Common Fixed Points in Partially Ordered Fuzzy Metric Spaces

Nuthawan Pimpila

A Thesis Proposal submitted in partial fulfillment of the requirements for the degree of Master of Science in Mathematics at Mahasarakham University

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The examining committee has unanimously approved this thesis, submitted by Miss Nuthawan Pimpila, as a partial fulfillment of the requirements for the Master of Science in Mathematics at Mahasarakham University.

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ACKNOWLEDGEMENTS

This thesis, funded awards ceremony for outstanding students in the capital Bhumibol. The dissertation would not have been accomplished if without the help from several people. First, I wish to express my deepest sincere gratitude to Asst. Prof. Dr. Daruni Boonchari for their initial idea, guidance and encouragement which enable me to carry out my study research successfully.

I would like to thank Asst. Prof. Dr. Somnuek Worawiset, Asst. Prof. Dr. Chokchai Viriyapong, Asst. Prof. Jeeranunt Khampakdee, Ph.D. and Asst. Prof. Dr. Kittisak Saegsura for their constructive comments and suggestions.

I would like to express my sincere gratitude to my parents for always encouraging.

Finally, I would like to thank all graduate students and staffs at the Department of Mathematics for supporting the preparation of this thesis.

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TITLE Common Fixed Points in Partially Ordered Fuzzy Metric Spaces

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DEGREE Master of Science **MAJOR** Mathematics

UNIVERSITY Mahasarakham University **YEAR** 2019

ABSTRACT

In this research, we get the results of fixed point on L-fuzzy metric spaces and Partially order L-fuzzy metric spaces using ϕ-Contractions. Moreover, we study the properties of open set, closed set, closure operator and interior operator in L-Fuzzy minimal structure and Minimal structure space, including characterization of sg-submaximal space.

Keywords : common fixed point, ϕ-contractions, sg-submaximal space.

CONTENTS

CHAPTER 1

INTRODUCTION

In 1987 Guo and Lakshmikan [16], the notion of couple fixed point was first introduced. Recently, In 2006 Bhaskar and Lakshmikan [3], established some coupled fixed point theorems in partially ordered metric space. The main idea [29],[31],[33], involves combining the ideas of iterative technique in the contraction mapping principle with those in the monotone technique. In 1965 Zadeh [48], the notion of fuzzy sets was introduce and various concepts of fuzzy metric space were considered by Erceg [10] in 1979 and Kramosil [22] in 1975. In 2007 we will adopt the usual terminology, notation and convention of L-fuzzy metric space introduced by Saadati et al. [39] which are a generalization of fuzzy metric space by George and Veeramani [14]. In 2010 Shakeri et al. [41] established common fixed point on Partially order L-fuzzy metric spaces by using monotone technique. After that, in 2011 Ha [18] and Ha et al. [19], established common coupled fixed point results in fuzzy metric space. In 2013 Choudhury et al. [6] established coupled coincidence point and fixed point results for compatible mapping in partially ordered fuzzy metric space. In 2005 Singh and Jain [42], generalized the notion of compatible maps by introducing the notion of weakly compatible maps and proved a common fixed point theorem for six self-maps under the k-contractive-type condition in metric spaces. Recently, in 2009 Fang [12], the notion of new common fixed point theorems for compatible maps and weakly compatible maps satisfying ϕ-contractive condition in Menger space with continuous t-norm Δ of H-type. For many results [6], [18], [19], [36] are obtained under the assumptions: (a) $\phi(t) = kt$ for all $t > 0$, where $k \in (0, 1)$ or (b) $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for all $t > 0$. It is obvious that condition (a) is special. In 2010 Ciric [7] has pointed out, the condition (b) is very strong and difficult for testing in practice. Then Ciric introduced the condition (CBW). Later, Jachymski [20] correct the condition (CBW). In order to weaken the condition of (CBW) further. In 2015 Fang [11], introduced the condition for each $t > 0$ there exists $r \geq t$ such that $\lim_{n\to\infty} \phi^n(r) = 0$ in the context of fuzzy metric space, Wang [45] the notion result under the condition for each $t > 0$

there exists $r \geq t$ such that $\lim_{n\to\infty} \phi^n(r) = 0$, we present some coincidence point and common fixed point for weakly compatible mappings in partially ordered fuzzy metric spaces. Recently, Wang et al. [46] established common fixed point in Menger space using the new notion of Φ_{w^*} denote the set of all functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following condition:

$$\text{for each } t_1, t_2 > 0 \text{ there exists } r \geq \max\{t_1, t_2\} \text{ and } N \in \mathbb{N}$$
$$\text{such that } \phi^n(r) < \min\{t_1, t_2\} \text{ for all } n > N.$$

Another part of this thesis, we talked about minimal structure space. In 1999 Maki [27], introduced the notion of minimal structure, also Popa and Noir [32], introduced the notion of m_X-open sets, m_X-closed sets and then characterized those sets using m_X-closure and m_X-interior operators, respectively. After that, in 2002 Cao et al. [4], defined some new types of open sets and closed sets in topological space obtained some results in topological space. In 1976 Lowen [24], defined open set and closed set in fuzzy topological space and in 2006 Ahmohammady and Roohi [1], defined open set and closed set in fuzzy minimal structure. Late, in 2009 Rosas [37], [38], introduced some new types of open set and closed set in minimal structure. The concept of relationships of generalized closed sets and some new characterizations of sg-submaximal were introduced by Gansteer [13] in 1987.

In this thesis, we have two main results. First, we introduced a new concept of common fixed point in L-fuzzy metric space and partially ordered L-fuzzy metric space, using Φ_{w^*} from Wang et al. [46]. Second, we study the properties of open set, closed set, closure and interior in L-fuzzy minimal structure space and minimal structure space, including the relationship between every type of closed sets. We provide the characterization of sg-submaximal space.

CHAPTER 2

PRELIMINARIES

In this chapter, we will give some definitions, notations, dealing with some preliminaries and some useful results that will be duplicated in later chapter.

2.1 Fuzzy metric space

This section discusses some basic concepts and preliminaries of metric space and fuzzy set including Other basic definitions.

Definition 2.1.1. [23] A pair (X, d) is said to be a *metric space* if X is a non-empty set, $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$ we have:

(1) $d(x, y) \geq 0$,

(2) $d(x, y) = 0$ if and only if $x = y$,

(3) $d(x, y) = d(y, x)$ (symmetry),

(4) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

Definition 2.1.2. [23] A sequence $\{x_n\}$ in a metric space (X, d) is said to be *convergent* if there is an $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0$$

x is called the limit of x_n and we write

$$\lim_{n\to\infty} x_n = x \text{ or, simply } x_n \to x$$

we say that $\{x_n\}$ converges to x.

Proposition 2.1.3. [23] A sequence $\{x_n\}$ convergent in a metric space (X, d) to a point x is equivalent to the condition that for each $\varepsilon > 0$ there is a natural number N such that $n \geq N$ implies $d(x_n, x) < \varepsilon$.

Definition 2.1.4. [40] A binary operation $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ is a *continuous t-norm* if Δ satisfies the following conditions:

(1) Δ is continuous,

(2) $a\Delta 1 = a$ (boundary condition),

(3) $\Delta(a, b) = \Delta(b, a)$ (commutative),

(4) $\Delta(a, \Delta(b, c)) = \Delta(\Delta(a, b), c)$ (associative),

(5) $a\Delta b \leq c\Delta d$ whenever $a \leq c$ and $b \leq d$ (monotonicity)

for all $a, b, c, d \in [0, 1]$.

Definition 2.1.5. [40] Let (X, d) be a metric spaces, and T is a mapping from X to X. T is *equicontinuous* at a point $x_0 \in X$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(T(x_0), T(x)) < \varepsilon$ for all $x \in X$ such that $d(x_0, x) < \delta$.
The family is pointwise equicontinuous if it is equicontinuous at each point of X.

Definition 2.1.6. [40] A t-norm Δ is said to be of *H-type* if the family of functions $\{\Delta^m(t)\}_{m=1}^{\infty}$ is equicontinuous at $t = 1$, that is, for all $\varepsilon \in (0, 1)$ there exists $\delta \in (0, 1)$ such that if $t \in (1 - \delta, 1]$, then $\Delta^m t > 1 - \varepsilon$, when

$$\Delta^1(t) = \Delta(t, t), \quad \Delta^m(t) = \Delta t(\Delta^{m-1}(t)), \quad m \in \mathbb{N}, t \in [0, 1](\Delta^0(t) = t).$$

Definition 2.1.7. [48] A *fuzzy set* on X is a mapping $A : X \to [0, 1]$

Definition 2.1.8. [12] A mapping $f : \mathbb{R} \to \mathbb{R}^+$ is called a *distribution* if it is non-decreasing left-continuous with $\sup_{t\in\mathbb{R}} f(t) = 1$ and $\inf_{t\in\mathbb{R}} f(t) = 0$.

We shall denote by $\mathcal{D}$ the set of all distribution functions. A spacial element of $\mathcal{D}$ is the function H defined by

$$H(t) = \begin{cases} 0, & \text{if} \quad t \leq 0, \\ 1, & \text{if} \quad t > 0. \end{cases}$$

Let $F_1, F_2 \in \mathcal{D}$. The algebraic sum $F_1 \oplus F_2$ of F_1 and F_2 is defined by

$$(F_1 \oplus F_2)(t) = \sup_{t_1+t_2=t} \min\{F_1(t_1), F_2(t_2)\} \text{ for all } t \in \mathbb{R}.$$

Obviously, $(F_1 \oplus F_2)(2t) = \min\{F_1(t_1), F_2(t_2)\}$ for all $t \geq 0$.

Definition 2.1.9. [12, 22] A tripe (X, M, Δ) is said to be a *fuzzy metric space* if X is a non-empty set, Δ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,

(FM-1) $M(x, y, t) > 0$.

(FM-2) $M(x, y, t) = 1$ if and only if $x = y$.

(FM-3) $M(x, y, t) = M(y, x, t)$.

(FM-4) $\Delta(M(x, y, t), M(y, z, s)) \leq M(x, z, t + s)$.

(FM-5) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

We shall consider a fuzzy metric space (X, M, Δ), which satisfies the following condition:

$$\lim_{t\to\infty} M(x, y, t) = 1, \text{ for all } x, y \in X.$$

Let (X, M, Δ) be a fuzzy metric space. For $t > 0$, the *open ball* $B(x, r, t)$ with a center $x \in X$ and a radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

A subset $A \subseteq X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subseteq A$. Let τ denote the family of all open subset of X. Then τ is called the topology on X.

Definition 2.1.10. [14] Let (X, M, Δ) be a fuzzy metric space. Then

(1) a sequence $\{x_n\}$ in X is said to be *convergent* to X if

$$\lim_{n\to\infty} M(x_n, x, t) = 1,$$

for all $t > 0$.

(2) A sequence $\{x_n\}$ in X is said to be a *Cauchy sequence* if for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that

$$M(x_n, x_m, t) > 1 - \varepsilon$$

for all $t > 0$ and $n, m \geq n_0$.

(3) A fuzzy metric space (X, M, Δ) is said to be *complete* if and only if every Cauchy sequence in X is convergent.

2.2 Topology and partially order space

This section discusses some basic concepts and preliminaries of topology space and partially order space including other basic definitions.

Definition 2.2.1. [43] Let X be a non-empty set. A class τ of subsets of X is a $topology$ on X iff τ satisfies the following axioms :

(1) X and $\emptyset$ belong to τ.

(2) The union of any number of sets in τ belongs to τ.

(3) The intersection of any two sets in τ belongs to τ.

The elements of τ are then called $open\ sets$ and there complements are called $closed\ sets$, the pair (X, τ) is called a $topological\ space$.

Definition 2.2.2. [34] A topological space (X, τ) is a $Hausdorff\ space$ if all distinct points in X are pairwise neighborhood-separable. Points x and y in a topological space X can be separated by neighbourhoods if there exists a neighbourhood U of x and a neighbourhood V of y such that U and V are disjoint $(U \cap V = \emptyset)$.

Definition 2.2.3. [21] Let X be non-empty set and a family $\mathbf{P}$ is called a $partition$ of X if and only if all of the following conditions hold:

(1) The $\mathbf{P}$ does not contain the empty set (that is $\emptyset \notin \mathbf{P}$).

(2) The union of the sets in $\mathbf{P}$ is equal to X (that is $\bigcup_{A \in \mathbf{P}} A = X$).

(3) The intersection of any two distinct sets in $\mathbf{P}$ is empty (that is $A \neq B \rightarrow A \cap B = \emptyset$ for all $A, B \in \mathbf{P}$).

Definition 2.2.4. [21] Let P be a set, the partial order state that the relation $\preceq$ is reflexive, antisymmetric, and transitive. That is, for all $a, b, c \in P$, it must satisfy:

(1) $a \preceq a$ (reflexivity),

(2) if $a \preceq b$ and $b \preceq a$, then $a = b$ (antisymmetry),

(3) if $a \preceq b$ and $b \preceq c$, then $a \preceq c$ (transitivity).

A set with a partial order is called a $partially\ ordered\ set$ (also called a poset) denoted by $(P, \preceq)$.

Definition 2.2.5. [21] Let $(P, \preceq)$ be a partially ordered set. a is a lower bound of a subset S of P such that $a \preceq x$ for all $x \in S$. A lower bound a is called an infimum (briefly inf) of S if for all lower bounds y of S then $y \preceq a$.
Similarly, b is a upper bound of a subset S of P such that $b \succeq x$ for all x in S. An upper bound b is called a supremum (briefly sup) of S if for all upper bounds z of S then $z \succeq b$.

Definition 2.2.6. [35] Let $\{A, B\}$ be a partition of the set $\Lambda_n = \{1, 2, ..., n\}$. If $(X, \preceq)$ is a partially ordered space(or pospace), for all $y, v \in X$ and $i \in \Lambda_n$

$$y \preceq_i v \Leftrightarrow \begin{cases} y \preceq v, & \text{if } \ i \in A, \\ y \succeq v, & \text{if } \ i \in B. \end{cases}$$

If $y \preceq_i v$ or $y \succeq_i v$, then two points y and v are *comparable* (denoted by $y \asymp v$).

Every pospace is a Hausdorff space. If we take equality $=$ as the partial order, this definition becomes the definition of a Hausdorff space.

2.3 Partially order L-fuzzy metric spaces

This section discusses the relationship between lattice and poset, including some properties of L-fuzzy metric spaces.

Proposition 2.3.1. [2] Let X be a set, $\wedge$ and $\vee$ two binary operations defined on X, and 0 and 1 two elements of X. Then $(X, \vee, \wedge, 0, 1)$ is a *lattice* if and only if the following axioms are satisfied:

(1) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$ (associative),

(2) $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$ (commutative),

(3) $x \wedge x = x \vee x$ (idempotent),

(4) $x \wedge (x \vee y) = x = x \vee (x \wedge y)$,

(5) $x \wedge 0 = 0, x \vee 1 = 1$.

A lattice is a poset $(X, \preceq)$ with the properties

(1) X has an upper bound 1 and a lower bound 0;

(2) for any two elements $x, y \in X$, there is a least upper bound and a greatest lower bound of the set $\{x, y\}$. In a lattice, we denote the least upper bound of $\{x, y\}$ by $x \vee y$, and the greatest lower bound by $x \wedge y$.

