A). Consequently, we obtain $\tau_1\tau_2$-βCl$(A) \subseteq \tau_1\tau_2$-Cl$(A)$. □

Lemma 3.1.15. If $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is punctually (τ_1, τ_2)-paracompact and punctually (τ_1, τ_2)-regular, then $\beta ClF^+_{\circledast}(V) = F^+(V)$ for every $\sigma_1\sigma_2$-open V of Y.

Proof. Let V be a $\sigma_1\sigma_2$-open set of Y and $x \in \beta$Cl$F^+_{\circledast}(V)$. Then, we have $\sigma_1\sigma_2$-βCl$(F(x))) \subseteq V$ and $F(x) \subseteq V$. Therefore, we have $x \in F^+(V)$, and hence βCl$F^+_{\circledast}(V) \subseteq F^+(V)$. On the other hand, let $x \in F^+(V)$. Then, $F(x) \subseteq V$ and by Lemma 3.1.10, there exists a $\sigma_1\sigma_2$-open set U of Y such that $F(x) \subseteq \sigma_1\sigma_2$-Cl$(U) \subseteq U \subseteq V$. By Lemma 3.1.14, we have $\sigma_1\sigma_2$-βCl$(F(x)) \subseteq \sigma_1\sigma_2$-Cl$(U) \subseteq V$. This shows that $x \in \beta$Cl$F^+_{\circledast}(V)$, and hence $F^+(V) \subseteq \beta$Cl$F^+_{\circledast}(V)$. Consequently, we obtain βCl$F^+_{\circledast}(V) = F^+(V)$. □

Theorem 3.1.16. Let $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be punctually (τ_1, τ_2)-paracompact and punctually (τ_1, τ_2)-regular. Then, F is upper $\beta(\tau_1, \tau_2)$-continuous if and only if βCl$F_{\circledast} : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is upper $\beta(\tau_1, \tau_2)$-continuous.

Proof. Suppose that F is upper $\beta(\tau_1, \tau_2)$-continuous. Let $x \in X$ and V be a $\sigma_1\sigma_2$-open set of Y such that βCl$F_{\circledast}(x) \subseteq V$. By Lemma 3.1.15, we have $x \in \beta$Cl$F^+_{\circledast}(V) = F^+(V)$. Since F is upper $\beta(\tau_1, \tau_2)$-continuous, there exists a $\tau_1\tau_2$-β-open set U containing x such that $F(U) \subseteq V$. Since $F(z)$ is $\sigma_1\sigma_2$-paracompact and $\sigma_1\sigma_2$-regular for each $z \in U$, by Lemma 3.1.10 there exists a $\sigma_1\sigma_2$-open set H such that $F(z) \subseteq H \subseteq \sigma_1\sigma_2$-Cl$(H) \subseteq V$. By Lemma 3.1.14, we have $\sigma_1\sigma_2$-βCl$(F(z)) \subseteq \sigma_1\sigma_2$-Cl$(H) \subseteq V$ for each $z \in U$, and hence βCl$F_{\circledast}(U) \subseteq V$. This shows that βCl$F_{\circledast}$ is upper $\beta(\tau_1, \tau_2)$-continuous.

Conversely, we suppose that βCl$F_{\circledast} : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is upper $\beta(\tau_1, \tau_2)$-continuous. Let $x \in X$ and V be a $\sigma_1\sigma_2$-open set of Y such that $F(x) \subseteq V$. By

Lemma 3.1.15, we have $x \in F^{+}(V) = \beta\text{Cl}F^{+}_{\circledast}(V)$, and hence $\beta\text{Cl}F_{\circledast}(x) \subseteq V$. Since $\beta ClF_{\circledast}$ is upper $\beta(\tau_1, \tau_2)$-continuous, there exists a $\tau_1\tau_2$-β-open set U of containing x such that $\beta\text{Cl}F_{\circledast}(U) \subseteq V$; hence, $F(U) \subseteq V$. This shows that F is upper $\beta(\tau_1, \tau_2)$-continuous. □

Lemma 3.1.17. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, it follows that for each $\sigma_1\sigma_2$-β-open set V of Y $\beta\text{Cl}F^{-}_{\circledast}(V) = F^{-}(V)$.

Proof. Suppose that V is a $\sigma_1\sigma_2$-β-open set Y. Let $x \in \beta\text{Cl}F^{-}_{\circledast}(V)$. Then, $\sigma_1\sigma_2$-$\beta\text{Cl}(F(x)) \cap V \neq \emptyset$. Hence, $F(x) \cap V \neq \emptyset$. Therefore, we obtain $x \in F^{-}(V)$. This shows that $\beta\text{Cl}F^{-}_{\circledast}(V) \subseteq F^{-}(V)$. On the other hand, let $x \in F^{-}(V)$. Then, we have $\emptyset \neq F(x) \cap V \subseteq \sigma_1\sigma_2$-$\beta\text{Cl}(F(x)) \cap V$. Thus, $x \in \beta\text{Cl}F^{-}_{\circledast}(V)$. This shows that $F^{-}(V) \subseteq \beta\text{Cl}F^{-}_{\circledast}(V)$. Consequently, we obtain $\beta\text{Cl}F^{-}_{\circledast}(V) = F^{-}(V)$. □

Theorem 3.1.18. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower $\beta(\tau_1, \tau_2)$-continuous if and only if $\beta ClF_{\circledast} : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower $\beta(\tau_1, \tau_2)$-continuous.

Proof. By utilizing Lemma 3.1.17, this can be proved similarly to that of Theorem 3.1.16. □

For a multifunction $F : X \to Y$, the graph multifunction $G_F : X \to X \times Y$ is defined as follows: $G_F(x) = \{x\} \times F(x)$ for every $x \in X$.

Lemma 3.1.19. [10] The following hold for a multifunction $F : X \to Y$:

(i) $G^{+}_{F}(A \times B) = A \cap F^{+}(B)$,

(ii) $G^{-}_{F}(A \times B) = A \cap F^{-}(B)$,

for any subsets $A \subseteq X$ and $B \subseteq Y$.

Lemma 3.1.20. Let (X, τ_1, τ_2) be a bitopological space. If A is $\tau_1\tau_2$-β-open and B is $\tau_1\tau_2$-open in X, then $A \cap B$ is $\tau_1\tau_2$-β-open.

Proof. Suppose that A is $\tau_1\tau_2$-β-open and B is $\tau_1\tau_2$-open in X. Then, we have $A \subseteq \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\text{-Cl}(A)))$ and $B = \tau_1\text{-Int}(B) = \tau_2\text{-Int}(B)$. By Proposition 2.1.6(1),

$$A \cap B \subseteq \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\text{-Cl}(A))) \cap B$$

$$\subseteq \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\text{-Cl}(A)) \cap B)$$
$$= \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\text{-Cl}(A) \cap B))$$
$$\subseteq \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\text{-Cl}(A \cap B))).$$

Consequently, we obtain $A \cap B$ is $\tau_1\tau_2$-β-open. □

Definition 3.1.21. [21] A bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-compact if every cover of X by $\tau_1\tau_2$-open sets of X has a finite subcover.

By ρ_i we denote the product topology $\tau_i \times \sigma_i$ for $i = 1, 2$.

Theorem 3.1.22. Let $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a multifunction such that $F(x)$ is $\sigma_1\sigma_2$-compact for each $x \in X$. Then, F is upper $\beta(\tau_1, \tau_2)$-continuous if and only if $G_F : (X, \tau_1, \tau_2) \to (X \times Y, \rho_1, \rho_2)$ is upper $\beta(\tau_1, \tau_2)$-continuous.

Proof. Suppose that $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is upper $\beta(\tau_1, \tau_2)$-continuous. Let $x \in X$ and W be a $\rho_1\rho_2$-open set of $X \times Y$ containing $G_F(x)$. For each $y \in F(x)$, there exist $\tau_1\tau_2$-open set $U(y)$ of X and $\sigma_1\sigma_2$-open set $V(y)$ of Y such that $(x, y) \in U(y) \times V(y) \subseteq W$. The family $\{V(y) \mid y \in F(x)\}$ is $\sigma_1\sigma_2$-open cover of $F(x)$ and there exists a finite number of points, say, $y_1, y_2, ..., y_n$ in $F(x)$ such that $F(x) \subseteq \cup\{V(y_i) \mid 1 \leq i \leq n\}$. Put

$$U = \cap\{U(y_i) \mid 1 \leq i \leq n\} \text{ and } V = \cup\{V(y_i) \mid 1 \leq i \leq n\}.$$

Then, we have U is $\tau_1\tau_2$-open in X and V is $\sigma_1\sigma_2$-open in Y such that $\{x\} \times F(x) \subseteq U \times V \subseteq W$. Since F is upper $\beta(\tau_1, \tau_2)$-continuous, there exists a $\tau_1\tau_2$-β-open set G containing x such that $F(G) \subseteq V$. By Lemma 3.1.19 and 2.2.4(3), we have $U \cap G \subseteq U \cap F^+(V) = G_F^+(U \times V) \subseteq G_F^+(W)$. By Lemma 3.1.20, $U \cap G$ is $\tau_1\tau_2$-β-open in X, and $G_F(U \cap G) \subseteq W$ by Lemma 2.2.4(1). This shows that G_F is upper $\beta(\tau_1, \tau_2)$-continuous.

Conversely, we suppose that $G_F : (X, \tau_1, \tau_2) \to (X \times Y, \rho_1, \rho_2)$ is upper $\beta(\tau_1, \tau_2)$-continuous. Let $x \in X$ and V be a $\sigma_1\sigma_2$-open containing $F(x)$. Since $X \times V$ is $\rho_1\rho_2$-open in $X \times Y$ and $G_F(x) \subseteq X \times V$, there exists a $\tau_1\tau_2$-β-open set U containing x such that $G_F(U) \subseteq X \times V$. Therefore, by Lemma 2.2.4(1) and 3.1.19, $U \subseteq G_F^+(X \times V) = F^+(V)$, and $F(U) \subseteq V$. This shows that F is upper $\beta(\tau_1, \tau_2)$-continuous. □

Theorem 3.1.23. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower $\beta(\tau_1, \tau_2)$-continuous if and only if $G_F : (X, \tau_1, \tau_2) \to (X \times Y, \rho_1, \rho_2)$ is lower $\beta(\tau_1, \tau_2)$-continuous.

Proof. Suppose that $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower $\beta(\tau_1, \tau_2)$-continuous. Let $x \in X$ and W be a $\rho_1\rho_2$-open set of $X \times Y$ such that $G_F(x) \cap W \neq \emptyset$. There exists $y \in F(x)$ such that $(x, y) \in W$, and hence $(x, y) \in U \times V \subseteq W$ for some $\tau_1\tau_2$-open set U of X and $\sigma_1\sigma_2$-open set V of Y. Since $F(x) \cap V \neq \emptyset$, there exists a $\tau_1\tau_2$-β-open set G containing x such that $F(z) \cap V \neq \emptyset$ for each $z \in G$; hence $G \subseteq F^-(V)$. By Lemma 3.1.19 and Lemma 3.1.20, we have $U \cap G \subseteq U \cap F^-(V) = G_F^-(U \times V) \subseteq G_F^-(W)$. Moreover, $U \cap G$ is a $\tau_1\tau_2$-β-open set containing x, and hence G_F is lower $\beta(\tau_1, \tau_2)$-continuous.

Conversely, we suppose that $G_F : (X, \tau_1, \tau_2) \to (X \times Y, \rho_1, \rho_2)$ is lower $\beta(\tau_1, \tau_2)$-continuous. Let $x \in X$ and V be a $\sigma_1\sigma_2$-open set of Y such that $F(x) \cap V \neq \emptyset$. Then, we have $X \times V$ is $\rho_1\rho_2$-open in $X \times Y$ and

$$G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset.$$

There exists a $\tau_1\tau_2$-β-open set U containing x such that $G_F(z) \cap (X \times V) \neq \emptyset$ for each $z \in U$. By Lemma 3.1.19, we obtain $U \subseteq G_F^-(X \times V) = F^-(V)$.
This shows that F is lower $\beta(\tau_1, \tau_2)$-continuous. □

3.2 Characterizations of upper and lower almost $\beta(\tau_1, \tau_2)$-continuous multifunctions

In this section, we introduce the concepts of upper and lower almost $\beta(\tau_1, \tau_2)$-continuous multifunctions. Moreover, several interesting characterizations of these multifunctions are discussed.

Definition 3.2.1. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be:

(1) *upper almost $\beta(\tau_1, \tau_2)$-continuous* at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists a $\tau_1\tau_2$-β-open set U containing x such that $F(U) \subseteq \sigma_1$-Int(σ_2-Cl(V));

(2) *lower almost $\beta(\tau_1, \tau_2)$-continuous* at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V

of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau_1\tau_2$-β-open set U containing x such that $F(z) \cap \sigma_1$-Int(σ_2-Cl$(V)) \neq \emptyset$ for every $z \in U$;

(3) *upper almost* (resp. *lower almost*) $\beta(\tau_1, \tau_2)$*-continuous* if F has this property at each point of X.