Definition 2.3.2. [15] A partially ordered set $(X, \preceq)$ is a *complete lattice* if every subset A of X has both the infimum (or called meet) and the supremum (or called join) in $(X, \preceq)$.

Definition 2.3.3. [15] Let $(L, \leq_L)$ be a complete lattice if L is a lattice and $\leq_L$ is an operators on L. $\mathcal{U}$ is a non-empty set called a universe. An L-*fuzzy set* $\mathfrak{L}$ on $\mathcal{U}$ is defined as a mapping $\mathfrak{L} : \mathcal{U} \to L$. Define first $0_L = \inf \ L$ and $1_L = \sup \ L$.

Lemma 2.3.4. [41] Consider the set L^* and the operation $\leq_{L^*}$ defined by

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0,1]^2, x_1 + x_2 \leq 1\},$$

$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Longleftrightarrow x_1 \leq y_1$, and $x_2 \geq y_2$, for every $(x_1, x_2), (y_1, y_2) \in L^*$. Then $(L^* \leq_{L^*})$ is a complete lattice.

Definition 2.3.5. [15] A negation on $(L, \leq_L)$ any strictly decreasing mapping $\mathcal{N} : L \to L$ satisfying $\mathcal{N}(0_L) = 1_L$ and $\mathcal{N}(1_L) = 0_L$. If $\mathcal{N}(\mathcal{N}(x)) = x$, for all $x \in L$, then $\mathcal{N}$ is called an *involutive negation.*

Definition 2.3.6. [15] A *triangular norm* (t-norm) on $(L, \leq_L)$ is a mapping $\Delta : L^2 \to L$ satisfying the following conditions:

(i) $(\forall x \in L)$, $\Delta(x, 1_L) = x$ (boundary condition);

(ii) $(\forall x, y \in L)$, $\Delta(x, y) = \Delta(y, x)$ (commutativity);

(iii) $(\forall x, y, z \in L)$, $\Delta(x, \Delta(y, z)) = \Delta(\Delta(x, y), z)$ (associativity);

(iv) $(\forall x, y, x', y' \in L)$, $x \leq_L x'$ and $y \leq_L y' \Rightarrow \Delta(x, y) \leq_L \Delta(x', y')$ (monotonicity).

A t-norm Δ on $(L, \leq_L)$ is said to be *continuous* if for and $x, y \in (L, \leq_L)$ and any sequences $\{x_n\}$ and $\{y_n\}$ which converge to x and y we have

$$\lim_{n\to\infty} \Delta(x_n, y_n) = \Delta(x, y).$$

Definition 2.3.7. [15] A t-norm can also be defined recursively as an $(n+1)$-ary operation $(n \in \mathbb{N})$ by $\Delta^1 = \Delta$ and

$$\Delta^n(x_1, ..., x_{n+1}) = \Delta(\Delta^{n-1}(x_1, ..., x_n), x_{n+1})$$

for $n \geq 2$ and $x_i \in L$.

A t-norm Δ is said to be of *H-type* if a family of $\{\Delta^n\}_{n\in\mathbb{N}}$ is equicontinuous at $x = 1_L$, that is,

$$\forall \varepsilon \in L\backslash\{0_L, 1_L\}, \exists \delta \in L\backslash\{0_L, 1_L\} : a >_L \mathcal{N}(\delta) \Rightarrow \Delta^n(a) >_L \mathcal{N}(\varepsilon)\ (n \geq 1).$$

Δ_M is a trivial example of a t-norm of H-type, but there exist t-norms of H-type weaker then Δ_M [17] where

$$\Delta_M(x,y) = \begin{cases} x, & \text{if } x \leq_L y, \\ y, & \text{if } y \leq_L x. \end{cases}$$

The t-norm Δ_M is the strongest t-norm, that is, $\Delta \leq \Delta_M$.

Definition 2.3.8. [41] The 3-tuple (X, M, Δ) is said to be an L-*fuzzy metric space* if X is an arbitrary(non-empty) set, Δ is a continuous t-norm on $(L, \leq_L)$ and M is an L-fuzzy set on $X \times X \times (0, \infty)$ satisfying the following conditions for every $x, y, z \in X$ and $t, s \in (0, \infty)$;

(L-1) $M(x, y, t) >_L 0_L$;

(L-2) $M(x, y, t) = 1_L$ for all $t > 0$ if and only if $x = y$;

(L-3) $M(x, y, t) = M(y, x, t)$;

(L-4) $\Delta(M(x, y, t), M(y, z, s)) \leq_L M(x, z, t + s)$;

(L-5) $M(x, y, \cdot) : (0, \infty) \to L$ is continuous.

If the L-fuzzy metric space (X, M, Δ) satisfies the condition:

(L-6) $\lim_{t\to\infty} M(x, y, t) = 1_L$.

Then (X, M, Δ) is said to be a *Menger* L-*fuzzy metric space* or for short a $\mathcal{M}L$-*fuzzy metric space*.

Let (X, M, Δ) be an L-fuzzy metric space. For $t \in (0, \infty)$, we define the *open ball* $B(x, r, t)$ with center $x \in X$ and radius $r \in L \setminus \{0_L, 1_L\}$, as

$$B(x, r, t) = \{y \in X : M(x, y, t) >_L \mathcal{N}(r)\}.$$

A subset $A \subseteq X$ is called *open* if for each $x \in A$, there exist $t > 0$ and $r \in L\setminus\{0_L, 1_L\}$ such that $B(x, r, t) \subseteq A$. Let τ_M denote the family of all open subsets of X. Then τ_M is called the *topology induced by the* $L-$ *fuzzy metric* M.

Example 2.3.9. [41] Let $X = \mathbb{N}$ define $\Delta(a, b) = (\max(0, a_1 + b_1 - 1), (a_2 + b_2 - a_2 b_2)$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in L^*, and let $M(x, y, t)$ on $X^2 \times (0, \infty)$ be defined as follows :

$$M(x,y,t) = \begin{cases} \left(\dfrac{x}{y}, \dfrac{y-x}{y}\right) & \text{if} \quad x \leq y; \\ \left(\dfrac{y}{x}, \dfrac{x-y}{x}\right) & \text{if} \quad y \leq x \end{cases}$$

for all $x, y \in x$ and $t > 0$ then (X, M, Δ) is an L- fuzzy metric space.

Lemma 2.3.10. [41] Let (X, M, Δ) be an L-fuzzy metric space. Then, $M(x, y, t)$ is non-decreasing with respect to t, for all x, y in X.

Definition 2.3.11. [41] A sequence $\{x_n\}_{n\in\mathbb{N}}$ in an L-fuzzy metric space (X, M, Δ) is called a *Cauchy sequence*, if for each $\varepsilon \in L\backslash\{0_L\}$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m \geq n \geq n_0 (n \geq m \geq n_0)$,

$$M(x_m, x_n, t) >_L \mathcal{N}(\varepsilon).$$

The sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be *convergent* to $x \in X$ in the L-fuzzy metric space (X, M, Δ) if $M(x_n, x, t) = M(x, x_n, t) \to 1_L$ whenever $n \to \infty$ for every $t > 0$. A L-fuzzy metric space is said to be *complete* if and only if every Cauchy sequence is convergent.

Definition 2.3.12. [41] Let (X, M, Δ) be an L-fuzzy metric space. M is said to be *continuous* on $X \times X \times (0, \infty)$ if

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t).$$

whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X \times X \times (0, \infty)$ converges to a point $(x, y, t) \in X \times X \times (0, \infty)$, that is, $\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(y_n, y, t) = 1_L$ and $\lim_{n\to\infty} M(x, y, t_n) = M(x, y, t)$.

Lemma 2.3.13. [41] Let (X, M, Δ) be an L-fuzzy metric space. Then, M is continuous function on $X \times X \times (0, \infty)$.

Definition 2.3.14. [41] Let T be self maps of an L-fuzzy metric space (X, M, Δ). T is said to be *weakly sequential continuous* if $x_n \to x$, then $T(x_n) \to T(x)$, whenever $x \in X$ and $\{x_n\}$ is a sequence in X.

Definition 2.3.15. [18] Let (T, G) be two self maps of an L-fuzzy metric space (X, M, Δ). (T, G) is said to be *compatible* if $\lim_{n\to\infty} M(T(G(x_n)), G(T(x_n)), t) = 1_L$ for all $t > 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} G(x_n) = u$ for some $u \in X$.

Definition 2.3.16. [16] Let (X, M, Δ) be an L-fuzzy metric space. A element $u \in X$ is called a *coincidence point* of T and G if $T(u) = G(u)$. Furthermore, if $T(u) = G(u) = u$, then we say that u is a *common fixed point* of T and G.

Definition 2.3.17. [19] Let (T, G) be two self maps of an L-fuzzy metric space (X, M, Δ). (T, G) is said to be *weakly compatible* if they commute at their coincidence point, that is if $T(u) = G(u)$ for some $u \in X$, then $T(G(u)) = G(T(u))$.

Definition 2.3.18. [41] A *partially ordered L-fuzzy metric space* is a quadruple $(X, M, \Delta, \preceq)$ such that (X, M, Δ) is a L-fuzzy metric space and $\preceq$ is a partial order on X.

2.4 ϕ-Contractions

This section discusses improvement of conditions ϕ-contractions including some properties of ϕ-contractions.

Definition 2.4.1. [47] Let (X, M, Δ) be a fuzzy metric space. A mapping $T : X \to X$ is called *k-contraction* if there exists a constant $k \in (0, 1)$ such that

$$M(T(x), T(y), kt) \leq M(x, y, t)$$

for all $x, y \in X$ and $t > 0$.

The mapping $T : X \to X$ satisfying above condition is usually called a k-contraction. A natural generalization of k-contraction is called *ϕ-contractions*. A mapping $T : X \to X$ is called a ϕ-contractions if it satisfies

$$M(T(x), T(y), \phi(t)) \leq M(x, y, t)$$

for all $x, y \in X$ and $t > 0$, where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$.

Definition 2.4.2. [12] Let X be a topology space and a mapping $f : X \to \mathbb{R}^+$. f is called *upper semi − continuous* at x if for every $\varepsilon > 0$ there exists a neighborhood U of x such that $f(x_n) \leq f(x) + \varepsilon$ for all x_n is sequence in U when $x_n \to x$.

Definition 2.4.3. [12] Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function and $\phi^n(t)$ be the nth iteration of $\phi(t)$. $(\phi - 1), (\phi - 2)$ and $(\phi - 3)$ respectively denote the following conditions:

$(\phi-1)$ ϕ is non-decreasing,

$(\phi-1)'$ ϕ is strictly increasing,

$(\phi-2)$ is upper semi-continuous from the right,

$(\phi-3)$ $\sum_{n=0}^{\infty}\phi^n(t)<+\infty$ for all $t>0$.

We define there classes of functions Φ_0, Φ and Φ_1 as follows:

(1) Φ_0 is the class of all functions ϕ satisfying conditions $(\phi-1)$ and $(\phi-3)$,

(2) Φ is the class of all functions ϕ satisfying conditions $(\phi-1)$-$(\phi-3)$,

(3) Φ_1 is the class of all functions ϕ satisfying conditions $(\phi-1)'$ and $(\phi-3)$.

Remark. [12] Obviously, $\Phi_0\subseteq\Phi$ and $\Phi_1\subseteq\Phi$. If $\phi\in\Phi_0$, then $\phi(t)<t$ for all $t>0$.

Of course, in the definitions above it is very strong and difficult for testing in practice. As a result, the following ϕ-contractions have been updated.

Definition 2.4.4. [20] Let Φ' denote the family of all function $\phi:\mathbb{R}^+\to\mathbb{R}^+$ verifying the condition, for each $t>0$ such that $0<\phi(t)<t$ and $\lim_{n\to\infty}\phi^n(t)=0$. Will this say conditions of Φ' that (CBW).

In order to weaken the condition (CBW) further, introduced the following condition.

Definition 2.4.5. [45] Let Φ_W denote the family of all function $\phi:\mathbb{R}^+\to\mathbb{R}^+$ verifying the condition, for each $t>0$ there exists $r\geq t$ such that $\lim_{n\to\infty}\phi^n(r)=0$. In is evident that condition Φ' (or CBW) implies Φ_W. However, the following example show that the reverse is not true in general. Hence $\Phi'\subseteq\Phi_W$.

Example 2.4.6. [11] Let the function $\phi:\mathbb{R}^+\to\mathbb{R}^+$ be defined by

$$\phi(t)=\begin{cases}\frac{t}{1+t}, & \text{if } 0\leq t<1,\\ -\frac{t}{3}+\frac{4}{3}, & \text{if } 1\leq t\leq 2,\\ t-\frac{4}{3}, & \text{if } 2<t<\infty.\end{cases}$$

Notice that $\phi\in\Phi_W$ but $\phi\notin\Phi'$.

Definition 2.4.7. [46] Let Φ_{w^*} denote the family of all function $\phi:\mathbb{R}^+\to\mathbb{R}^+$ satisfying the condition, for each $t_1,t_2>0$ there exists $r\geq\max\{t_1,t_2\}$ and $N\in\mathbb{N}$ such that $\phi^n(r)<\min\{t_1,t_2\}$ for all $n>N$.

Obviously, the condition of Φ_{w^*} implies condition for each $t > 0$ there exists $r \geq t$ and $N \in \mathbb{N}$ such that $\phi^n(r) < t$ for all $n > N$.

It is easy to see that for each $\phi \in \Phi_W, \phi \in \Phi_{w^*}$. In fact, if $\phi \in \Phi_W$, then for each $t_1, t_2 > 0$, there exist $r_1 \geq t_1$ and $r_2 \geq t_2$ such that $\lim_{n\to\infty} \phi^n(r_1) = \lim_{n\to\infty} \phi^n(r_2) = 0$. Assume that $t_1 \leq t_2$. Then there exists $N \in \mathbb{N}$ such that $\phi^n(r_2) < t_1$ for all $n > N$. Thus $\phi \in \Phi_{w^*}$. However, if $\phi \in \Phi_{w^*}$, then it is unnecessary that $\phi \in \Phi_W$.

Example 2.4.8. [46] Let the function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\phi(t) = t$ for all $t \in [0,1]$, $\phi(t) = t - 1$ for all $t \in (1, \infty)$. Then $\phi \in \Phi_{w^*}$. In fact, for each $t_1, t_2 \in (0, \infty)$, there exists $N \in \mathbb{N}$ such that $r = 1+N+\varepsilon > \max\{t_1, t_2\}$, where $\varepsilon \in (0, \min\{t_1, t_2, 1\})$. Then we have $\phi^n(r) = \varepsilon < \min\{t_1, t_2\}$ for all $n > N + 1$. So $\phi \in \Phi_{w^*}$. However, since $\lim_{n\to\infty} \phi^n(r) \neq 0$ for all $r > 0$, $\phi \notin \Phi_W$.

Remark. [46] From Example 2.4.8 we see tat Φ_{w^*} is a proper subfamily of Φ_W. On Φ_{w^*}, Φ_W and Φ', we have $\Phi' \subseteq \Phi_W \subseteq \Phi_{w^*}$.