Remark. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following implication holds:

upper $\beta(\tau_1, \tau_2)$-continuity $\Rightarrow$ upper almost $\beta(\tau_1, \tau_2)$-continuity.

The converse of the implication is not true in general. We give an example for the implication as follows.

Example 3.2.2. Let $X = \{a, b, c\}$ with topologies $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{a, b\}, X\}$. Let $Y = \{1, 2, 3, 4, 5\}$ with topologies
$\sigma_1 = \{\emptyset, \{1\}, \{2, 3, 4, 5\}, Y\}$ and $\sigma_2 = \{\emptyset, \{2\}, \{3\}, \{2, 3\}, \{2, 3, 4, 5\}, Y\}$.
A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is defined as follow:
$F(a) = \{1\}$, $F(b) = \{2, 3\}$, $F(c) = \{4, 5\}$. Then, F is upper almost $\beta(\tau_1, \tau_2)$-continuous, but F is not upper $\beta(\tau_1, \tau_2)$-continuous.

Remark. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following implication holds:

lower $\beta(\tau_1, \tau_2)$-continuity $\Rightarrow$ lower almost $\beta(\tau_1, \tau_2)$-continuity.

The converse of the implication is not true in general. We give an example for the implication as follows.

Example 3.2.3. Let $X = \{a, b\}$ with topologies $\tau_1 = \{\emptyset, \{b\}, X\}$ and
$\tau_2 = \{\emptyset, X\}$. Let $Y = \{1, 2, 3, 4\}$ with topologies $\sigma_1 = \{\emptyset, \{1, 2\}, Y\}$ and $\sigma_2 = \{\emptyset, \{1\}, \{1, 2\}, Y\}$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is defined as follow: $F(a) = \{1, 2\}$, $F(b) = \{3\}$. Then, F is lower almost $\beta(\tau_1, \tau_2)$-continuous, but F is not lower $\beta(\tau_1, \tau_2)$-continuous.

Theorem 3.2.4. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is upper almost $\beta(\tau_1, \tau_2)$-continuous at $x \in X$ if and only if $x \in \tau_1\tau_2$-βInt$(F^+(\sigma_1\sigma_2$-sCl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y containing $F(x)$.

Proof. Let V be a $\sigma_1\sigma_2$-open set containing $F(x)$. Then, there exists a $\tau_1\tau_2$-β-open set U containing x such that $F(U) \subseteq \sigma_1\sigma_2$-sCl$(V)$. Then, $x \in U \subseteq F^+(\sigma_1\sigma_2$-sCl$(V))$. Therefore, $x \in \tau_1\tau_2$-βInt$(F^+(\sigma_1\sigma_2$-sCl$(V)))$.

Conversely, let V be a $\sigma_1\sigma_2$-open set containing $F(x)$. Moreover, we have $x \in \tau_1\tau_2$-βInt$(F^+(\sigma_1\sigma_2$-sCl$(V)))$. There exists a $\tau_1\tau_2$-β-open set G containing x such that $G \subseteq F^+(\sigma_1\sigma_2$-sCl$(V))$, and hence $F(G) \subseteq \sigma_1\sigma_2$-sCl$(V) = \sigma_1$-Int$(\sigma_2$-Cl$(V))$. This shows that F is upper almost $\beta(\tau_1, \tau_2)$-continuous at x. □

Theorem 3.2.5. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower almost $\beta(\tau_1, \tau_2)$-continuous at $x \in X$ if and only if $x \in \tau_1\tau_2$-βInt$(F^-(\sigma_1\sigma_2$-sCl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$.

Proof. The proof is similar to that of Theorem 3.2.4. □

Theorem 3.2.6. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) F is upper almost $\beta(\tau_1, \tau_2)$-continuous;

(2) for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists a $\tau_1\tau_2$-β-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2$-sCl$(V)$;

(3) $F^+(V) \subseteq \tau_1\tau_2$-$\beta$Int$(F^+(\sigma_1\sigma_2$-sCl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(4) $\tau_1\tau_2$-βCl$(F^-(\sigma_1\sigma_2$-sInt$(K))) \subseteq F^-(K)$ for every $\sigma_1\sigma_2$-closed set K of Y.

Proof. (1) $\Rightarrow$ (2): The proof follows from Definition 3.2.1(1).

(2) $\Rightarrow$ (3): Let V be a $\sigma_1\sigma_2$-open set of Y and $x \in F^+(V)$. Then, $F(x) \subseteq V$ and there exists a $\tau_1\tau_2$-β-open set U containing x such that $F(U) \subseteq \sigma_1$-Int$(\sigma_2$-Cl$(V)) = \sigma_1\sigma_2$-sCl(V). Therefore, we have $x \in U \subseteq F^+(\sigma_1\sigma_2$-sCl$(V))$. Thus,

$$x \in \tau_1\tau_2\text{-}\beta\text{Int}(F^+(\sigma_1\sigma_2\text{-sCl}(V))).$$

Consequently, we obtain $F^+(V) \subseteq \tau_1\tau_2\text{-}\beta\text{Int}(F^+(\sigma_1\sigma_2\text{-sCl}(V)))$.

(3) $\Rightarrow$ (4): Let K be a $\sigma_1\sigma_2$-closed set of Y. Since $Y - K$ is $\sigma_1\sigma_2$-open, by Lemma 2.2.4(5) we obtain

$$\begin{aligned} X - F^-(K) = F^+(Y - K) & \\ &\subseteq \tau_1\tau_2\text{-}\beta\text{Int}(F^+(\sigma_1\sigma_2\text{-sCl}(Y - K))) \\ &= \tau_1\tau_2\text{-}\beta\text{Int}(F^+(Y - \sigma_1\sigma_2\text{-sInt}(K))) \\ &= \tau_1\tau_2\text{-}\beta\text{Int}(X - F^-(\sigma_1\sigma_2\text{-sInt}(K))) \\ &= X - \tau_1\tau_2\text{-}\beta\text{Cl}(F^-(\sigma_1\sigma_2\text{-sInt}(K))). \end{aligned}$$

Therefore, we obtain $\tau_1\tau_2\text{-}\beta\text{Cl}(F^-(\sigma_1\sigma_2\text{-sInt}(K))) \subseteq F^-(K)$.

(4) $\Rightarrow$ (3): The proof is obvious.

(3) $\Rightarrow$ (1): Let $x \in X$ and V be a $\sigma_1\sigma_2$-open set containing $F(x)$. Then, we have $x \in \tau_1\tau_2\text{-}\beta\text{Int}(F^+(\sigma_1\sigma_2\text{-sCl}(V)))$. Therefore, by Theorem 3.2.4, F is upper almost $\beta(\tau_1, \tau_2)$-continuous at x. □

Theorem 3.2.7. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) F is lower almost $\beta(\tau_1, \tau_2)$-continuous;

(2) for each $x \in X$ and each $\sigma_1\sigma_2$-open set V containing $F(x)$, there exists a $\tau_1\tau_2$-β-open set U containing x such that $F(z) \cap \sigma_1\sigma_2\text{-sCl}(V)$ for each $z \in U$;

(3) $F^-(V) \subseteq \tau_1\tau_2\text{-}\beta\text{Int}(F^-(\sigma_1\sigma_2\text{-sCl}(V)))$ for every $\sigma_1\sigma_2$-open set V;

(4) $\tau_1\tau_2\text{-}\beta\text{Cl}(F^+(\sigma_1\sigma_2\text{-sInt}(K))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$-closed set K.

Proof. By utilizing Theorem 3.2.5, this can be similar to Theorem 3.2.6. □

Theorem 3.2.8. If a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower almost $\beta(\tau_1, \tau_2)$-continuous, then $\beta ClF_{\circledast} : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower almost $\beta(\tau_1, \tau_2)$-continuous.

Proof. Suppose that F is lower almost $\beta(\tau_1, \tau_2)$-continuous. Let $x \in X$ and V be a $\sigma_1\sigma_2$-open set of Y such that $\beta\text{Cl}F_{\circledast}(x) \cap V \neq \emptyset$. By Lemma 3.1.17, we have

$x \in \beta\text{Cl}F^-_{\circledast}(V) = F^-(V)$. Since F is lower almost $\beta(\tau_1, \tau_2)$-continuous, there exists a $\tau_1\tau_2$-β-open set U containing x such that

$$F(z) \cap \sigma_1\text{-Int}(\sigma_2\text{-Cl}(V)) \neq \emptyset \text{ for each } z \in U.$$

Therefore, $\sigma_1\sigma_2$-$\beta Cl(F(z)) \cap \sigma_1$-Int$(\sigma_2$-Cl$(V) \neq \emptyset$ for each $z \in U$, and hence

$$\beta\text{Cl}F_{\circledast}(z) \cap \sigma_1\text{-Int}(\sigma_2\text{-Cl}(V) \neq \emptyset \text{ for each } z \in U.$$

This shows that $\beta\text{Cl}F_{\circledast}$ is lower almost $\beta(\tau_1, \tau_2)$-continuous. □

Definition 3.2.9. Let (X, τ_1, τ_2) be a bitopological space. The β-frontier of a subset A of X, denoted by $\tau_1\tau_2$-$\beta Fr(A)$, is defined by

$$\begin{aligned} \tau_1\tau_2\text{-}\beta\text{Fr}(A) &= \tau_1\tau_2\text{-}\beta\text{Cl}(A) \cap \tau_1\tau_2\text{-}\beta\text{Cl}(X - A) \\ &= \tau_1\tau_2\text{-}\beta\text{Cl}(A) - \tau_1\tau_2\text{-}\beta\text{Int}(A). \end{aligned}$$

Theorem 3.2.10. The set of all points x of X at which a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is not upper $\beta(\tau_1, \tau_2)$-continuous is identical with the union of the $\tau_1\tau_2$-β-frontier of the upper inverse images of $\sigma_1\sigma_2$-open sets containing $F(x)$.

Proof. Let $x \in X$ at which F is not upper $\beta(\tau_1, \tau_2)$-continuous. There exists a $\sigma_1\sigma_2$-open set V containing $F(x)$ such that $U \cap (X - F^+(V)) \neq \emptyset$ for every $\tau_1\tau_2$-β-open set U containing x. Then, we have

$$x \in \tau_1\tau_2\text{-}\beta\text{Cl}(X - F^+(V)) = X - \tau_1\tau_2\text{-}\beta\text{Int}(F^+(V))$$

and $x \in F^+(V)$. Hence, we obtain $x \in \tau_1\tau_2$-βFr$(F^+(V))$.

Conversely, we suppose that V is a $\sigma_1\sigma_2$-open set containing $F(x)$ such that $x \in \tau_1\tau_2$-βFr$(F^+(V))$. If F is upper $\beta(\tau_1, \tau_2)$-continuous at x, there exists a $\tau_1\tau_2$-β-open set U containing x such that $U \subseteq F^+(V)$. This implies that $x \in \tau_1\tau_2$-βInt$(F^+(V))$. This is a contradiction; hence, F is not upper $\beta(\tau_1, \tau_2)$-continuous. □

Theorem 3.2.11. The set of all points x of X at which a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is not lower $\beta(\tau_1, \tau_2)$-continuous is identical with the union of the $\tau_1\tau_2$-β-frontier of the lower inverse images of $\sigma_1\sigma_2$-open sets meeting $F(x)$.

Proof. The proof is similar to that of Theorem 3.2.10. □

CHAPTER 4

WEAKLY $\beta(\tau_1, \tau_2)$-CONTINUOUS MULTIFUNCTIONS

4.1 Weakly $\beta(\tau_1, \tau_2)$-continuous multifunctions

In this section, we introduce and investigate the notions of upper and lower weakly $\beta(\tau_1, \tau_2)$-continuous multifunctions. Furthermore, the relationships weak $\beta(\tau_1, \tau_2)$-continuity and the other types of continuity are investigated.

Definition 4.1.1. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be:

(1) *upper weakly* $\beta(\tau_1, \tau_2)$*-continuous* at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists a $\tau_1\tau_2$-β-open set U containing x such that $F(U) \subseteq \sigma_1\sigma_2$-Cl$(V)$;

(2) *lower weakly* $\beta(\tau_1, \tau_2)$*-continuous* at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau_1\tau_2$-β-open set U containing x such that $F(z) \cap \sigma_1\sigma_2$-Cl$(V) \neq \emptyset$ for every $z \in U$;

(3) *upper weakly* (resp. *lower weakly*) $\beta(\tau_1, \tau_2)$*-continuous* if F has this property at each point of X.

Theorem 4.1.2. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is upper weakly $\beta(\tau_1, \tau_2)$-continuous at $x \in X$ if and only if $x \in \tau_1\tau_2$-βInt$(F^+(\sigma_1\sigma_2\text{Cl}(V)))$ for every $\sigma_1\sigma_2$-open set V of Y containing $F(x)$.