Lemma 2.4.9. [46] Let $\phi \in \Phi_{w^*}$. Then for each $t > 0$, there exists $r \geq t$ such that $\phi(r) < t$.

CHAPTER 3

COMMON FIXED POINTS ON L-FUZZY METRIC SPACE

In this section, we study the concept and get results of fixed points in L-fuzzy metric space and partially ordered L-fuzzy metric space.

3.1 L-fuzzy metric space

In this section, we get results of fixed points in L-fuzzy metric space.

Lemma 3.1.1. Let (X, M, Δ) be a $\mathcal{M}L$-fuzzy metric space and $x, y \in X$. If there exists a function $\phi \in \Phi_{w^*}$ such that

$$M(x, y, \phi(t)) \geq_L M(x, y, t), \tag{3.1}$$

for all $t > 0$, then $x = y$.

Proof. For all $t > 0$, we give function $\phi \in \Phi_{w^*}$ such that $M(x, y, \phi(t)) \geq_L M(x, y, t)$ and since (3.1) implies that $\phi(t) > 0$ it follows that $\phi^m(t) > 0$ for all $m \in \mathbb{N}$. By induction, suppose that $M(x, y, \phi^m(t)) \geq_L M(x, y, t)$ for some $m \in \mathbb{N}$, then,

$$\begin{aligned} M(x, y, \phi^{m+1}(t)) &= M(x, y, \phi(\phi^m(t))) \\ &\geq_L M(x, y, \phi^m(t)) \\ &\geq_L M(x, y, t). \end{aligned}$$

We will that

$$M(x, y, \phi^m(t)) \geq_L M(x, y, t), \tag{3.2}$$

for all $t > 0$ and $m \in \mathbb{N}$.

Next we show that $M(x, y, t) = 1_L$ for all $t > 0$, by assume that $M(x, y, t) <_L 1_L$. In fact, if there exists $t_0 > 0$ such that $M(x, y, t_0) <_L 1_L$, then since X is a $\mathcal{M}L$-fuzzy metric space, it follows that $\lim_{t\to\infty} M(x, y, t) = 1_L$ there exists $t_1 > t_0$ such that $M(x, y, t) >_L M(x, y, t_0)$ for all $t \geq t_1$.

Since $\phi \in \Phi_{w^*}$, there exists $t_2 \geq \max\{t_0, t_1\}$ and $n_0 \in \mathbb{N}$ such that $\phi^m(t_2) <$

$\min\{t_0, t_1\}$ for all $m > n_0$. By Lemma 2.3.10 and (3.2), it follows that for each $m > n_0$,

$$\begin{aligned} M(x,y,t_0) &\geq_L M(x,y,\phi^m(t_2)) \\ &\geq_L M(x,y,t_2) \\ &\geq_L M(x,y,t_1) \\ &>_L M(x,y,t_0). \end{aligned}$$

It is a contradiction. Therefore $M(x,y,t) = 1_L$ for all $t > 0$, that is $x = y$. □

Lemma 3.1.2. Let (X, M, Δ) be an L-fuzzy metric space and $x, y \in X$. Let $n \geq 1, g_1, g_2, \ldots, g_n$ be self maps on complete lattice $(L, \leq_L)$ and for some $\phi \in \Phi_{w^*}$,

$$M(x,y,\phi(t)) \geq_L \Delta_M\{g_1(t), g_2(t), \ldots, g_n(t), M(x,y,t)\} \text{ for all } t > 0.$$

Then

$$M(x,y,\phi(t)) \geq_L \Delta_M\{g_1(t), g_2(t), \ldots, g_n(t)\}$$

for all $t > 0$.

Proof. If $\Delta_M\{g_1(t), g_2(t), \ldots, g_n(t), M(x,y,t)\} <_L M(x,y,t)$,
then $M(x,y,\phi(t)) \geq_L \Delta_M\{g_1(t), g_2(t), \ldots, g_n(t)\}$ for all $t > 0$.
If $\Delta_M\{g_1(t), g_2(t), \ldots, g_n(t), M(x,y,t)\} = M(x,y,t)$,
then $M(x,y,\phi(t)) \geq_L M(x,y,t)$ for all $t > 0$. By Lemma 3.1.1, we see that $M(x,y,t) = 1_L$ for all $t > 0$. Thus $g_1(t) = g_2(t) = \ldots = g_n(t) = 1_L$ for all $t > 0$. Then $M(x,y,\phi(t)) \geq_L \Delta_M\{g_1(t), g_2(t), \ldots, g_n(t)\}$ for all $t > 0$. □

Lemma 3.1.3. Let (X, M, Δ) be a $\mathcal{M}L$-fuzzy metric space such that Δ is a t-norm of H-type. Let $\{x_n\}$ be a sequence in (X, M, Δ). If there exists a function $\phi \in \Phi_{w^*}$ satisfying;

(i) $\phi(t)$ for all $t > 0$;

(ii) $M(x_n, x_m, \phi(t)) \geq_L M(x_{n-1}, x_{m-1}, t)$ for all $m, n \in \mathbb{N}$ and $t > 0$.

Then $\{x_n\}$ is a Cauchy sequence.

Proof. We proceed with the following steps:

Step 1. We claim that for any $t > 0$,

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1_L. \tag{3.3}$$

Since X is a $\mathcal{M}L$-fuzzy metric space, it follows that $\lim_{t\to\infty} M(x_0, x_1, t) = 1_L$, for any $\varepsilon \in L\backslash\{0_L, 1_L\}$, there exists $t_0 > 0$ such that $M(x_0, x_1, t_0) >_L \mathcal{N}(\varepsilon)$. Since $\phi \in \Phi_{w^*}$, there exist $t_1 \geq \max\{t, t_0\}$ and $n_0 \in \mathbb{N}$ such that $\phi^n(t_1) < \min\{t, t_0\}$ for all $n \geq n_0$. It is evident that (ii) implies that

$$M(x_n, x_{n+1}, \phi(t)) \geq_L M(x_{n-1}, x_n, t) \tag{3.4}$$

for all $n \in \mathbb{N}$ and $t > 0$. It follows from (i) that $\phi^n(t) > 0$ for all $n \in \mathbb{N}$ and $t > 0$. By induction we will that, $M(x_1, x_2, \phi^1(t)) \geq_L M(x_0, x_1, t)$ is true, and suppose that $M(x_n, x_{n+1}, \phi^n(t)) \geq_L M(x_{n-1}, x_n, t)$ is also true for some $n \in \mathbb{N}$ and for all $t > 0$. To show that $M(x_{n+1}, x_{n+2}, \phi^{n+1}(t)) \geq_L M(x_n, x_{n+1}, t)$ for all $t > 0$ and $n \in \mathbb{N}$.

$$\begin{aligned} M(x_{n+1}, x_{n+2}, \phi^{n+1}(t)) &= M(x_{n+1}, x_{n+2}, \phi(\phi^n(t))) \\ &\geq_L M(x_{n+1}, x_{n+2}, \phi^n(t) \\ &\geq_L M(x_n, x_{n+1}, t), \end{aligned}$$

it implies that $M(x_n, x_{n+1}, \phi^n(t)) \geq_L M(x_{n-1}, x_n, t)$ for all $n \in \mathbb{N}$ and $t > 0$. So, we have

$$M(x_n, x_{n+1}, \phi^n(t)) \geq_L M(x_0, x_1, t) \tag{3.5}$$

for all $n \in \mathbb{N}$ and $t > 0$. So, by (3.5) and Lemma 2.3.10 we have

$$\begin{aligned} M(x_n, x_{n+1}, t) &\geq_L M(x_n, x_{n+1}, \phi^n(t_1)) \\ &\geq_L M(x_0, x_1, t_1) \\ &\geq_L M(x_0, x_1, t_0) \\ &>_L \mathcal{N}(\varepsilon), \end{aligned}$$

for all $n \geq n_0$ and $t > 0$. Thus, we conclude that (3.3) holds.

Step 2. we claim that for all $t > 0$,

$$M(x_n, x_m, t) \geq_L \Delta^{m-n} M(x_n, x_{n+1}, t - \phi(r)) \tag{3.6}$$

for all $m \geq n+1$, where $r \geq t$. Since $\phi \in \Phi_{w^*}$, by Lemma 2.4.9, for all $t > 0$ there exists $r \geq t$ such that $\phi(r) < t$. Net, we prove by induction, since $M(x_n, x_{n+1}, t) \geq_L M(x_n, x_{n+1}, t - \phi(r)) = \Delta^1 M(x_n, x_{n+1}, t - \phi(r))$, then (3.6) holds for $m = n+1$. suppose now that $M(x_n, x_m, t) \geq_L \Delta^{m-n} M(x_n, x_{n+1}, t - \phi(r))$ holds for some fixed $m \geq n+1$.
By (L-4), (ii) and the monotonicity of Δ, we get

$$\begin{aligned}
M(x_n, x_{n+1}, t) &= M(x_n, x_{n+1}, t - \phi(r) + \phi(r)) \\
&\geq_L \Delta\{M(x_n, x_{n+1}, t - \phi(r)), M(x_{n+1}, x_{m+1}, \phi(r))\} \\
&\geq_L \Delta\{M(x_n, x_{n+1}, t - \phi(r)), M(x_n, x_m, r)\} \\
&\geq_L \Delta\{M(x_n, x_{n+1}, t - \phi(r)), M(x_n, x_m, t)\} \\
&\geq_L \Delta\{M(x_n, x_{n+1}, t - \phi(r)), (\Delta^{m-n} M(x_n, x_{n+1}, t - \phi(r)))\} \\
&= \Delta^{m+1-n} M(x_n, x_{n+1}, t - \phi(r)).
\end{aligned}$$

Thus, we prove that if (3.6) holds for some $m \geq n+1$, then it also holds for $m+1$. We conclude that (3.6) holds for all $m \geq n+1$.
Step 3. We claim that $\{x_n\}$ is a Cauchy sequence. As Δ is a t-norm of H-type, for any $\varepsilon \in L \backslash \{0_L, 1_L\}$ there exists $\delta \in L \backslash \{0_L, 1_L\}$ such that if $a >_L \mathcal{N}(\delta)$, then $\Delta^n(a) >_L \mathcal{N}(\varepsilon)$ for all $n \in \mathbb{N}$. It follows from (3.3) $\lim_{n \to \infty} M(x_n, x_{n+1}, t) = 1_L$ that there exists $n_1 \in \mathbb{N}$ such that

$$M(x_n, x_{n+1}, t - \phi(r)) >_L \mathcal{N}(\delta),$$

for all $n \geq n_1$. So, we have

$$\Delta^{m-n} M(x_n, x_{n+1}, t - \phi(r)) >_L \mathcal{N}(\varepsilon), \tag{3.7}$$

for all $m > n \geq n_1$.

By (3.6) and (3.7), we see that for each $t > 0$ and $\varepsilon \in L \backslash \{0_L\}, M(x_n, x_m, t) >_L \mathcal{N}(\varepsilon)$

for all $m > n \geq n_1$, which implies that $\{x_n\}$ is a Cauchy sequence. □

Theorem 3.1.4. Let (X, M, Δ) be a complete $\mathcal{M}L$-fuzzy metric space with a continuous t-norm Δ of H-type and let P, Q, S and T be self-maps on X. If the following conditions are satisfied:

(i) $T(X) \subseteq Q(X), S(X) \subseteq P(X)$;

(ii) either P or T is weakly sequential continuous;

(iii) (T, P) is compatible and (S, Q) is weakly compatible;

(iv) there exists $\phi \in \Phi_{w^*}$ such that

$$\begin{aligned} M(T(x), S(y), \phi(t)) \geq_L \Delta_M\{&M(P(x), T(x), t), M(Q(y), S(y), t), M(P(x), Q(y), t),\\ &M(Q(y), T(x), \beta t), [M(P(x), \xi, (2-\beta)t) \oplus\\ &M(\xi, S(y), (2-\beta)t)]\} \end{aligned} \tag{3.8}$$

for all $x, y, \xi \in X,\ \beta \in (0, 2)$ and $t > 0$. Then P, Q, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$. From condition (i) there exists $x_1, x_2 \in X$ such that $T(x_0) = Q(x_1) = y_0$ and $S(x_1) = P(x_2) = y_1$. So, for all every $m \in \mathbb{N}_0$, there exists $x_{m+1} \in X$ such that $T(x_m) = Q(x_{m+1}) = y_m$ and $S(x_{m+1}) = P(x_{m+2}) = y_{m+1}$.
Assume that $\phi \in \Phi_{w^*}$ such that (3.8) holds. Putting $x = x_m,\ y = x_{m+1}$ and $\xi = y_m$ in (3.8), we get

$$\begin{aligned} M(y_m, y_{m+1}, \phi(t)) &= M(T(x_m), S(x_{m+1}), \phi(t))\\ &\geq_L \Delta_M\{M(P(x_m), T(x_m), t), M(Q(x_{m+1}), S(x_{m+1}), t),\\ &\quad M(P(x_m), Q(x_{m+1}), t), M(Q(x_{m+1}), T(x_m), \beta t)\\ &\quad [M(P(x_m), \xi, (2-\beta)t) \oplus M(\xi, S(x_{m+1}), (2-\beta)t)]\}\\ &= \Delta_M\{M(y_{m-1}, y_m, t), M(y_m, y_{m+1}, t), M(y_{m-1}, y_m, t),\\ &\quad M(y_m, y_m, \beta t), [M(y_{m-1}, y_m, (2-\beta)t) \oplus M(y_m, y_{m+1},\\ &\quad (2-\beta)t)]\}\\ &\geq_L \Delta_M\{M(y_{m-1}, y_m, t), M(y_m, y_{m+1}, t), M(y_{m-1}, y_m, (2-\beta)\frac{t}{2}),\\ &\quad M(y_m, y_{m+1}, (2-\beta)\frac{t}{2})\}. \end{aligned}$$

Take $\beta \to \infty$, we get

$$M(y_m, y_{m+1}, \phi(t)) \geq_L \Delta_M\{M(y_{m-1}, y_m, t), M(y_m, y_{m+1}, t)\},$$

for all $m \in \mathbb{N}$. By Lemma 3.1.2 we get,

$$M(y_m, y_{m+1}, \phi(t)) \geq_L M(y_{m-1}, y_m, t)$$

for all $t > 0$ and $m \in \mathbb{N}$. It implies that (ii) in Lemma 3.1.3 holds. Obviously, the inequality (3.8) implies that $\phi(t) > 0$ for all $t > 0$. Thus $\{y_m\}$ is a Cauchy sequence in X.

Since X is complete, $\lim_{m\to\infty} y_m = z$ for some $z \in X$, and so

$$\lim_{m\to\infty} T(x_m) = \lim_{m\to\infty} P(x_m) = \lim_{m\to\infty} Q(x_{m+1}) = \lim_{m\to\infty} S(x_{m+1}) = z. \tag{3.9}$$

Now, we prove z is common fixed point of P, Q, S and T.