Proof. Let V be a $\sigma_1\sigma_2$-open set containing $F(x)$. Therefore, there exists a $\tau_1\tau_2$-β-open set U containing x such that $F(U) \subseteq \sigma_1\sigma_2$-Cl$(V)$. Therefore, $x \in U \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$. Since U is $\tau_1\tau_2$-β-open, we have $x \in \tau_1\tau_2$-βInt$(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$. Conversely, let V be a $\sigma_1\sigma_2$-open set of Y containing $F(x)$, and we have $x \in \tau_1\tau_2$-βInt$(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$. There exists a $\tau_1\tau_2$-β-open set G of X containing x such that $G \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$; hence, $F(G) \subseteq \sigma_1\sigma_2$-Cl$(V)$. This shows that F is upper weakly $\beta(\tau_1, \tau_2)$-continuous at x. □

Theorem 4.1.3. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower weakly $\beta(\tau_1, \tau_2)$-continuous at $x \in X$ if and only if $x \in \tau_1\tau_2$-βInt$(F^-(\sigma_1\sigma_2\text{Cl}(V)))$ for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$.

Proof. The proof is similar to that of Theorem 4.1.2. □

Remark. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following implication hold:

upper (τ_1, τ_2)-β-continuity

$\Downarrow$

upper almost (τ_1, τ_2)-β-continuity

$\Downarrow$

upper weak (τ_1, τ_2)-β-continuity.

The converse of the implications are not true in general. We give an example for the implication as follows.

Example 4.1.4. Let $X = \{1, 2, 3\}$ with topologies $\tau_1 = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, X\}$ and $\tau_2 = \{\emptyset, \{2, 3\}, X\}$. Let $Y = \{a, b, c\}$ with topologies
$\sigma_1 = \{\emptyset, \{a\}, \{c\}, \{a, c\}, Y\}$ and $\sigma_2 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, Y\}$.
A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is defined as follow: $F(1) = \{b\}$, $F(2) = \{c\}$, $F(3) = \{a, b\}$. Then, F is upper(lower) weakly $\beta(\tau_1, \tau_2)$-continuous, but F is not upper almost $\beta(\tau_1, \tau_2)$-continuous and upper $\beta(\tau_1, \tau_2)$-continuous.

The following theorems give some characterizations of upper and lower weakly $\beta(\tau_1, \tau_2)$-continuous multifunctions.

Theorem 4.1.5. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) F is upper weakly $\beta(\tau_1, \tau_2)$-continuous;

(2) $F^+(V) \subseteq \tau_1\tau_2$-$\beta$Int$(F^+(\sigma_1\sigma_2\text{Cl}(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(3) $\tau_1\tau_2$-βCl$(F^-(\sigma_1\sigma_2\text{-Int}(K)))) \subseteq F^-(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $\tau_1\tau_2$-βCl$(F^-(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(B)))))) \subseteq F^-(\sigma_1\sigma_2$-Cl$(B))$ for every subset B of Y;

(5) $F^+(\sigma_1\sigma_2$-Int$(B)) \subseteq \tau_1\tau_2$-$\beta$Int$(F^+(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$- Int$(B))))$ for every subset B of Y;

(6) $F^+(V) \subseteq \tau_1\tau_2$-$\beta$Int$(F^+(\sigma_1\sigma_2$-Cl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(7) $\tau_1\tau_2$-$\beta Cl(F^-(V)) \subseteq F^-(\sigma_1\sigma_2$-Cl$(V))$ for every $\sigma_1\sigma_2$-open set V of Y.

Proof. (1) $\Rightarrow$ (2): Let V be a $\sigma_1\sigma_2$-open set of Y and $x \in F^+(V)$. Then, $F(x) \subseteq V$ and by Theorem 4.1.2, $x \in \tau_1\tau_2$-βInt$(F^+(\sigma_1\sigma_2$Cl$(V)))$. Therefore, we obtain $F^+(V) \subseteq \tau_1\tau_2$-$\beta$Int$(F^+(\sigma_1\sigma_2Cl(V)))$.

(2) $\Rightarrow$ (3): Let K be a $\sigma_1\sigma_2$-closed set of Y. Therefore, $Y - K$ is a $\sigma_1\sigma_2$-opened set. By (2),

$$\begin{aligned} X - F^-(K) &= F^+(Y - K) \\ &\subseteq \tau_1\tau_2\text{-}\beta\text{Int}(F^+(\sigma_1\sigma_2\text{Cl}(Y - K))) \\ &= X - \tau_1\tau_2\text{-}\beta\text{Cl}(F^-(\sigma_1\sigma_2\text{Int}(K))). \end{aligned}$$

Consequently, we obtain $\tau_1\tau_2$-βCl$(F^-(\sigma_1\sigma_2$Int$(K))) \subseteq F^-(K)$.

(3) $\Rightarrow$ (4): Let B be a subset of Y. Then, $\sigma_1\sigma_2$-Cl(B) is $\sigma_1\sigma_2$-closed in Y. Thus, we obtain $\tau_1\tau_2$-βCl$(F^-(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(B)))) \subseteq F^-(\sigma_1\sigma_2$-Cl$(B))$.

(4) $\Rightarrow$ (5): Let B be a subset of Y. By (4), we obtain

$$\begin{aligned} F^+(\sigma_1\sigma_2\text{-Int}(B)) &= X - F^-(\sigma_1\sigma_2\text{-Cl}(Y - B)) \\ &\subseteq X - \tau_1\tau_2\text{-}\beta\text{Cl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - B)))) \\ &= \tau_1\tau_2\text{-}\beta\text{Int}(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))). \end{aligned}$$

(5) $\Rightarrow$ (6): The proof is obvious.

(6) $\Rightarrow$ (7): Let V be a $\sigma_1\sigma_2$-open set of Y. By (6), we have

$$\begin{aligned} \tau_1\tau_2\text{-}\beta\text{Cl}(F^-(V)) &\subseteq \tau_1\tau_2\text{-}\beta\text{Cl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \\ &= \tau_1\tau_2\text{-}\beta\text{Cl}(X - F^+(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \end{aligned}$$

$$= X - \tau_1\tau_2\text{-}\beta\text{Int}(F^+(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$$
$$= X - \tau_1\tau_2\text{-}\beta\text{Int}(F^+(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))))$$
$$\subseteq X - F^+(Y - \sigma_1\sigma_2\text{-Cl}(V))$$
$$= F^-(\sigma_1\sigma_2\text{-Cl}(V)).$$

Consequently, we obtain $\tau_1\tau_2\text{-}\beta\text{Cl}(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$.