Case 1. Suppose that P is weakly sequential continuous. By (3.9) we have $P(T(x_m)) \to P(z)$ and $P(P(x_m)) \to P(z)$. Since (T, P) is compatible, we have

$$\lim_{m\to\infty} M(P(T(x_m)), T(P(x_m)), t) = 1_L$$

for all $t > 0$, and we have

$$M(T(P(x_m)), P(z), t) \geq_L \Delta\{M(T(P(x_m)), P(T(x_m)), \frac{t}{2}), M(P(T(x_m)), P(z), \frac{t}{2})\}$$

taking limit $m \to \infty$

$$\lim_{m\to\infty} M(T(P(x_m)), P(z), t) \geq_L M(P(z), P(z), \frac{t}{2}) = 1_L.$$

That is $\lim_{m\to\infty} T(P(x_m)) = P(z)$.

We first prove that z is a common fixed point of T and P. Since $\phi \in \Phi_{w^*}$, putting

$x = P(x_m),\ y = x_{m+1},\ \xi = T(P(x_m))$ and $\beta = 1$ in (3.8), we get

$$\begin{aligned} M(T(P(x_m)), S(x_{m+1}), \phi(t)) \geq_L\ & \Delta_M\{M(P(P(x_m)), T(P(x_m)), t), M(Q(x_{m+1}), \\ & S(x_{m+1}), t), M(P(P(x_m)), Q(x_{m+1}), t), M(Q(x_{m+1}), \\ & T(P(x_m)), t)[M(P(P(x_m)), T(P(x_m)), t) \oplus \\ & M(T(P(x_m)), S(x_{m+1}), t)]\} \\ \geq_L\ & \Delta_M\{M(P(P(x_m)), T(P(x_m)), t), M(Q(x_{m+1}), \\ & S(x_{m+1}), t), M(P(P(x_m)), Q(x_{m+1}), t), M(Q(x_{m+1}), \\ & T(P(x_m)), t), [M(P(P(x_m)), T(P(x_m)), \varepsilon), \\ & M(T(P(x_m)), S(x_{m+1}), t - \varepsilon)]\} \end{aligned}$$

taking limit $m \to \infty$ and $\varepsilon \in (0, t)$

$$\begin{aligned} M(P(z), z, \phi(t)) \geq_L\ & \Delta_M\{M(P(z), P(z), t), M(z, z, t), M(P(z), z, t), \\ & M(z, P(z), t), M(P(z), P(z), \varepsilon), M(P(z), z, t - \varepsilon)\} \\ \geq_L\ & \Delta_M\{M(p(z), z, t), M(P(z), z, t - \varepsilon)\} \\ =\ & M(P(z), z, t - \varepsilon). \end{aligned}$$

Then $M(P(z), z, \phi(t)) \geq_L M(P(z), z, t - \varepsilon)$. Also, letting $\varepsilon \to 0$, we get $M(P(z), z, \phi(t)) \geq_L M(P(z), z, t)$ for all $t > 0$, by Lemma 3.1.1 which implies that $P(z) = z$.
By $\phi \in \Phi_{w^*}$, putting $x = \xi = z,\ y = x_{m+1}$ and $\beta = 1$ in (3.8), we get

$$\begin{aligned} M(T(z), S(x_{m+1}), \phi(t)) \geq_L\ & \Delta_M\{M(P(z), T(z), t), M(Q(x_{m+1}), S(x_{m+1}), t), \\ & M(P(z), Q(x_{m+1}), t), M(Q(x_{m+1}), T(z), t) \\ & [M(P(z), z, t) \oplus M(z, S(x_{m+1}), t)]\} \\ =\ & \Delta_M\{M(z, T(z), t), M(Q(x_{m+1}), S(x_{m+1}), t), \\ & M(z, Q(x_{m+1}), t), M(Q(x_{m+1}), T(z), t) \\ & [M(z, z, t) \oplus M(z, S(x_{m+1}), t)]\} \end{aligned}$$

taking limit $m \to \infty$

$$\begin{aligned} M(T(z), z, \phi(t)) &\geq_L \Delta_M\{M(z, T(z), t), M(z, z, t), M(z, z, t), M(z, T(z), t), \\ &\qquad [M(z, z, t) \oplus M(z, z, t)]\} \\ &= M(T(z), z, t), \end{aligned}$$

for all $t > 0$. By Lemma 3.1.1 which implies that $T(z) = z$. Therefore z is a common fixed point of T and P.

Next, from $T(z) = z$ and (3.9), we can prove that z is also a common fixed point of S and Q. Since $T(X) \subseteq Q(X)$, there exists $v \in X$ such that $z = T(z) = Q(v)$. By $\phi \in \Phi_{w^*}$, putting $x = x_m,\ y = v,\ \xi = z$ and $\beta = 1$ in (3.8), we get

$$\begin{aligned} M(T(x_m), S(v), \phi(t)) &\geq_L \Delta_M\{M(P(x_m), T(x_m), t), M(Q(v), S(v), t), \\ &\qquad M(P(x_m), Q(v), t), M(Q(v), T(x_m), t) \\ &\qquad [M(P(x_m), z, t) \oplus M(z, S(v), t)]\} \\ &\geq_L \Delta_M\{M(P(x_m), T(x_m), t), M(Q(v), S(v), t), \\ &\qquad M(P(x_m), Q(v), t), M(Q(v), T(x_m), t) \\ &\qquad M(P(x_m), z, \varepsilon), M(z, S(v), t - \varepsilon)\} \end{aligned}$$

taking limit $m \to \infty$ and $\varepsilon \in (0, t)$

$$\begin{aligned} M(z, S(v), \phi(t)) &\geq_L \Delta_M\{M(z, z, t), M(Q(v), S(v), t), M(z, Q(v), t), \\ &\qquad M(Q(v), z, t), M(z, z, \varepsilon), M(z, S(v), t - \varepsilon)\} \\ &\geq_L \Delta_M\{M(p(z), z, t), M(P(z), z, t - \varepsilon)\} \\ &= M(z, S(v), t - \varepsilon), \end{aligned}$$

for all $t > 0$ and $\varepsilon \in (0, t)$. Letting $\varepsilon \to \infty$, we have $M(z, S(v), \phi(t)) \geq_L M(z, S(v), t)$ for all $t > 0$. By Lemma 3.1.1 which implies that $S(v) = z$.

So we have $Q(v) = z = S(v)$ is a coincidence point of Q and S. Since (Q, S) is weakly compatible, that is $S(Q(v)) = Q(S(v))$, and so $S(z) = Q(z)$.

By $\phi \in \Phi_{w^*}$, putting $x = x_m,\ y = \xi = z$, and $\beta = 1$ in (3.8), we get

$$\begin{aligned} M(T(x_m), S(z), \phi(t)) \geq_L \Delta_M\{&M(P(x_m), T(x_m), t), M(Q(z), S(z), t),\\ &M(P(x_m), Q(z), t), M(Q(z), T(x_m), t)\\ &[M(P(x_m), z, t) \oplus M(z, S(z), t)]\} \end{aligned}$$

taking limit $m \to \infty$

$$\begin{aligned} M(z, S(z), \phi(t)) \geq_L \Delta_M\{&M(z, z, t), M(Q(z), S(z), t), M(z, Q(z), t), M(Q(z), z, t),\\ &M(z, z, \varepsilon), M(z, S(z), t - \varepsilon)]\} \end{aligned}$$

Letting $\varepsilon \to \infty$, we have $M(z, S(z), \phi(t)) \geq_L M(z, S(z), t)$ for all $t > 0$. By Lemma 3.1.1 which implies that $s(z) = z$. Hence $Q(z) = S(z) = z$. Therefore z is a common fixed point of P, Q, S and T.

Case 2. Suppose that T is weakly sequential continuous. By (3.9), note that $\lim_{m\to\infty} T(x_m) = \lim_{m\to\infty} P(x_m) = z$. We have $T(T(x_m)) \to T(z)$ and $T(P(x_m)) \to T(z)$. Since (T, P) is compatible, we have $\lim_{m\to\infty} M(P(T(x_m)), T(P(x_m)), t) = 1_L$ for all $t > 0$. From this fact, it is easy to prove that $\lim_{m\to\infty} P(T(x_m)) = T(z)$. Since $\phi \in \Phi_{w^*}$, putting $x = T(x_m),\ y = x_{m+1},\ \xi = T(z)$ and $\beta = 1$ in (3.8), we get

$$\begin{aligned} M(T(T(x_m)), S(x_m), \phi(t)) &\geq_L \Delta_M\{M(P(T(x_m)), T(T(x_m)), t), M(Q(x_{m+1}), S(x_{m+1}), t),\\ &\quad M(P(T(x_m)), Q(x_{m+1}), t), M(Q(x_{m+1}), T(T(x_m)), t)\\ &\quad [M(P(T(x_m)), T(z), t) \oplus M(T(z), S(x_{m+1}), t)]\}\\ &\geq_L \Delta_M\{M(P(T(x_m)), T(T(x_m)), t), M(Q(x_{m+1}), S(x_{m+1}), t),\\ &\quad M(P(T(x_m)), Q(x_{m+1}), t), M(Q(x_{m+1}), T(T(x_m)), t)\\ &\quad M(P(T(x_m)), T(z), \varepsilon), M(T(z), S(x_{m+1}), t - \varepsilon)\} \end{aligned}$$

for all $t > 0$ and $\varepsilon \in (0, t)$. Letting $m \to \infty$, we get

$$\begin{aligned} M(T(z), z, \phi(t)) \geq_L \Delta_M\{&M(T(z), T(z), t), M(z, z, t), M(T(z), z, t), M(z, T(z), t)\\ &M(T(z), T(z), \varepsilon), M(T(z), z, t - \varepsilon)\}\\ = {}& M(T(z), z, t - \varepsilon). \end{aligned}$$

Letting $\varepsilon \to 0$. if follows that $M(T(z), z, \phi(t)) \geq_L M(T(z), z, t)$ for all $t > 0$, which implies that $T(z) = z$. In Case 1, from $T(z) = z$, it is not difficult to prove that $S(z) = Q(z) = z$. In the following, we to show that $P(z) = z$.

Since $S(X) \subseteq P(X)$, there exists $w \in X$ such that $z = S(z) = P(w)$. By $\phi \in \Phi_{w^*}$, putting $x = w,\ y = x_{m+1},\ \xi = z$ and $\beta = 1$ in (3.8), we get

$$\begin{aligned} M(T(w), S(x_{m+1}), \phi(t)) \geq_L \Delta_M\{&M(P(w), T(w), t), M(Q(x_{m+1}), S(x_{m+1}), t),\\ &M(P(w), Q(x_{m+1}), t), M(Q(x_{m+1}), T(w), t)\\ &[M(P(w), z, t) \oplus M(z, S(x_{m+1}), t)]\} \end{aligned}$$

letting $m \to \infty$, we get

$$\begin{aligned} M(T(w), z, \phi(t)) &\geq_L \Delta_M\{M(z, T(w), t), M(z, z, t), M(z, z, t), M(z, T(w), t)\\ &\qquad [M(z, z, t) \oplus M(z, z, t)]\}\\ &= M(T(w), z, t) \end{aligned}$$

for all $t > 0$, which implies that $T(w) = z = P(w)$. Note that (T, P) is compatible, and so it is also weakly compatible. Hence $P(z) = P(T(w)) = T(P(w)) = T(z) = z$. This show that z is a common fixed point of P, Q, S and T.

Finally, we show the uniqueness of common fixed point. Let $z^{'}$ be another common fixed point of P, Q, S and T. Then $T(z^{'}) = S(z^{'}) = P(z^{'}) = Q(z^{'}) = z^{'}$. Thus, by $\phi \in \Phi_{w^*}$, putting $x = \xi = z,\ y = z^{'}$ and $\beta = 1$ in (3.8), we get

$$\begin{aligned} M(z, z^{'}, \phi(t)) &= M(T(z), S(z^{'}), \phi(t))\\ &\geq_L \Delta_M\{M(P(z), T(z), t), M(Q(z^{'}), S(z^{'}), t), M(P(z), Q(z^{'}), t),\\ &\qquad M(Q(z^{'}), T(z), t), [M(P(z), z, t) \oplus M(z, S(z^{'}), t)]\}\\ &\geq_L \Delta_M\{M(z, z, t), M(z^{'}, z^{'}, t), M(z, z^{'}, t), M(z^{'}, z, t)\\ &\qquad [M(z, z, t) \oplus M(z, z^{'}, t)]\}\\ &= M(z, z^{'}, t) \end{aligned}$$

for all $t > 0$. This implies that $z = z^{'}$. Therefore, z is a unique common fixed point of P, Q, S and T. □

From the above Theorem 3.1.4 we will see that conditions (ii), if P or T is continuous it implies that P or T is weakly sequential continuous but on the other hand is not true. For conditions (iv) clearly $\Phi_W \subseteq \Phi_{w^*}$ so we will have corollary as follows.

Corollary 3.1.5. Let (X, M, Δ) be a complete $\mathcal{M}$-fuzzy metric space with a continuous t-norm Δ of H-type and let P, Q, S and T be self-maps on X. If the following conditions are satisfied:

(i) $T(X) \subseteq Q(X), S(X) \subseteq P(X)$;

(ii) either P or T is continuous;

(iii) (T, P) is compatible and (S, Q) is weakly compatible;

(iv) there exists $\phi \in \Phi$ or $\phi \in \Phi_1$ such that

$$\begin{aligned} M(T(x), S(y), \phi(t)) \geq \Delta_M\{&M(P(x), T(x), t), M(Q(y), S(y), t), M(P(x), Q(y), t), \\ &M(Q(y), T(x), \beta t), [M(P(x), \xi, (2-\beta)t) \oplus \\ &M(\xi, S(y), (2-\beta)t)]\} \end{aligned}$$

for all $x, y, \xi \in X$, $\beta \in (0, 2)$ and $t > 0$. Then P, Q, S and T have a unique common fixed point in X.

3.2 Partially ordered L-fuzzy metric space

In this section, we study the concept of partially ordered and get results of fixed points in partially ordered L-fuzzy metric space.

Definition 3.2.1. [44] Suppose that $(X, \preceq)$ is a partially ordered set and $T, G : X \to X$ are mappings of X into itself. We say that T is $G-$nondecreasing if for $x, y \in X$,

$$G(x) \preceq G(y) \Rightarrow T(x) \preceq T(y).$$

Definition 3.2.2. [8] A triple $(X, \tau, \preceq)$ is called a partially ordered topological space if τ is a Hausdorff topology on X and $\preceq$ is a partial order on X. A partially ordered topological space $(X, \tau, \preceq)$ is said to have the sequential g-monotone paroperty if it verifies:

(1) If $\{x_m\}$ is a nondecrasing sequence and $\{x_m\} \to x$, then $g(x_m) \preceq g(x)$ for all m.

(2) If $\{y_m\}$ is a nonincreasing sequence and $\{y_m\} \to y$, then $g(y_m) \succeq g(y)$ for all m.

If g is the identity mapping, then X is said to have the sequential monotone property.

Theorem 3.2.3. Let $(X, M, \Delta, \preceq)$ be a complete partially ordered $\mathcal{M}L$-fuzzy metric space such that Δ is a t-norm of H-type. Let T and G be self-maps on X. such that T is G-nondecreasing mapping and $T(X) \subseteq G(X)$. Assume that exists $\phi \in \Phi_{w^*}$ such that, for all $t > 0$ and $y, x \in X$ with $G(y) \preceq G(x)$,

$$M(T(y), T(x), \phi(t)) \geq_L M(G(y), G(x), t). \tag{3.10}$$

Also suppose that either

(a) G is weakly sequential continuous and (T, G) is compatible or

(b) $(X, \tau_M, \preceq)$ has the sequential monotone property and $G(X)$ is closed. If there exists $y_0 \in X$ such that $G(y_0) \asymp T(y_0)$. Then T and G have a coincidence point.

Proof. Let $y_0 \in X$ such that $G(y_0) \asymp T(y_0)$. Since $T(X) \subseteq G(X)$, there exists $y_1 \in X$ such that $G(y_1) = T(y_0)$. So, for every $m \in \mathbb{N}_0$, there exists $y_{m+1} \in X$ such that $G(y_{m+1}) = T(y_m)$. Set $z_0 = G(y_0)$ and $z_{m+1} = G(y_{m+1}) = T(y_m)$ for every $m \in \mathbb{N}_0$ Since $G(y_0) \asymp T(y_0)$, suppose that $G(y_0) \preceq T(y_0)$, that is $z_0 \preceq z_1$ ($G(y_0) \succeq T(y_0)$ is treated similarly).

To show that $\{z_m\}$ is nondecrasing by induction, we get assume that $z_{m-1} \preceq z_m$ for some $m \in \mathbb{N}_0$, that is $G(y_{m-1}) \preceq G(y_m)$. Since T is G-nondecreasing mapping, we get $z_m = T(y_{m-1}) \preceq T(y_m) = z_{m+1}$. Thus $z_{m-1} \preceq z_m$ holds for all $m \in \mathbb{N}_0$.

Hence the sequence $\{z_m\}$ is nondecreasing. By (3.10) and monotonicity of $\{z_m\}$, we get

$$\begin{aligned} M(z_n, z_m, \phi(t)) &= M(T(y_{n-1}), T(y_{m-1}), \phi(t)) \\ &\geq_L M(G(y_{n-1}), G(y_{m-1}), \phi(t)) \\ &= M(z_{n-1}, z_{m-1}, t) \end{aligned}$$

for all $m, n \in \mathbb{N}$ and $t > 0$. Obviously, the inequality (3.10) implies that $\phi(t) > 0$ for

all $t > 0$ So, by Lemma 3.1.3 we will that $\{z_m\}$ is a Cauchy sequence.
Now, suppose that the condition (a) holds.
Since $(X, M, \Delta, \preceq)$ is complete, there exists $z \in X$ such that $\lim_{m\to\infty} z_m = z$, that is

$$\lim_{m\to\infty} T(y_m) = \lim_{m\to\infty} G(y_m) = z. \tag{3.11}$$

Since T and G are compatible, we have

$$\lim_{m\to\infty} M(G(G(y_{m+1})), T(G(y_m)), t) = \lim_{m\to\infty} M(G(T(y_m)), T(G(y_m)), t) = 1_L \tag{3.12}$$

for all $t > 0$. Since G is weakly sequential continuous, we get

$$\lim_{m\to\infty} G(G(y_{m+1}) = G(z). \tag{3.13}$$

By Lemma 2.3.13, we find M is a continuous function on $X \times X \times (0, \infty)$. By, the continuous of M and (3.11)-(3.13), we have

$$1_L = \lim_{m\to\infty} M(G(G(y_{m+1}), T(G(y_m)), t) = M(G(z), T(z), t)$$

for all $t > 0$. which implies that $T(z) = G(z)$ and z is a coincidence point of T and G. Now, suppose that the condition (b) hold.
Since $(X, M, \Delta, \preceq)$ is complete, there exists $y \in X$ such that

$$\lim_{m\to\infty} T(y_m) = \lim_{m\to\infty} G(y_m) = y.$$

Since $G(X)$ is closed, we give $y = G(z)$ for some $z \in X$. Thus

$$\lim_{m\to\infty} T(y_m) = \lim_{m\to\infty} G(y_m) = G(z).$$

Since $(X, \tau_M, \preceq)$ has the sequential monotone property, we have $G(y_m) \preceq G(z)$ for all $m \in \mathbb{N}_0$. Since $\phi \in \Phi_{w^*}$, by Lemma 2.4.9, for each $t > 0$ there exists $r \geq t$ such that $\phi(r) < t$. So, by (3.10) and the monotonicity of $M(x, y, t)$ with respect to t from

Lemma 2.3.10, we have

$$\begin{aligned} M(T(y_m), T(z), t) &\geq_L M(T(y_m), T(z), \phi(r)) \\ &\geq_L M(G(y_m), G(z), r) \\ &\geq_L M(G(y_m), G(z), t). \end{aligned}$$

for all $t > 0$ and $m \in \mathbb{N}_0$. taking limit $m \to \infty$.

$$\lim_{m\to\infty} M(T(y_m), T(z), t) \geq_L \lim_{m\to\infty} M(G(y_m), G(z), t) = 1_L,$$

that is $\lim_{m\to\infty} T(y_m) = T(z)$ for all $t > 0$ and $m \in \mathbb{N}_0$. Since $G(z) = \lim_{m\to\infty} T(y_m) = T(z)$, we conclude that $G(z) = T(z)$ and z is a coincidence point of T and G. □

Theorem 3.2.4. In addition to the hypotheses of Theorem 3.2.3. Suppose that for all coincidence point $y, v \in X$ of mapping T and G there exists $u \in X$ such that $G(u)$ is comparable to $G(y)$ and $G(v)$. Also suppose that (G, T) is weakly compatible if assume (b) holds. Then T and G have a unique common fixed pint.

Proof. Let $u \in X$, put $u_0 = u$ and define a sequence $\{G(u_m)\}$ by $G(u_{m+1}) = T(u_m)$ for all $m \in \mathbb{N}$. We have $G(y) \preceq G(u_0)$. Since T is a G-nondecreasing mapping, we have $G(y) = T(y) \preceq T(u_0) = G(u_1)$. By induction we obtain $G(y) \preceq G(u_m)$ for all $m \in \mathbb{N}_0$. Since $\lim_{t\to\infty} M(G(u), G(y), t) = 1_L$, for any $\varepsilon \in L\backslash\{0_L, 1_L\}$, there exists $t_2 > 0$ such that $M(G(u_0), G(y), t_2) >_L \mathcal{N}(\varepsilon)$.
Since $\phi \in \Phi_{w^*}$ there exists $t_3 \geq t_2$ such that $\lim_{m\to\infty} \phi^m(t_3) = 0$, from Lemma 2.4.9 Thus, for each $t > 0$ there exists $m_0 \in \mathbb{N}$ such that $\phi^m(t_3) < t$ for all $m \geq m_0$. So, by (3.10) and the monotonicity of $M(x, y, t)$ with respect to t from Lemma 2.3.10, we

get for all $m \geq m_0$ and $t > 0$,

$$\begin{aligned} M(G(u_m), G(y), t) &\geq_L M(G(u_m), G(y), \phi^m(t_3)) \\ &= M(T(u_{m-1}), T(y), \phi^m(t_3)) \\ &\geq_L M(G(u_{m-1}), G(y), \phi^{m-1}(t_3)) \\ &\cdots \\ &\geq_L M(G(u_0), G(y), t_3) \\ &\geq_L M(G(u_0), G(y), t_2) \\ &>_L \mathcal{N}(\varepsilon). \end{aligned}$$

We deduce that $\lim_{m\to\infty} M(G(u_m), G(y), t) = 1_L$. That is $\lim_{m\to\infty} G(u_m) = G(y)$. Similarly, we find that $\lim_{m\to\infty} G(u_m) = G(v)$. The uniqueness of the limit prove that $G(y) = G(v)$.
Case 1. suppose that the condition (a) holds.
Denote $w \in X$ which setting $w = G(y) = T(y)$. By (3.11) we know that $\lim_{m\to\infty} G(y_m) = \lim_{m\to\infty} T(y_m) = y$. That is $G(y_m) \preceq y$ using (3.10), we get

$$\begin{aligned} M(T(y_m), T(y), \phi(t)) &\geq_L M(G(y_m), G(y), t) \\ M(y, w, \phi(t)) &\geq_L M(y, w), t). \end{aligned}$$

By Lemma 3.1.1, we get $y = w$ Thus w is also a coincidence point and $w = G(w) = T(w)$ is a fixed point of T and G.
Nets, we show the uniqueness of common fixed point. Let $w^{'}$ be another common fixed point of G and T. Then $w^{'} = G(w^{'}) = G(w) = w$. Therefore, w is a unique common fixed point of G and T.
Case 2. suppose that the condition (b) holds and let (G, T) be weakly compatible
Denote $w \in X$ which setting $w = G(y) = T(y)$. Since (G, T) be weakly compatible, we have $T(w) = T(G(y)) = G(T(y)) = G(w)$. So, w is also a coincidence point of T and G. Therefore $G(w) = G(y) = w$. So, w is a common fixed point of T and G.
Finally, we show the uniqueness of common fixed point. Let $w^{'}$ be another common fixed point of G and T. Then $w^{'} = G(w^{'}) = G(w) = w$. This completes the prove. □

Clearly $\Phi_W \subseteq \Phi_{w^*}$ and if lattices $L = [0, 1]$ in Theorems 3.2.3 and 3.2.4, then the following corollary is obtained immediately.

Corollary 3.2.5. Let $(X, M, \Delta, \preceq)$ be a complete partially ordered $\mathcal{M}$-fuzzy metric space such that Δ is a t-norm of H-type. Let T and G be self-maps on X. such that T is G-nondecreasing mapping and $T(X) \subseteq G(X)$. Assume that exists $\phi \in \Phi_w$ such that, for all $t > 0$ and $y, x \in X$ with $G(y) \preceq G(x)$,

$$M(T(y), T(x), \phi(t)) \geq M(G(y), G(x), t).$$

Also suppose that either

(a) T and G is continuous and compatible and $M(x, y, \cdot) : \mathbb{R}^+ \to \mathbb{I}$ is continuous

or

(b) $(X, \tau_M, \preceq)$ has the sequential monotone property and $G(X)$ is closed. If there exists $y_0 \in X$ such that $G(y_0) \asymp T(y_0)$. Then T and G have a coincidence point.

Corollary 3.2.6. In addition to the hypotheses of Corollary 3.2.5 Suppose that for all coincidence point $y, v \in X$ of mapping T and G there exists $u \in X$ such that $G(u)$ is comparable to $G(y)$ and $G(v)$. Also suppose that (G, T) be weakly compatible if assume (b) holds. Then T and G have a unique common fixed pint.

CHAPTER 4

MINIMAL STRUCTURE SPACE

4.1 L-fuzzy minimal structure space

This section discusses basic properties of L-fuzzy minimal structure space and basic concepts of closure and interior.

Definition 4.1.1. [27] Let X be a non-empty set and $P(X)$ be the power set of X. A subfamily m_X of $P(X)$ is called a *minimal structure* (briefly $m-structure$) on X if $\emptyset \in m_X$ and $X \in m_X$.

The pair (X, m_X) is called a *minimal structure space* (briefly $m-space$). Each member of m_X is said to be m_X-open and the complement of an m_X-open set is said to be m_X-closed set.

Definition 4.1.2. [5] Let A and B be a fuzzy sets. We denote

(1) $A \subseteq B \Leftrightarrow A(x) \leq B(x)$ for all $x \in X$,

(2) $(\bigcup_{\alpha\in\Lambda} A_\alpha)(x) = \sup_{\alpha\in\Lambda} A_\alpha(x)$ for all $x \in X$,

(3) $(\bigcap_{\alpha\in\Lambda} A_\alpha)(x) = \inf_{\alpha\in\Lambda} A_\alpha(x)$ for all $x \in X$.

Let I be the unit interval $[0, 1]$ of the real number line. A member A of I^X is called a fuzzy set of X. By $\widetilde{0}$ and $\widetilde{1}$ we denote constant maps on X with value 0 and 1, respectively. For any $A \in I^X$, A^C denotes the complement $\widetilde{1} - A$.

Definition 4.1.3. Let A and B be an L-fuzzy sets. We denote

(1) $A \leq B \Leftrightarrow A(x) \leq_L B(x)$ for all $x \in X$,

(2) $(\bigvee_{\alpha\in\Lambda} A_\alpha)(x) = \sup_{\alpha\in\Lambda} A_\alpha(x)$ for all $x \in X$,

(3) $(\bigwedge_{\alpha\in\Lambda} A_\alpha)(x) = \inf_{\alpha\in\Lambda} A_\alpha(x)$ for all $x \in X$.

Let L be a lattice. A member A of L^X is called a L-fuzzy set of X. By $\widetilde{\emptyset}$ and $\widetilde{X}$ we denote constant maps on X with 0_L and 1_L, respectively. For any $A \in L^X$, A^C denotes the complement $\widetilde{X} - A$. All other notations are standard notations of L-fuzzy set theory.

Definition 4.1.4. Let X be a non-empty set and $r \in L\backslash\{0_L, 1_L\}$. A L-fuzzy family $\mathfrak{L} : L^X \to L$ on X is said to have a L-fuzzy minimal structure if the family

$$\mathfrak{L}_r = \{A \in L^X : \mathfrak{L}(A) >_L \mathcal{N}(r)\}$$

contains $\widetilde{\emptyset}$ and $\widetilde{X}$.

Then the pair $(X, \mathfrak{L})$ is called a *L-fuzzy minimal structure space*. Every member of $\mathfrak{L}_r$ is called a *L-fuzzy open set*. A fuzzy set A is called a *L-fuzzy closed set* if the complement of A (simply, A^C) is a L-fuzzy open set.

Let $(X, \mathfrak{L})$ be an L-fuzzy minimal structure space and $r \in L\backslash\{0_L, 1_L\}$. The L-fuzzy closure of A, denote by $Cl_{\mathfrak{L}}(A)$, is define as

$Cl_{\mathfrak{L}}(A) = \bigwedge\{B \in L^X : B^C \in \mathfrak{L}_r \text{ and } A \leq B\}$.

$Cl_{\mathfrak{L}}(A)(x) = \inf\limits_{\alpha\in\Lambda}\{B_\alpha(x) \in I^X : B_\alpha^C \in \mathfrak{L}_r \text{ and } A(x) \leq_L B(x)\}$ for all $x \in X$.

The L-fuzzy interior of A, denote by $Int_{\mathfrak{L}}(A)$, is define as

$Int_{\mathfrak{L}}(A) = \bigvee\{B \in L^X : B \in \mathfrak{L}_r \text{ and } B \leq A\}$

$Int_{\mathfrak{L}}(A)(x) = \sup\limits_{\alpha\in\Lambda}\{B_\alpha(x) \in L^X : B_\alpha \in \mathfrak{L}_r \text{ and } B(x) \leq_L A(x)\}$ for all $x \in X$.