(7) $\Rightarrow$ (1): Let $x \in X$ and V be a $\sigma_1\sigma_2$-open set containing $F(x)$. By (7), we have

$$x \in F^+(V) \subseteq F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$$
$$= X - F^-(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V)))$$
$$\subseteq X - \tau_1\tau_2\text{-}\beta\text{Cl}(F^-(Y - \sigma_1\sigma_2\text{-Cl}(V)))$$
$$= \tau_1\tau_2\text{-}\beta\text{Int}(F^+(\sigma_1\sigma_2\text{-Cl}(V))).$$

Therefore, $x \in \tau_1\tau_2\text{-}\beta\text{Int}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$, and hence F is upper weakly $\beta(\tau_1, \tau_2)$-continuous by Theorem 4.1.2. □

Theorem 4.1.6. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) F is lower weakly $\beta(\tau_1, \tau_2)$-continuous;

(2) $F^-(V) \subseteq \tau_1\tau_2\text{-}\beta\text{Int}(F^-(\sigma_1\sigma_2\text{Cl}(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(3) $\tau_1\tau_2\text{-}\beta\text{Cl}(F^+(\sigma_1\sigma_2\text{-Int}(K)))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $\tau_1\tau_2\text{-}\beta\text{Cl}(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B))))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(B))$ for every subset B of Y;

(5) $F^-(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \tau_1\tau_2\text{-}\beta\text{Cl}(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{- Int}(B))))$ for every subset B of Y;

(6) $F^-(V) \subseteq \tau_1\tau_2\text{-}\beta\text{Int}(F^-(\sigma_1\sigma_2\text{-Cl}(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(7) $\tau_1\tau_2\text{-}\beta\text{Cl}(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$ for every $\sigma_1\sigma_2$-open set V of Y.

Proof. The proof is similar to that of Theorem 4.1.5. □

Definition 4.1.7. Let A be a subset of a bitopological space (X, τ_1, τ_2). A point $x \in X$ is called $\tau_1\tau_2$-θ-*cluster point* of A if $\tau_1\tau_2$-Cl$(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $\tau_1\tau_2$-θ-cluster point of A is called $\tau_1\tau_2$-θ-*closure* of A and is denoted by $\tau_1\tau_2$-Cl$_\theta(A)$.

A subset A of X is said to be $\tau_1\tau_2$-θ-*closed* if $A = \tau_1\tau_2$-Cl$_\theta(A)$. The complement of a $\tau_1\tau_2$-θ-closed set is said to be $\tau_1\tau_2$-θ-*open*. The union of all $\tau_1\tau_2$-θ-open sets contained in A is called $\tau_1\tau_2$-θ-*interior* of A and is denoted by $\tau_1\tau_2$-Int$_\theta(A)$.

Lemma 4.1.8. For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:

(1) If A is $\tau_2\tau_2$-open in X, then $\tau_1\tau_2$-Cl$(A) = \tau_1\tau_2$- Cl$_\theta(A)$.

(2) $\tau_1\tau_2$-Cl$_\theta(A)$ is $\tau_1\tau_2$-closed in X.

Proof. (1) In general, this holds that $\tau_1\tau_2$-Cl$(A) \subseteq \tau_1\tau_2$-Cl$_\theta(A)$. Suppose that $x \notin \tau_1\tau_2$-Cl(A). Then, there exists a $\tau_1\tau_2$-open set U containing x such that $U \cap A = \emptyset$; hence $\tau_1\tau_2$-Cl$(U) \cap A = \emptyset$. This shows that $x \notin \tau_1\tau_2$-Cl$_\theta(A)$. Therefore, we obtain $\tau_1\tau_2$-Cl$_\theta(A) \subseteq \tau_1\tau_2$-Cl$(A)$. Consequently, $\tau_1\tau_2$-Cl$(A) = \tau_1\tau_2$-Cl$_\theta(A)$.

(2) Let $x \in X - \tau_1\tau_2$-Cl$_\theta(A)$. Then, $x \notin \tau_1\tau_2$-Cl$_\theta(A)$. There exists a $\tau_1\tau_2$-open set U_x containing x such that $\tau_1\tau_2$-Cl$(U_x) \cap A = \emptyset$. Then, we have $\tau_1\tau_2$-Cl$_\theta(A) \cap U_x = \emptyset$ and so $x \in U_x \subseteq X - \tau_1\tau_2$-Cl$_\theta(A)$. Therefore, we obtain $X - \tau_1\tau_2$-Cl$_\theta(A) = \bigcup_{x \in X - \tau_1\tau_2\text{-Cl}_\theta(A)} U_x$. This shows that $\tau_1\tau_2$-Cl$_\theta(A)$ is $\tau_1\tau_2$-closed. □

Definition 4.1.9. A subset A of a bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)r-closed (resp. (τ_1, τ_2)s-open, (τ_1, τ_2)p-open, $(\tau_1, \tau_2)\beta$-*open*) if $A = \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))))$

Theorem 4.1.10. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) F is upper weakly $\beta(\tau_1, \tau_2)$-continuous;

(2) $\tau_1\tau_2$-βCl$(F^-(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$_\theta(B)))) \subseteq F^-(\sigma_1\sigma_2$-Cl$_\theta(B))$ for every subset B of Y;

(3) $\tau_1\tau_2$-βCl$(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}_\theta(B))$ for every subset B of Y;

(4) $\tau_1\tau_2$-βCl$(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(5) $\tau_1\tau_2$-βCl$(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$ for every (σ_1, σ_2)p-open set V of Y;

(6) $\tau_1\tau_2$-βCl$(F^-(\sigma_1\sigma_2\text{-Int}(K))) \subseteq F^-(K)$ for every (σ_1, σ_2)r-closed set K of Y;

(7) $\tau_1\tau_2$-βCl$(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y;

(8) $\tau_1\tau_2$-βCl$(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$ for every (σ_1, σ_2)s-open set V of Y.

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. Then, $\sigma_1\sigma_2$-Cl$_\theta(B)$ is $\sigma_1\sigma_2$-closed in Y. Therefore, by Theorem 4.1.5(3) we obtain

$$\tau_1\tau_2\text{-}\beta\text{Cl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}_\theta(B)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}_\theta(B)).$$

(2) $\Rightarrow$ (3): This is obvious since $\sigma_1\sigma_2$-Cl$(B) \subseteq \sigma_1\sigma_2$-Cl$_\theta(B)$ for every subset B of Y.

(3) $\Rightarrow$ (4): This is obvious since $\sigma_1\sigma_2$-Cl$(V) = \sigma_1\sigma_2$-Cl$_\theta(V)$ for every $\sigma_1\sigma_2$-open set V of Y.

(4) $\Rightarrow$ (5): Let V be a (σ_1, σ_2)p-open set of Y. Then, we have $V \subseteq \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$, and hence

$$\sigma_1\sigma_2\text{-Cl}(V) = \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))).$$

Now, put $G = \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$, then G is $\sigma_1\sigma_2$-open in Y and $\sigma_1\sigma_2$-Cl$(G) = \sigma_1\sigma_2$-Cl(V). Therefore, by (4) we have

$$\tau_1\tau_2\text{-}\beta\text{Cl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V)).$$

(5) $\Rightarrow$ (6): Let K be any (σ_1, σ_2)r-closed set of Y. Then, we have $\sigma_1\sigma_2$-Int(K) is (σ_1, σ_2)p-open in Y and by (5), we obtain

$$\begin{aligned}\tau_1\tau_2\text{-}\beta\text{Cl}(F^-(\sigma_1\sigma_2\text{-Int}(K))) &= \tau_1\tau_2\text{-}\beta\text{Cl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) = F^-(K).\end{aligned}$$

(6) $\Rightarrow$ (7): Let V be any $(\sigma_1, \sigma_2)\beta$-open set of Y. Then, we have

$$V \subseteq \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))).$$

Since $\sigma_1\sigma_2$-Cl(V) is (σ_1, σ_2)r-closed in Y, by (6)

$$\tau_1\tau_2\text{-}\beta\text{Cl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V)).$$

(7) $\Rightarrow$ (8): This is obvious since every (σ_1, σ_2)s-open set is $(\sigma_1, \sigma_2)\beta$-open.

(8) $\Rightarrow$ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Since V is (σ_1, σ_2)s-open set in Y, by (8) we have

$$\tau_1\tau_2\text{-}\beta\text{Cl}(F^-(V)) \subseteq \tau_1\tau_2\text{-}\beta\text{Cl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V)).$$

By Theorem 4.1.5, we obtain F is upper weakly (τ_1, τ_2)-β-continuous. □

Theorem 4.1.11. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) F is lower weakly $\beta(\tau_1, \tau_2)$-continuous;

(2) $\tau_1\tau_2$-βCl$(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}_\theta(B)))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}_\theta(B))$ for every subset B of Y;

(3) $\tau_1\tau_2$-βCl$(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}_\theta(B))$ for every subset B of Y;

(4) $\tau_1\tau_2$-βCl$(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(5) $\tau_1\tau_2$-βCl$(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$ for every (σ_1, σ_2)p-open set V of Y;

(6) $\tau_1\tau_2$-βCl$(F^+(\sigma_1\sigma_2\text{-Int}(K))) \subseteq F^+(K)$ for every (σ_1, σ_2)r-closed set K of Y;

(7) $\tau_1\tau_2$-βCl$(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y;

(8) $\tau_1\tau_2$-βCl$(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$ for every (σ_1, σ_2)s-open set V of Y.

Proof. The proof is similar to that of Theorem 4.1.10. □

CHAPTER 5

CONCLUSIONS

5.1 Conclusions

The purposes of this thesis are to introduce the notions of $\beta(\tau_1, \tau_2)$-continuous multifunctions, almost $\beta(\tau_1, \tau_2)$-continuous multifunctions, and weakly $\beta(\tau_1, \tau_2)$-continuous multifunctions; moreover, some characterizations of these multifunctions are obtained. The results are as follows:

1. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be:

(1) *upper* $\beta(\tau_1, \tau_2)$*-continuous* at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists a $\tau_1\tau_2$-β-open set U containing x such that $F(U) \subseteq V$;

(2)*lower* $\beta(\tau_1, \tau_2)$*-continuous* at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau_1\tau_2$-β-open set U containing x such that $F(z) \cap V \neq \emptyset$ for every $z \in U$;

(3) *upper* (resp. *lower*) $\beta(\tau_1, \tau_2)$*-continuous* if F has this property at each point of X.

From the above definition, the following theorems are derived:

1.1 A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is upper $\beta(\tau_1, \tau_2)$-continuous at $x \in X$ if and only if $x \in \tau_1\tau_2$-βInt$(F^+(V))$ for every $\sigma_1\sigma_2$-open set V of Y containing $F(x)$.

1.2 A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower $\beta(\tau_1, \tau_2)$-continuous at $x \in X$ if and only if $x \in \tau_1\tau_2$-βInt$(F^-(V))$ for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$.

1.3 For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) F is upper $\beta(\tau_1, \tau_2)$-continuous;

(2) $F^+(V)$ is $\tau_1\tau_2$-β-open in X for every $\sigma_1\sigma_2$-open set V of Y;

(3) $F^-(K)$ is $\tau_1\tau_2$-β-closed in X for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $\tau_1\tau_2$-βCl$(F^-(B)) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(B))$ for every subset B of Y;

(5) τ_1-Int(τ_2-Cl(τ_1-Int($F^-(B)$))) $\subseteq F^-(\sigma_1\sigma_2$-Cl$(B))$ for every subset B.

1.4 For a multifunction $F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) F is lower $\beta(\tau_1, \tau_2)$-continuous;

(2) $F^-(V)$ is $\tau_1\tau_2$-β-open in X for every $\sigma_1\sigma_2$-open set V of Y;

(3) $F^+(K)$ is $\tau_1\tau_2$-β-closed in X for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $\tau_1\tau_2$-βCl$(F^+(B)) \subseteq F^+(\sigma_1\sigma_2$-Cl$(B))$ for every subset B of Y;

(5) τ_1-Int(τ_2-Cl(τ_1-Int($F^+(B)$))) $\subseteq F^+(\sigma_1\sigma_2$-Cl$(B))$ for every subset B.

2. A multifunction $F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be:

(1) *upper almost $\beta(\tau_1, \tau_2)$-continuous* at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists a $\tau_1\tau_2$-β-open set U containing x such that $F(U) \subseteq \sigma_1$-Int(σ_2-Cl(V));

(2) *lower almost $\beta(\tau_1, \tau_2)$-continuous* at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau_1\tau_2$-β-open set U containing x such that $F(z) \cap \sigma_1$-Int(σ_2-Cl(V)) $\neq \emptyset$ for every $z \in U$;

(3) *upper almost* (resp. *lower almost*) *$\beta(\tau_1, \tau_2)$-continuous* if F has this property at each point of X.

From the above definition, the following theorems are derived:

2.1 A multifunction $F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is upper almost $\beta(\tau_1, \tau_2)$-continuous at $x \in X$ if and only if $x \in \tau_1\tau_2$-βInt$(F^+(\sigma_1\sigma_2$-sCl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y containing $F(x)$.

2.2 A multifunction $F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is lower almost $\beta(\tau_1, \tau_2)$-continuous at $x \in X$ if and only if $x \in \tau_1\tau_2$-βInt$(F^-(\sigma_1\sigma_2$-sCl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$.

2.3 For a multifunction $F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) F is upper almost $\beta(\tau_1, \tau_2)$-continuous;

(2) for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists a $\tau_1\tau_2$-β-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2$-sCl$(V)$;

(3) $F^+(V) \subseteq \tau_1\tau_2$-$\beta$Int$(F^+(\sigma_1\sigma_2$-sCl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(4) $\tau_1\tau_2$-βCl$(F^-(\sigma_1\sigma_2\text{-sInt}(K))) \subseteq F^-(K)$ for every $\sigma_1\sigma_2$-closed set K.