Example 4.1.5. Let $([0, 1], \leq_L)$ be a complete lattice and A be an L-fuzzy sets define as follows:

$$A(x) = \begin{cases} x + \frac{1}{2}, & ; 0 \leq x \leq \frac{1}{4}, \\ \frac{1}{3}(x-1) + \frac{1}{2}, & ; \frac{1}{4} \leq x \leq 1. \end{cases}$$

Let us consider a L-fuzzy minimal structure as follows:

$$\mathfrak{L}(\mu) = \begin{cases} \frac{2}{3}, & \text{if} \quad \mu = \widetilde{\emptyset}, \widetilde{X}, \\ 0, & \text{if} \quad \text{otherwise}. \end{cases}$$

Let $r = \frac{2}{3} \Rightarrow \mathcal{N}(r) = 1 - r = \frac{1}{3}$, $\mathfrak{L}_{\frac{2}{3}} = \{A \in L^{([0,1],\leq_L)} : \mathfrak{L}(A) > \frac{1}{3}\}$. $\mathfrak{L}_{\frac{2}{3}} = \{\widetilde{\emptyset}, \widetilde{X}\}$. Then

$(X, \mathfrak{L})$ is L-fuzzy minimal structure spaces. Then $\widetilde{\emptyset}$ and $\widetilde{X}$ are L-fuzzy open and $\widetilde{X} - A$ are L-fuzzy closed sets.

Theorem 4.1.6. Let $(X, \mathfrak{L})$ be an L-fuzzy minimal structure spaces. and $A, B \in L^X$.

(1) $Int_{\mathfrak{L}}(A) \leq A$ and if $A \in \mathfrak{L}_r$, then $Int_{\mathfrak{L}}(A) = A$.

(2) $A \leq Cl_{\mathfrak{L}}(A)$ and if $A^C \in \mathfrak{L}_r$, then $Cl_{\mathfrak{L}}(A) = A$.

(3) If $A \leq B$, then $Int_{\mathfrak{L}}(A) \leq Int_{\mathfrak{L}}(B)$ and $Cl_{\mathfrak{L}}(A) \leq Cl_{\mathfrak{L}}(B)$.

(4) $Int_{\mathfrak{L}}(A) \wedge Int_{\mathfrak{L}}(B) \geq Int_{\mathfrak{L}}(A \wedge B)$ and $Cl_{\mathfrak{L}}(A) \vee Cl_{\mathfrak{L}}(B) \leq Cl_{\mathfrak{L}}(A \vee B)$.

(5) $Int_{\mathfrak{L}}(Int_{\mathfrak{L}}(A)) = Int_{\mathfrak{L}}(A)$ and $Cl_{\mathfrak{L}}(Cl_{\mathfrak{L}}(A)) = Cl_{\mathfrak{L}}(A)$.

(6) $\widetilde{X} - Cl_{\mathfrak{L}}(A) = Int_{\mathfrak{L}}(\widetilde{X} - A)$ and $\widetilde{X} - Int_{\mathfrak{L}}(A) = Cl_{\mathfrak{L}}(\widetilde{X} - A)$.

Proof. (1) Let B_α be an $\mathcal{L}$-fuzzy open set such that $B_\alpha \leq A$ for all $\alpha \in \Lambda$. Then, for any $x \in X$, $B_\alpha(x) \leq_L A(x)$ for all $\alpha \in \Lambda$. Thus $(\bigvee_{\alpha\in\Lambda} B_\alpha)(x) = \sup_{\alpha\in\Lambda} B_\alpha(x) \leq_L A(x)$ for all $x \in X$. This implies that $(Int_{\mathfrak{L}}(A))(x) \leq_L A(x)$ for all $x \in X$.
Hence $Int_{\mathfrak{L}}(A) \leq A$.
Next We show that $Int_{\mathfrak{L}}(A) = A$. Let $A \in \mathfrak{L}_r$. Then $A \in \{B_\alpha \in L^X : B_\alpha \in \mathfrak{L}_r$ and $B_\alpha \leq A$ for all $\alpha \in \Lambda\}$. Thus for any $x \in X$, $A(x) \in \{B_\alpha(x) : B_\alpha \leq A$ for all $\alpha \in \Lambda\}$. Thus $A(x) \leq_L \sup_{\alpha\in\Lambda}\{B_\alpha(x) : B_\alpha \leq A\} = (\bigvee_{\alpha\in\Lambda} B_\alpha)(x)$ and so $A \leq Int_{\mathfrak{L}}(A)$. Since $Int_{\mathfrak{L}}(A) \leq A$, we get that $Int_{\mathfrak{L}}(A) = A$.
(2) Let B_α be an $\mathcal{L}$-fuzzy closed such that $A \leq B_\alpha$ for all $\alpha \in \Lambda$. Then, for any $x \in X$, $A(x) \leq_L B_\alpha(x)$ for all $\alpha \in \Lambda$. Thus $(\bigwedge_{\alpha\in\Lambda} B_\alpha)(x) = \inf_{\alpha\in\Lambda} B_\alpha(x) \geq_L A(x)$ for all $x \in X$. This implies that $A(x) \leq_L (Cl_{\mathfrak{L}}(A))(x)$ for all $x \in X$. Hence $A \leq Cl_{\mathfrak{L}}(A)$.
Next we show that if $A^C \in \mathfrak{L}_r$, then $Cl_{\mathfrak{L}}(A) = A$. Let $A^C \in \mathfrak{L}_r$. Then $A \in \{B_\alpha \in L^X : B_\alpha^C \in \mathfrak{L}_r$ and $A \leq B_\alpha$ for all $\alpha \in \Lambda\}$. Thus, for any $x \in X$, $A(x) \in \{B_\alpha(x) : B_\alpha^C \in \mathfrak{L}_r$ and $A \leq B_\alpha$ for all $\alpha \in \Lambda\}$. Thus $A(x) \geq_L \inf_{\alpha\in\Lambda}\{B_\alpha(x) : B_\alpha^C \in \mathfrak{L}_r$ and $A \leq B_\alpha\} = (\bigwedge_{\alpha\in\Lambda} B_\alpha)(x)$ and so $Cl_{\mathfrak{L}}(A) \leq A$. Since $A \leq Cl_{\mathfrak{L}}(A)$, we have $Cl_{\mathfrak{L}}(A) = A$.
(3) Let $A \leq B$, then $A(x) \leq_L B(x)$ for all $x \in X$. Let $B_\beta \in \mathfrak{L}_r$ such that $B_\beta \leq A$ for all $\beta \in \Lambda$. Since $A \leq B$, we have $B_\beta \leq B$ for all $\beta \in \Lambda$.
Thus $B_\beta \in \{F_\alpha \in L^X : F_\alpha \leq B, F_\alpha \in \mathfrak{L}_r$ for all $\alpha \in \Lambda\}$. So, for any $x \in X$, $B_\beta(x) \leq_L \sup_{\alpha\in\Lambda}\{F_\alpha(x) : F_\alpha \in \mathfrak{L}_r$ and $F_\alpha \leq B\}$. Thus $\sup_{\alpha\in\Lambda}\{F_\alpha(x) : F_\alpha \in \mathfrak{L}_r$ and $F_\alpha \leq B\}$ is an upper bound of $\{B_\beta(x) : B_\beta \in \mathfrak{L}_r$ and $B_\beta \leq A$ for all $\beta \in \Lambda\}$. Hence $\sup_{\beta\in\Lambda}\{B_\beta(x) : B_\beta \in \mathfrak{L}_r$ and $B_\beta \leq A\} \leq_L \sup_{\alpha\in\Lambda}\{F_\alpha(x) : F_\beta \in \mathfrak{L}_r$ and $F_\beta \leq B\}$ for all $x \in X$. This implies that $Int_{\mathfrak{L}}(A) \leq Int_{\mathfrak{L}}(B)$.
Next we show that $Cl_{\mathfrak{L}}(A) \leq Cl_{\mathfrak{L}}(B)$. Let $A \leq B$, then $A(x) \leq_L B(x)$ for all $x \in X$. Let B_β be an $\mathcal{L}$-fuzzy closed such that $A \leq B_\beta$ for all $\beta \in \Lambda$.

Since $A \leq B$, we have $B \leq B_\beta$ for all $\beta \in \Lambda$. Thus $B_\beta \in \{F_\alpha \in L^X : B \leq F_\alpha, F_\alpha^C \in \mathfrak{L}_r \}$. So, for any $x \in X$, $B_\beta(x) \geq_L \inf\limits_{\alpha \in \Lambda}\{F_\alpha(x) : F_\alpha^C \in \mathfrak{L}_r$ and $F_\alpha \leq B\}$.
Thus $\inf\limits_{\alpha \in \Lambda} F_\alpha(x) : F_\alpha^C \in \mathfrak{L}_r$ and $F_\alpha \leq B\}$ is a lower bound of $\{B_\beta(x) : B_\beta^C \in \mathfrak{L}_r$ and $A \leq B_\beta$ for all $\beta \in \Lambda\}$. and so $\inf\limits_{\beta \in \Lambda}\{B_\beta(x) : B_\beta^C \in \mathfrak{L}_r$ and $A_\alpha \leq B_\beta\} \geq_L \inf\limits_{\alpha \in \Lambda}\{F_\alpha(x) : F_\alpha^C \in \mathfrak{L}_r$ and $F_\alpha \leq B\}$ for all $x \in X$. Hence $Cl_{\mathfrak{L}}(A) \leq Cl_{\mathfrak{L}}(B)$.
(4) Since $A \wedge B \leq A$, $A \wedge B \leq B$ and using (3), we have $Int_{\mathfrak{L}}(A \wedge B) \leq Int_{\mathfrak{L}}(A)$ and $Int_{\mathfrak{L}}(A \wedge B) \leq Int_{\mathfrak{L}}(B)$. Therefore $Int_{\mathfrak{L}}(A \wedge B) \leq Int_{\mathfrak{L}}(A) \wedge Int_{\mathfrak{L}}(B)$.
We show that $Cl_{\mathfrak{L}}(A) \vee Cl_{\mathfrak{L}}(B) \leq Cl_{\mathfrak{L}}(A \vee B)$. Since $A \leq A \vee B$, $B \leq A \vee B$ and by (3), we get that $Cl_{\mathfrak{L}}(A) \leq Cl_{\mathfrak{L}}(A \vee B)$ and $Cl_{\mathfrak{L}}(B) \leq Cl_{\mathfrak{L}}(A \vee B)$. Therefore $Cl_{\mathfrak{L}}(A) \vee Cl_{\mathfrak{L}}(B) \leq Cl_{\mathfrak{L}}(A \vee B)$.
(5) By (1) and (3), we have $Int_{\mathfrak{L}}(Int_{\mathfrak{L}}(A)) \leq Int_{\mathfrak{L}}(A)$. For any $G_\beta \in L^X$ be such that $G_\beta \in \mathfrak{L}_r$ and $G_\beta \leq A$, we have $G_\beta \leq Int_{\mathfrak{L}}(A)$. Thus $G_\beta \leq \bigvee\{G_\alpha : G_\alpha \in \mathfrak{L}_r, G_\alpha \leq Int_{\mathfrak{L}}(A)$ for all $\alpha \in \Lambda\}$. This implies that

$$\begin{aligned} Int_{\mathfrak{L}}(A) &= \bigvee\{G_\beta : G_\beta \in \mathfrak{L}_r, G_\beta \leq A \text{ for all } \beta \in \Lambda\} \\ &\leq \bigvee\{G_\alpha : G_\alpha \in \mathfrak{L}_r, G_\alpha \leq Int_{\mathcal{L}}(A) \text{ for all } \alpha \in \Lambda\} \\ &= Int_{\mathfrak{L}}(Int_{\mathfrak{L}}(A)). \end{aligned}$$

So $Int_{\mathfrak{L}}(A) = Int_{\mathfrak{L}}(Int_{\mathfrak{L}}(A))$. We show that $Cl_{\mathfrak{L}}(Cl_{\mathfrak{L}}(A)) = Cl_{\mathfrak{L}}(A)$.
It follows from (2) and (3), we get that $Cl_{\mathfrak{L}}(A) \leq Cl_{\mathfrak{L}}(Cl_{\mathfrak{L}}(A))$. For any $F_\beta \in L^X$ be such that $F_\beta^C \in \mathfrak{L}_r$ and $A \leq F_\beta$, we have $Cl_{\mathfrak{L}}(A) \leq F_\beta$. Thus $\bigwedge\{F_\alpha : F_\alpha^C \in \mathfrak{L}_r, Cl_{\mathfrak{L}}(A) \leq F_\alpha$ for all $\alpha \in \Lambda\} \leq F_\beta$. This implies that

$$\begin{aligned} Cl_{\mathfrak{L}}(A) &= \bigwedge\{F_\beta : F_\beta^C \in \mathfrak{L}_r, A \leq F_\beta \text{ for all } \beta \in \Lambda\} \\ &\geq \bigwedge\{F_\alpha : F_\alpha^C \in \mathfrak{L}_r, Cl_{\mathfrak{L}}(A) \leq F_\alpha \text{ for all } \alpha \in \Lambda\} \\ &= Cl_{\mathfrak{L}}(Cl_{\mathfrak{L}}(A)). \end{aligned}$$

So $Cl_{\mathfrak{L}}(A) = Cl_{\mathfrak{L}}(Cl_{\mathfrak{L}}(A))$.

(6) We show that $Cl_{\mathfrak{L}}(\widetilde{X}-A)=\widetilde{X}-Int_{\mathfrak{L}}(A)$. For each $x\in X$ and $\alpha\in\Lambda$, we have

$$\begin{aligned}(\widetilde{X}-Int_{\mathfrak{L}}(A))(x) &= \widetilde{X}(x)-Int_{\mathfrak{L}}(A)(x)\\ &= \widetilde{X}(x)-(\bigvee\{G_\alpha\in L^X : G_\alpha\leq A, G_\alpha\in\mathfrak{L}_r, \alpha\in\Lambda\})(x)\\ &= \widetilde{X}(x)-\sup_{\alpha\in\Lambda}\{G_\alpha(x) : G_\alpha\leq A, G_\alpha\in\mathfrak{L}_r\}\\ &\leq_L \widetilde{X}(x)-G_\alpha(x)\\ &= (\widetilde{X}-G_\alpha)(x).\end{aligned}$$

Thus

$$\begin{aligned}(\widetilde{X}-Int_{\mathfrak{L}}(A))(x) &\leq_L \inf_{\alpha\in\Lambda}\{(\widetilde{X}-G_\alpha)(x) : (\widetilde{X}-G_\alpha)^C\in\mathfrak{L}_r, \widetilde{X}-A\leq\widetilde{X}-G_\alpha\}\\ &\leq_L \inf_{\alpha\in\Lambda}\{F_\alpha(x) : F_\alpha^C\in\mathfrak{L}_r, \widetilde{X}-A\leq F_\alpha\}\\ &= (\bigwedge_{\alpha\in\Lambda}\{F_\alpha : F_\alpha^C\in\mathfrak{L}_r, \widetilde{X}-A\leq F_\alpha\})(x)\\ &= (Cl_{\mathfrak{L}}(\widetilde{X}-A))(x).\end{aligned}$$

So $\widetilde{X}-Int_{\mathfrak{L}}(A)\leq Cl_{\mathfrak{L}}(\widetilde{X}-A)$.

Consider, For any $\alpha\in\Lambda$, $G_\alpha\in L^X, G_\alpha\leq(A)$, we have

$$\begin{aligned}G_\alpha(x) &= (\widetilde{X}-(\widetilde{X}-G_\alpha))(x)\\ &= \widetilde{X}(x)-(\widetilde{1}-G_\alpha)(x)\\ &\leq_L \widetilde{X}(x)-\inf_{\alpha\in\Lambda}\{(\widetilde{X}-G_\alpha)(x) : \widetilde{X}-A\leq\widetilde{X}-G_\alpha, (\widetilde{X}-G_\alpha)^C\in\mathfrak{L}_r\}\\ &\leq_L \widetilde{X}(x)-\inf_{\alpha\in\Lambda}\{(F_\alpha)(x) : \widetilde{X}-A\leq F_\alpha, F_\alpha^C\in\mathfrak{L}_r\}\\ &= Cl_{\mathfrak{L}}(\widetilde{X}(x)-A)(x)\\ &= (\widetilde{X}-Cl_{\mathfrak{L}}(A))(x).\end{aligned}$$

Then $\widetilde{X}-Cl_{\mathfrak{L}}(\widetilde{X}-A)(x)\geq_L \sup_{\alpha\in\Lambda}\{(G_\alpha)(x) : G_\alpha\leq A, G_\alpha\in\mathfrak{L}_r$

$$= Int_{\mathfrak{L}}(A))(x).$$

This implies that $Cl_{\mathfrak{L}}(\widetilde{X}-A)\leq\widetilde{X}-Int_{\mathfrak{L}}(A)$. Hence $Cl_{\mathfrak{L}}(\widetilde{X}-A)=Int_{\mathfrak{L}}(A)$.

Next we show that $Int_{\mathfrak{L}}(\widetilde{X}-A)=\widetilde{X}-Cl_{\mathfrak{L}}(A)$.

We have
$$\widetilde{X} - Cl_{\mathfrak{L}}(A) = \widetilde{X} - Cl_{\mathfrak{L}}(\widetilde{X} - (\widetilde{X} - A))$$
$$= \widetilde{X} - (\widetilde{X} - Int_{\mathfrak{L}}(\widetilde{X} - A))$$
$$= Int_{\mathfrak{L}}(\widetilde{X} - A).$$

Hence $Int_{\mathfrak{L}}(\widetilde{X} - A) = \widetilde{X} - Cl_{\mathfrak{L}}(A)$.

□

4.2 Topology and minimal structure space

This section discusses basic properties of topology and minimal structure space and basic concepts of open set, closed set, closure and interior.

Definition 4.2.1. [43] Let (X, τ) be a topological space and $A \subseteq X$. The *interior* of A and the *closure* of A are defined as follow:

(1) $Int(A) = \bigcup\{U : U \subseteq A, U \in \tau\}$.

(2) $Cl(A) = \bigcap\{F : A \subseteq F, X \setminus F \in \tau\}$.

Definition 4.2.2. [25, 26, 9] Let (X, τ) be a topological space and $A \subseteq X$. Then A is called:

(1) *semi open* if and only if $A \subseteq Cl(Int(A))$.

(2) *pre open* if and only if $A \subseteq Int(Cl(A))$.

(3) *b open* if and only if $A \subseteq Int(Cl(A)) \cup Cl(Int(A))$.

(4) *regular open* if and only if $A = Int(Cl(A))$.

The complement of semi open (resp. pre open, b open and regular open) set is called semi closed (resp. pre closed, b closed and regular closed) set, as follows.

Definition 4.2.3. [25, 26, 9] Let (X, τ) be a topological space and $A \subseteq X$. Then A is called:

(1) *semi closed* if and only if $Int(Cl(A)) \subseteq A$.

(2) *pre closed* if and only if $Cl(Int(A)) \subseteq A$.

(3) *b closed* if and only if $Int(Cl(A)) \cap Cl(Int(A)) \subseteq A$.

(4) *regular closed* if and only if $A = Cl(Int(A))$.

The family of all semi open (resp. pre open, b open and regular open) set in topological space is denoted by $SO(X,\tau)$ (resp. $PO(X,\tau)$, $BO(X,\tau)$, $RO(X,\tau)$) and the family of all semi closed (resp. pre closed, b closed and regular closed) set in topological space is denoted by $SC(X,\tau)$ (resp. $PC(X,\tau)$, $BC(X,\tau)$, $RC(X,\tau)$).

Definition 4.2.4. [25] Let (X,τ) be a topological space and $A \subseteq X$. The semi closed of a subset A, denoted by $scl(A)$ is the intersection of all semi closed subsets of (X,τ) that contain A.

Definition 4.2.5. [26] Let (X,τ) be a topological space and $A \subseteq X$. The pre closed of a subset A, denoted by $pcl(A)$ is the intersection of all pre closed subsets of (X,τ) that contain A.

Net, we introduce the $m-$structure and the $m-$operator notions. Also, we define some important subsets associated to these concepts.

Definition 4.2.6. [32] Let X be a non-empty set and m_X an $m-$structure on X. For a subset A of X, the $m_X - interior$ of A denoted by $mInt(A)$ and the $m_X - closure$ of A denoted by $mCl(A)$ are defined as follows:

(1) $mInt(A) = \bigcup\{B \subseteq X : B \in m_X \text{ and } B \subseteq A\}$.

(2) $mCl(A) = \bigcap\{B \subseteq X : X - B \in m_X \text{ and } A \subseteq B\}$.

Lemma 4.2.7. [32] Let X be a non-empty set and m_X an $m-$structure on X. For any subsets A and B of X, the following properties hold:

(1) $mInt(A) \subseteq A$. If $A \in m_X$, then $mInt(A) = A$.

(2) $A \subseteq mCl(A)$. If $X - A \in m_X$, then $mCl(A) = A$.

(3) $mCl(\emptyset) = \emptyset, mCl(X) = X, mInt(\emptyset) = \emptyset$ and $mInt(X) = X$.

(4) If $A \subseteq B$, then $mCl(A) \subseteq mCl(B)$ and $mInt(A) \subseteq mInt(B)$.

(5) $mCl(mCl(A)) = mCl(A)$ and $mInt(mInt(A)) = mInt(A)$.

(6) $mInt(A \cap B) = mInt(A) \cap mInt(B)$ and $mInt(A) \cup mInt(B) \subseteq mInt(A \cup B)$.

(7) $mCl(A \cup B) = mCl(A) \cup mCl(B)$ and $mCl(A \cap B) \subseteq mCl(A) \cap mCl(B)$.

(8) $X - mCl(A) = mInt(X - A)$ and $X - mInt(A) = mCl(X - A)$.

Definition 4.2.8. [30, 37, 38] Let (X, m_X) be an $m - space$ and $A \subseteq X$. Then A is called:

(1) m_X-semiopen if and only if $A \subseteq mCl(mInt(A))$.

(2) m_X-preopen if and only if $A \subseteq mInt(mCl(A))$.

(3) m_X-bopen if and only if $A \subseteq mInt(mCl(A)) \cup mCl(mInt(A))$.

(4) m_X-regular open if and only if $A = mInt(mCl(A))$.

The complement of m_X-semiopen (resp. m_X-preopen, m_X-bopen and m_X-regular open) set is called m_X-semiclosed (resp. m_X-preclosed, m_X-bclosed and m_X-regular closed) set, as follows.

Definition 4.2.9. [30, 37, 38] Let (X, m_X) be an $m - space$ and $A \subseteq X$. Then A is called:

(1) m_X-semiclosed if and only if $mInt(mCl(A)) \subseteq A$.

(2) m_X-preclosed if and only if $mCl(mInt(A)) \subseteq A$.

(3) m_X-bclosed if and only if $mInt(mCl(A)) \cap mCl(mInt(A)) \subseteq A$.

(4) m_X-regular closed if and only if $A = mCl(mInt(A))$.

The family of all m_X-semiopen (resp. m_X-preopen, m_X-bopen and m_X-regular open) set in topological space is denoted by $m_XSO(X)$ (resp. $m_XPO(X)$, $m_XBO(X)$, $m_XRO(X)$) and the family of all m_X-semiclosed (resp. m_X-preclosed, m_X-bclosed and m_X-regularclosed) set in $m - space$ is denoted by $m_XSC(X)$ (resp. $m_XPC(X)$, $m_XBC(X)$, $m_XRC(X)$).

Definition 4.2.10. [30, 37] Let (X, m_X) be an $m - space$ and $A \subseteq X$. Then A is called:

(1) $sCl(A) = \bigcap\{B \subseteq X : B$ is m_X-semiclosed set and $A \subseteq B\}$.

(2) $pCl(A) = \bigcap\{B \subseteq X : B$ is m_X-preclosed set and $A \subseteq B\}$.

(3) $bCl(A) = \bigcap\{B \subseteq X : B$ is m_X-bclosed set and $A \subseteq B\}$.

Lemma 4.2.11. [37] Let (X, m_X) be an $m - space$ and $A \subseteq X$. We have:

(1) $sCl(A) = A \cup mInt(mCl(A))$.

(2) $pCl(A) = A \cup mCl(mInt(A))$.

Definition 4.2.12. [37] Let (X, m_X) be an $m - space$ and $A \subseteq X$. Then A is called m_X-semi general closed (briefly m_X-sg closed) if $sCl(A) \subseteq U$ whenever $A \subseteq U$ and $U \in m_XSO(X)$.

The complement of m_X-sg closed set is called m_X-sg open set.

Definition 4.2.13. [28] Let (X, m_X) be an $m-space$ and $A \subseteq X$. Then A is called:

(1) m_X-*nowhere dense* if and only if $mInt(mCl(A)) = \emptyset$.

(2) m_X-*dense* if and only if $mCl(A) = X$.

(3) m_X-*codense* if and only if $mInt(A) = \emptyset$.

Definition 4.2.14. [37] Let (X, m_X) be an $m-space$ and let $X_1, X_2 \subseteq X$ defined by $X_1 = \{x \in X : \{x\}$ is m_X-*nowhere dense*$\}$ and $X_2 = \{x \in X : \{x\}$ is m_X-preopen $\}$. It is easy to see that $\{X_1, X_2\}$ is a decomposition of X (i.e. $X = X_1 \cup X_2$).

4.3 sg-submaximal space

This section discusses the characterization of sg-submaximal space.

Definition 4.3.1. An $m-space$ (X, m_X) is said to sg-submaximal if every m_X-*codense* subset of X is m_X-sg closed.

Example 4.3.2. Let $X = \{a, b, c\}$. Define the m-structure on X by $m_X = \{\emptyset, \{c\}, \{a, b\}, X\}$. Then $\emptyset, \{a\}, \{b\}$ are m_X-*codense*. Moreover, we get $m_X SO(X) = \{\emptyset, \{c\}, \{a, b\}, X\}$. So $\emptyset, \{a\}, \{b\}$ are m_X-sg closed. Hence (X, m_X) is sg-submaximal of X.

Theorem 4.3.3. Let (X, m_X) be an $m-space$ and $A \subseteq X$. Then A is m_X-sg closed if and only if $X_1 \cap sCl(A) \subseteq A$.

Proof. ($\Rightarrow$) Let $x \in X_1 \cap sCl(A)$, then $\{x\}$ is an $m-space$. Assume that $x \notin A$ then $A \subseteq X - \{x\}$. Thus $sCl(A) \subseteq X - \{x\}$, a contradiction. Therefore $x \in A$ that is $X_1 \cap sCl(A) \subseteq A$.

($\Leftarrow$) Suppose that $X_1 \cap sCl(A) \subseteq A$. Let $U \in m_X SO(X)$ such that $A \subseteq U$ and let $x \in sCl(A)$. If $x \in X_1$ then $x \in X_1 \cap sCl(A) \subseteq A$. So $sCl(A) \subseteq A$.

Assume now $x \in X_2$. Suppose that $x \notin U$. This implies that $X - U$ is m_X-semiclosed and $x \in X - U$. Since $\{x\}$ is m_X-preopen, we have

$$\begin{aligned} sCl(\{x\}) &= \{x\} \cup mInt(mCl(\{x\})) \\ &= mInt(mCl(\{x\})) \\ &\subseteq mInt(mCl(X - U)) \\ &\subseteq X - U. \end{aligned}$$

Since $\{x\}$ is m_X-preopen and we get that $mInt(mCl(\{x\})) \cap A \neq \emptyset$, then let $y \in mInt(mCl(\{x\})) \cap A$ we get that $y \in mInt(mCl(\{x\})) \cap A \subseteq (X - U) \cap U = \emptyset$, contradiction. Thus $x \in U$ and $sCl(A) \subseteq U$. Hence A is m_X-sg closed. □

Lemma 4.3.4. If A is m_X-regular open and $mInt(A)$ is m_X-open, then A is m_X-open.

Proof. Let A be m_X-regular open, then $A = mInt(mCl(A))$. Thus $mInt(A) = mInt(mInt(mCl(A))) = mInt(mCl(A)) = A$. It implies that A is m_X-open. □

Lemma 4.3.5. If A is m_X-sg closed set and B be m_X-closed sets, then $A \cup B$ is m_X-sg closed.

Proof. Let A is m_X-sg closed set and B be m_X-closed sets. Then $X_1 \cap sCl(A) \subseteq A$. Consider,

$$\begin{aligned} X_1 \cap sCl(A \cup B) &\subseteq X_1 \cap (sCl(A) \cup sCl(B)) \\ &= sCl(A) \cup (X_1 \cap sCl(B)) \\ &= (A \cup mInt(mCl(A))) \cup (X_1 \cap sCl(B)) \\ &= A \cup B, \end{aligned}$$

therefore by Theorem 4.2.3 $A \cup B$ is m_X-sg closed set. □

Lemma 4.3.6. Let (X, m_X) be an $m - space$ and $A, B \subseteq X$. If A is m_X-semiclosed set and B is m_X-sg closed set, then $A \cap B$ is m_X-sg closed set.

Proof. Let A is m_X-semiclosed set and B is m_X-sg closed set, then $mInt(mCl(A)) \subseteq A$ and $X_1 \cap sCl(A) \subseteq A$. Consider,

$$\begin{aligned} X_1 \cap sCl(A \cap B) &\subseteq X_1 \cap (sCl(A) \cap sCl(B)) \\ &= sCl(A) \cap (X_1 \cap sCl(B)) \\ &= (A \cup mInt(mCl(A))) \cap (X_1 \cap sCl(B)) \\ &= A \cap B, \end{aligned}$$

therefore by Theorem 4.2.3 $A \cap B$ is m_X-sg closed set. □

Lemma 4.3.7. Let (X, m_X) be $m - space$ and $A \in X$. Then $bCl(A) = sCl(A) \cap pCl(A)$.

Proof. Consider,

$$bCl(A) = A \cup (mInt(mCl(A)) \cap mCl(mInt(A)))$$
$$= (A \cup mInt(mCl(A))) \cap (A \cup mCl(mInt(A))),$$

by Lemma 4.1.12, $bCl(A) = sCl(A) \cap pCl(A)$. □

We now consider the property of sg-submaximal. First we wiil give some elementary characterizations of sg-submaximal spaces.