2.4 For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) F is lower almost $\beta(\tau_1, \tau_2)$-continuous;

(2) for each $x \in X$ and each $\sigma_1\sigma_2$-open set V containing $F(x)$, there exists a $\tau_1\tau_2$-β-open set U containing x such that $F(z) \cap \sigma_1\sigma_2$-sCl$(V)$ for each $z \in U$;

(3) $F^-(V) \subseteq \tau_1\tau_2$-$\beta$Int$(F^-(\sigma_1\sigma_2\text{-sCl}(V)))$ for every $\sigma_1\sigma_2$-open set V;

(4) $\tau_1\tau_2$-βCl$(F^+(\sigma_1\sigma_2\text{-sInt}(K))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$-closed set K.

2.5 If a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower almost $\beta(\tau_1, \tau_2)$-continuous, then $\beta ClF_{\circledast} : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower almost $\beta(\tau_1, \tau_2)$-continuous.

3. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be:

(1) *upper weakly $\beta(\tau_1, \tau_2)$-continuous* at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists a $\tau_1\tau_2$-*beta*-open set U containing x such that $F(U) \subseteq \sigma_1\sigma_2$-Cl$(V)$;

(2) *lower weakly $\beta(\tau_1, \tau_2)$-continuous* at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau_1\tau_2$-β-open set U containing x such that $F(z) \cap \sigma_1\sigma_2$-Cl$(V) \neq \emptyset$ for every $z \in U$;

(3) *upper weakly* (resp. *lower weakly*) $\beta(\tau_1, \tau_2)$-*continuous* if F has this property at each point of X.

3.1 A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is upper weakly $\beta(\tau_1, \tau_2)$-continuous at $x \in X$ if and only if $x \in \tau_1\tau_2$-βInt$(F^+(\sigma_1\sigma_2\text{Cl}(V)))$ for every $\sigma_1\sigma_2$-open set V containing $F(x)$.

3.2 A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower weakly $\beta(\tau_1, \tau_2)$-continuous at $x \in X$ if and only if $x \in \tau_1\tau_2$-βInt$(F^-(\sigma_1\sigma_2\text{Cl}(V)))$ for every $\sigma_1\sigma_2$-open set V such that $F(x) \cap V \neq \emptyset$.

3.3 For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) F is upper weakly $\beta(\tau_1, \tau_2)$-continuous;

(2) $F^+(V) \subseteq \tau_1\tau_2$-$\beta$Int$(F^+(\sigma_1\sigma_2\text{Cl}(V)))$ for every $\sigma_1\sigma_2$-open set V;

(3) $\tau_1\tau_2$-βCl$(F^-(\sigma_1\sigma_2\text{-Int}(K)))) \subseteq F^-(K)$ for every $\sigma_1\sigma_2$-closed set K;

(4) $\tau_1\tau_2$-βCl$(F^-(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(B)))))) \subseteq F^-(\sigma_1\sigma_2$-Cl$(B))$ for every subset B of Y;

(5) $F^+(\sigma_1\sigma_2$-Int$(B)) \subseteq \tau_1\tau_2$-$\beta$Int$(F^+(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$- Int$(B))))$ for every subset B of Y;

(6) $F^+(V) \subseteq \tau_1\tau_2$-$\beta$Int$(F^+(\sigma_1\sigma_2$-Cl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(7) $\tau_1\tau_2$-$\beta Cl(F^-(V)) \subseteq F^-(\sigma_1\sigma_2$-Cl$(V))$ for every $\sigma_1\sigma_2$-open set V of Y.

3.4 For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) F is lower weakly $\beta(\tau_1, \tau_2)$-continuous;

(2) $F^-(V) \subseteq \tau_1\tau_2$-$\beta$Int$(F^-(\sigma_1\sigma_2Cl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(3) $\tau_1\tau_2$-βCl$(F^+(\sigma_1\sigma_2$-Int$(K)))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$-closed set K;

(4) $\tau_1\tau_2$-βCl$(F^+(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(B)))))) \subseteq F^+(\sigma_1\sigma_2$-Cl$(B))$ for every subset B of Y;

(5) $F^-(\sigma_1\sigma_2$-Int$(B)) \subseteq \tau_1\tau_2$-$\beta$Cl$(F^-(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$- Int$(B))))$ for every subset B;

(6) $F^-(V) \subseteq \tau_1\tau_2$-$\beta$Int$(F^-(\sigma_1\sigma_2$-Cl$(V)))$ for every $\sigma_1\sigma_2$-open set V;

(7) $\tau_1\tau_2$-βCl$(F^+(V)) \subseteq F^+(\sigma_1\sigma_2$-Cl$(V))$ for every $\sigma_1\sigma_2$-open set V.

3.5 For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following implication hold:

upper (τ_1, τ_2)-β-continuity

$\Downarrow$

upper almost (τ_1, τ_2)-β-continuity

$\Downarrow$

upper weak (τ_1, τ_2)-β-continuity.

The converse of the implications are not true in general.

5.2 Recommendations

To this end, even though we have found several characterizations presented in this thesis, there is another type of continuity that interests us to investigate. The multifunction is defined as:
A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be:

(1) *upper* (τ_1, τ_2)*-continuous* at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists a $\tau_1\tau_2$-open set U containing x such that $F(U) \subseteq V$;

(2) *lower* (τ_1, τ_2)*-continuous* at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau_1\tau_2$-open set U containing x such that $F(z) \cap V \neq \emptyset$ for every $z \in U$;

(3) *upper* (resp. *lower*) (τ_1, τ_2)*-continuous* if F has this property at each point of X.

Therefore, there are many interesting questions involving several characterizations of this multifunction yet to be answered. Moreover, the relationships between (τ_1, τ_2)-continuity and $\beta(\tau_1, \tau_2)$-continuity are noticeable to study in the future. Besides, The condition, which make the converse of the relationships true in general, is a further interesting study.

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BIOGRAPHY

BIOGRAPHY

NAME	Miss Kaewta Laprom
DATE OF BIRTH	October 26, 1994
PLACE OF BIRTH	Saraburi Province, Thailand
ADDRESS	188, Nongplalai sub-district, Mueang Saraburi district, Saraburi provice 18000, Thailand kaewta.lap@msu.ac.th
EDUCATION	
2012	High School in Suankularb Wittayalai Saraburi School, Saraburi, Thailand
2017	Bachelor of Education in Mathematics Education Mahasarakham University, Thailand
2020	Master of Science in Mathematics Education, Mahasarakham University, Thailand