Theorem 4.3.8. Let X be an $m - space$, the following properties are equivalent:

(1) X is sg-submaximal,

(2) For any subset A of X, $A = mCl(A) \cap G$ where G is an m_X-sg open subset of X,

(3) For any subset A of X, $A = mInt(A) \cup F$ where F is an m_X-sg closed subset of X,

(4) every $m_X - codense$ subset A of X is m_X-sg closed,

(5) $mCl(A) - A$ is m_X-sg closed for every subset A of X.

Proof. $(1) \Rightarrow (2)$: Let $A \subseteq X$. we consider,

$$mInt(mCl(A) - A) = mInt(mCl(A)) \cap (X - A))$$
$$\subseteq mInt(mCl(A)) \cap mInt(X - A)$$
$$= mInt(mCl(A)) \cap [X - mCl(A)]$$
$$\subseteq mCl(A) \cap [X - mCl(A)]$$
$$= \emptyset.$$

This implies that $mCl(A) - A$ is $m_X - codense$. By (1) we get $mCl(A) - A$ is m_X-sg closed. Then $(X - mCl(A)) \cup A = X - (mCl(A) \cap (X - A)) = X - (mCl(A) - A)$ is m_X-sg open. Therefore

$$[(X - mCl(A)) \cup A] \cap mCl(A) = [(X - mCl(A)) \cap mCl(A)] \cup [A \cap mCl(A)]$$
$$= A.$$

Hence conclude that (2) is true.

(2) $\Rightarrow$ (3) : Let $A \subseteq X$. Then there exist m_X-sg open subset G of X such that $X - A = mCl(X - A) \cap G$. Thus

$$\begin{aligned} A &= X - [X - A] \\ &= X - [mCl(X - A) \cap G] \\ &= (X - mCl(X - A)) \cup (X - G) \\ &= mInt(A) \cup (X - G). \end{aligned}$$

This implies that $X - G$ is m_X-sg closed subset of X. Hence the statement that (3) is true.

(3) $\Rightarrow$ (4) : Let A be $m_X - codense$, that is $mInt(A) = \emptyset$. By (3), there exists m_X-sg closed subset F of X such that $A = mInt(A) \cup F$. Hence $A = mInt(A) \cup F = \emptyset \cup F = F$. So A is m_X-sg closed.

(4) $\Rightarrow$ (5) : Let $A \subseteq X$. We consider,

$$\begin{aligned} mInt(mCl(A) - A) &= mInt(mCl(A) \cap (X - A)) \\ &\subseteq mInt(mCl(A)) \cap mInt(X - A) \\ &= mInt(mCl(A)) \cap [X - mCl(A)] \\ &\subseteq mCl(A) \cap [X - mCl(A)] \\ &= \emptyset. \end{aligned}$$

This implies that $mCl(A) - A$ is $m_X - codense$, therefore $mCl(A) - A$ is m_X-sg closed.

(5) $\Rightarrow$ (1) : Let A be $m_X - codense$ of X, that is $mInt(A) = \emptyset$. By (5) we get that $mCl(X - A) - (X - A)$ is m_x-sg closed. We also have that

$$\begin{aligned} mCl(X - A) - (X - A) &= mCl(X - A) \cap A \\ &= [X - mInt(A)] \cap A \\ &= X \cap A \\ &= A. \end{aligned}$$

Hence A is m_X-sg closed. Therefore X is sg-submaximal. □

Example 4.3.9. Let $X = \{a, b, c\}$. Define the m-structure on X by $m_X = \{\emptyset, \{a\}, \{a, b\}, X\}$. Then $\emptyset, \{c\}, \{b, c\}$ are m_X-*codense*. Moreover, we get $m_X SO(X) = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$. So $\emptyset, \{c\}, \{b\}$ are m_X-sg closed. Hence (X, m_X) is sg-submaximal of X. It is clear that (1) and (4) are equivalent. For (2), (3), (5) it is not difficult to show how they are equivalent.

Theorem 4.3.10. Let X be an $m-space$, and let $mInt(E)$ be open set when $E \subseteq X$, the following properties are equivalent:

(1) every m_X-bclosed set is m_X-sg closed,

(2) every m_X-preclosed set is m_X-sg closed,

(3) X is sg-submaximal.

Proof. (1) $\Rightarrow$ (2) : Let A be m_X-preclosed, that is $mCl(mInt(A)) \subseteq A$. Then

$$\begin{aligned} mCl(mInt(A)) \cap mInt(mCl(A)) &\subseteq mCl(mInt(A)) \cap mCl(mCl(A)) \\ &= mCl(mInt(A)) \cap mCl(A) \\ &= A \cap mCl(A) \\ &= A. \end{aligned}$$

This implies that A is m_X-bclosed, therefore A is m_X-sg closed.
(2) $\Rightarrow$ (1) : Let A be m_X-bclosed, then $A = bCl(A)$. By Lemma 4.2.7, we get $bCl(A) = sCl(A) \cap pCl(A)$. We can easily see that $sCl(A)$ is m_X-semiclosed and $pCl(A)$ is m_X-preclosed. Therefore $pCl(A)$ is m_X-sg closed. By Lemma 4.2.6, implies that $A = bCl(A) = sCl(A) \cap pCl(A)$. Hence A is m_X-sg closed.
(2) $\Rightarrow$ (3) : Let A be $m_X - codense$, then $mInt(A) = \emptyset$. Since $mCl(mInt(A)) = mCl(A) = \emptyset \subseteq A$. Thus $mCl(mInt(A)) \subseteq A$, such that A is m_X-preclosed. Therefore A is m_X-sg closed. Hence X is sg-submaximal.
(3) $\Rightarrow$ (2) : Let A be m_X-preclosed, then $X - A$ is m_X-preopen and we will get $X - A \subseteq mInt(mCl(X - A))$. Let $G = mInt(mCl(X - A))$. Then we get $mCl(X -$

$A) \subseteq mCl(G)$. Consider

$$mCl(G) = mCl(mInt(mCl(X - A)))$$
$$= mCl(X - A).$$

Thus $mCl(G) = mCl(X - A)$. This implies that $G = mInt(mCl(G))$, i.e. G is m_X-regular open. Since $mCl(G) \subseteq X$, then $mInt(G)$ is open set. By Lemma 4.2.4, G is open set. Assume that

$$\begin{aligned} mCl(D) &= mCl[(X - A) \cup (X - G)] \\ &= mCl(X - A) \cup mCl(X - G) \\ &= mCl(G) \cup X - G \\ &= mCl(X) \\ &= X, \end{aligned}$$

therefore D is $m_x - dense$. Consider,

$$\begin{aligned} D \cap G &= [(X - A) \cup (A - G)] \cap G \\ &= [(X - A) \cap G] \cup [(X - G) \cap G] \\ &= [(X - A) \cap G] \cup \emptyset \\ &= X - A, \end{aligned}$$

thus $A = (X - D) \cup (X - G)$. Consider $X - D$ we will get $mInt(X - D) = X - mCl(D) = X - X = \emptyset$, thus $X - D$ is $m_X - codense$. Since X is sg-submaximal, then $X - D$ is m_X-sg closed. Since $X - G$ is closed set and by Lemma 4.2.5, $A = (X - D) \cup (X - G)$ is m_X-sg closed. □

Example 4.3.11. Let $X = \{a, b, c\}$. Define the m-structure on X by $m_X = \{\emptyset, \{a\}, \{b\}, X\}$. Then $\emptyset, \{c\}$ are m_X-$codense$. Moreover, we can find that $m_X SC(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$, $m_X BC(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$ and $m_X PC(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ So $\emptyset, \{c\}$ are m_X-sg closed. Hence (X, m_X) is sg-submaximal of X and (1)-(3) are equivalent.

CHAPTER 5

CONCLUSIONS

The aim of this thesis is to introduce the results of common fixed point in L-fuzzy metric spaces and partially ordered L-fuzzy metric spaces. And we study characterization of sg-submaximal space on minimal structure space. The results are as follows:

1) Let (X, M, Δ) be a $\mathcal{M}L$-fuzzy metric space and $x, y \in X$. If there exists a function $\phi \in \Phi_{w^*}$ such that

$$M(x, y, \phi(t)) \geq_L M(x, y, t),$$

for all $t > 0$, then $x = y$.

2) Let (X, M, Δ) by a L-fuzzy metric space and $x, y \in X$.
Let $n \geq 1, g_1, g_2, \ldots, g_n$ be self maps on complete lattice $(L, \leq_L)$ and for some $\phi \in \Phi_{w^*}$,

$$M(x, y, \phi(t)) \geq_L \Delta_M\{g_1(t), g_2(t), \ldots, g_n(t), M(x, y, t)\} \text{ for all } t > 0.$$

Then

$$M(x, y, \phi(t)) \geq_L \Delta_M\{g_1(t), g_2(t), \ldots, g_n(t)\}$$

for all $t > 0$.

3) Let (X, M, Δ) be a $\mathcal{M}L$-fuzzy metric space such that Δ is a t-norm of H-type. Let $\{x_n\}$ be a sequence in (X, M, Δ). If there exists a function $\phi \in \Phi_{w^*}$ satisfying;
(i) $\phi(t)$ for all $t > 0$;
(ii) $M(x_n, x_m, \phi(t)) \geq_L M(x_{n-1}, x_{m-1}, t)$ for all $m, n \in \mathbb{N}$ and $t > 0$.
Then $\{x_n\}$ is a Cauchy sequence.

4) Let (X, M, Δ) be a complete $\mathcal{M}L$-fuzzy metric space with a continuous t-norm Δ of H-type and let P, Q, S and T be self-maps on X. If the following conditions are satisfied:

(i) $T(X) \subseteq Q(X), S(X) \subseteq P(X)$;

(ii) either P or T is weakly sequential continuous;

(iii) (T,P) is compatible and (S,Q) is weakly compatible;

(iv) there exists $\phi \in \Phi_{w^*}$ such that

$$M(T(x),S(y),\phi(t)) \geq_L \Delta_M\{M(P(x),T(x),t), M(Q(y),S(y),t), M(P(x),Q(y),t), M(Q(y),T(x),\beta t),[M(P(x),\xi,(2-\beta)t)\oplus M(\xi,S(y),(2-\beta)t)]\}$$

for all $x,y,\xi \in X$, $\beta \in (0,2)$ and $t > 0$. Then P,Q,S and T have a unique common fixed point in X.

5) Let $(X,M,\Delta,\preceq)$ be a complete partially ordered $\mathcal{M}L$-fuzzy metric space such that Δ is a t-norm of H-type. Let T and G be self-maps on X. such that T is G-nondecreasing mapping and $T(X) \subseteq G(X)$. Assume that exists $\phi \in \Phi_{w^*}$ such that, for all $t > 0$ and $y,x \in X$ with $G(y) \preceq G(x)$,

$$M(T(y),T(x),\phi(t)) \geq_L M(G(y),G(x),t).$$

Also suppose that either

(a) G is weakly sequential continuous and (T,G) is compatible or

(b) $(X,\tau_M,\preceq)$ has the sequential monotone property and $G(X)$ is closed. If there exists $y_0 \in X$ such that $G(y_0) \asymp T(y_0)$. Then T and G have a coincidence point.

6) In addition to the hypotheses of 5) Suppose that for all coincidence point $y,v \in X$ of mapping T and G there exists $u \in X$ such that $G(u)$ is comparable to $G(y)$ and $G(v)$. Also suppose that (G,T) is weakly compatible if assume (b) holds. Then T and G have a unique common fixed pint.

7) Let $(X,\mathfrak{L})$ be an L-fuzzy minimal structure spaces. and $A,B \in L^X$.

(1) $Int_{\mathfrak{L}}(A) \leq A$ and if $A \in \mathfrak{L}_r$, then $Int_{\mathfrak{L}}(A) = A$.

(2) $A \leq Cl_{\mathfrak{L}}(A)$ and if $A^C \in \mathfrak{L}_r$, then $Cl_{\mathfrak{L}}(A) = A$.

(3) If $A \leq B$, then $Int_{\mathfrak{L}}(A) \leq Int_{\mathfrak{L}}(B)$ and $Cl_{\mathfrak{L}}(A) \leq Cl_{\mathfrak{L}}(B)$.

(4) $Int_{\mathfrak{L}}(A) \wedge Int_{\mathfrak{L}}(B) \geq Int_{\mathfrak{L}}(A \wedge B)$ and $Cl_{\mathfrak{L}}(A) \vee Cl_{\mathfrak{L}}(B) \leq Cl_{\mathfrak{L}}(A \vee B)$.

(5) $Int_{\mathfrak{L}}(Int_{\mathfrak{L}}(A)) = Int_{\mathfrak{L}}(A)$ and $Cl_{\mathfrak{L}}(Cl_{\mathfrak{L}}(A)) = Cl_{\mathfrak{L}}(A)$.

(6) $\widetilde{X} - Cl_{\mathfrak{L}}(A) = Int_{\mathfrak{L}}(\widetilde{X} - A)$ and $\widetilde{X} - Int_{\mathfrak{L}}(A) = Cl_{\mathfrak{L}}(\widetilde{X} - A)$. From the above definitions, we have the following theorems are derived

7.1) Let $(X, \mathfrak{L})$ be an L-fuzzy minimal structure space. The L-fuzzy closure and L-fuzzy interior of A, denoted by $Cl_{\mathfrak{L}}(A)$ and $Int_{\mathfrak{L}}(A)$, respectively, are defined as follows:

(i) $Cl_{\mathfrak{L}}(A) = \bigwedge\{B \in L^X : B^C \in \mathfrak{L}_r$ and $A \leq B\}$.

(ii) $Int_{\mathfrak{L}}(A) = \bigvee\{B \in L^X : B \in \mathfrak{L}_r$ and $B \leq A\}$.

8) Let (X, m_X) be an $m - space$ and $A \subseteq X$. Then A is m_X-sg closed if and only if $X_1 \cap sCl(A) \subseteq A$.

9) If A is m_X-regular open and $mInt(A)$ is m_X-open, then A is m_X-open.

10) If A is m_X-sg closed set and B be m_X-closed sets, then $A \cup B$ is m_X-sg closed.

11) Let (X, m_X) be an $m - space$ and $A, B \subseteq X$. If A is m_X-semiclosed set and B is m_X-sg closed set, then $A \cap B$ is m_X-sg closed set.

12) Let (X, m_X) be $m - space$ and $A \in X$. Then $bCl(A) = sCl(A) \cap pCl(A)$.

13) Let X be an $m - space$, the following properties are equivalent:

(i) X is sg-submaximal,

(ii) For any subset A of X, $A = mCl(A) \cap G$ where G is an m_X-sg open subset of X,

(iii) For any subset A of X, $A = mInt(A) \cup F$ where F is an m_X-sg closed subset of X,

(iv) every $m_X - codense$ subset A of X is m_X-sg closed,

(v) $mCl(A) - A$ is m_X-sg closed for every subset A of X.

14) Let X be an $m - space$, and let $mInt(E)$ be open set when $E \subseteq X$, the following properties are equivalent:

(i) every m_X-bclosed set is m_X-sg closed,

(ii) every m_X-preclosed set is m_X-sg closed,

(iii) X is sg-submaximal.

From the above definitions, I have the following theorems are derived

14.1) Let (X, m_X) be an $m-space$ is said to sg-submaximal if every m_X-$codense$ subset of X is m_X-sg closed.

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