Generalized Open Sets in Bi-Weak Structure Spaces

JIRAPORN NOIYAI

A thesis submitted in partial fulfillment of the requirements for
the degree of Master of Science in Mathematics Education
at Mahasarakham University
April 2023

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The examining committee has unanimously approved this thesis, submitted by Miss Jiraporn Noiyai, as a partial fulfillment of the requirements for the Master of Science in Mathematics Education at Mahasarakham University.

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Jiraporn Noiyai

ชื่อเรื่อง เซตเปิดวางนัยทั่วไปในปริภูมิสองโครงสร้างอ่อน
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บทคัดย่อ

งานวิจัยนี้ผู้วิจัยได้นำเสนอแนวคิดเกี่ยวกับเซตกึ่งเปิด $w(k,l)$ ในปริภูมิสองโครงสร้างอ่อน โดยที่ $k,l \in \{1,2\}$ ซึ่ง $k \neq l$ และได้ศึกษาสมบัติบางประการของเซตกึ่งเปิด $w(k,l)$ รวมทั้งได้ให้แนวคิดเกี่ยวกับความกึ่งต่อเนื่อง $w(k,l)$ โดยใช้เซตกึ่งเปิด $w(k,l)$ นอกจากนี้ยังได้แนะนำแนวคิดของเซตเปิด α-$w(k,l)$ เซตก่อนเปิด $w(k,l)$ และเซตเปิด β-$w(k,l)$ พร้อมทั้งศึกษาสมบัติบางประการของเซตข้างต้น

คำสำคัญ : ปริภูมิสองโครงสร้างอ่อน, เซตกึ่งเปิด $w(k,l)$, เซตเปิด α-$w(k,l)$, เซตก่อนเปิด $w(k,l)$, เซตเปิด β-$w(k,l)$

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ABSTRACT

In this research, the concept of $w(k,l)$-semi-open sets in bi-weak structure spaces, where $k,l \in \{1,2\}$ with $k \neq l$, is introduced. Some properties of $w(k,l)$-semi-open sets are obtained. Furthermore, the notion of $w(k,l)$-semi-continuity using $w(k,l)$-semi-open sets are introduced. Moreover, the concepts of $w(k,l)$-α-open sets, $w(k,l)$-preopen sets and $w(k,l)$-β-open sets in bi-weak structure spaces are introduced. Some properties of such sets are obtained.

Keywords : bi-weak structure space, $w(k,l)$-semi-open sets, $w(k,l)$-α-open sets, $w(k,l)$-preopen sets, $w(k,l)$-β-open sets.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Background

In mathematics, topology is a structure on a non-empty set. A non-empty set with a topology on such set is called a topological space. The elements of a topology are called open sets. The complements of open sets are called closed sets. Open and closed sets play an important role in a topological space. There are also two important operators, called that closure and interior, in a topological space by using closed sets and open sets, respectively. Later, mathematicians have studied many generalized sets of open sets. In 1963, Levine [5] introduced a generalized set of open set, which is called a semi-open set. In addition, Levine also studied semi-continuity on the topological spaces by using semi-open sets. In 1965, Njastad [7] introduced the concept of α-open sets in the topological spaces. And he have proven that the class of all α-open sets is a topology that is bigger than the original topology. In 1982, Mahhour [6] studied the concept of preopen sets and studied pre-continuity on the topological spaces. In 1983, Abd-El-Monsef [1] introduced the concepts of β-open sets and β-continuity on the topological spaces. Such semi-open sets, α-open sets, preopen sets and β-open sets are generalizations of open sets in the topological spaces.

In 2000, Popa and Noiri [8] introduced the concept of the minimal structure. In 2002, Császár [2] introduced the concept of the generalized topology. Later, in 2011, Császár [3] introduced the concept of the weak structure. Such structures are generalizations of topology. Which has expanded the concepts of semi-open sets, α-open sets, β-open sets and preopen sets in the above structures, especially in the weak structure, Császár [3] uses the symbol $\sigma(w)$, $\alpha(w)$, $\beta(w)$ and $\pi(w)$ represent classes of semi-open sets, α-open sets, β-open sets and preopen sets in the weak structure, respectively. And he also showed that these classes a generalized topology. Later in the year 2012, Ekici [4] presented a $r(w)$ class in a weak structure and he also studied the properties of $r(w)$. In 2017, Puiwong et al [9] introduced a new space, which consists of a nonempty set X and two weak structures w^1, w^2 on X. is called the bi-weak structure space. Also, they studied open sets, closed sets, and separation axioms on this space.

Therefore, we interested in expanding the concepts of semi-open sets, preopen sets, α-open sets, β-open sets and regular open sets on a topological space to a bi-weak structure space. Which is called a $w(k,l)$-semi-open sets, $w(k,l)$-preopen sets, $w(k,l)$-α-open sets, $w(k,l)$-β-open sets and $w(k,l)$-regular open sets in a bi-weak structure space, where $k,l \in \{1,2\}$ which $k \neq l$. As well as studying the properties and relationships of those sets.

1.2 Objective of the research

The purposes of the research are:

1. To construct and investigate the properties of $w(k,l)$-semi-open sets, $w(k,l)$-α-open sets, $w(k,l)$-preopen sets, $w(k,l)$-β-open sets and $w(k,l)$-regular open sets in bi-weak structure spaces.

2. To study the relationship of $w(k,l)$-semi-open sets, $w(k,l)$-α-open sets, $w(k,l)$-preopen sets, $w(k,l)$-β-open sets and $w(k,l)$-regular open sets in bi-weak structure spaces.

1.3 Objective of the research

The research procedure of this thesis consists of the following steps:

1. Criticism and possible extension of the literature review.

2. Doing research to investigate the main results.

3. Applying the results from 1.3.1 and 1.3.2 to the main results.

1.4 Scope of the study

The scopes of the study are: studying some properties of $w(k,l)$-semi-open sets, $w(k,l)$-preopen sets, $w(k,l)$-α-open sets, $w(k,l)$-β-open sets, and $w(k,l)$-regular open sets in bi-weak structure spaces.

CHAPTER 2

PRELIMINARIES

In this chapter, we will give some definitions, notations, dealing with some preliminaries and some useful results that will be duplicated in later chapter.

2.1 Weak Structures

Definition 2.1.1. [3] Let X be a nonempty set and $P(X)$ the power set of X. A subfamily w of $P(X)$ is called a *weak structure* (briefly WS) on X iff $\emptyset \in w$.

By (X, w) we denote a nonempty set X with a WS w on X and it is called a w-*space*. The elements of w are called w-*open sets* and the complements of w-open sets are called w-*closed sets*.

Let w be a weak structure on X and $A \subset X$. We define $i_w(A)$ as the union of all w-open subsets of A and $c_w(A)$ as the intersection of all w-closed sets containing A.

Theorem 2.1.2. [3] If w is a WS on X and $A, B \subset X$. Then

1. $A \subset c_w(A)$ and $i_w(A) \subset A$;
2. if $A \subset B$, then $c_w(A) \subset c_w(B)$ and $i_w(A) \subset i_w(B)$;
3. $c_w(c_w(A)) = c_w(A)$ and $i_w(i_w(A)) = i_w(A)$;
4. $c_w(X - A) = X - i_w(A)$ and $i_w(X - A) = X - c_w(A)$.

Theorem 2.1.3. [3] If w is a WS on X, $x \in i_w(A)$ if and only if there is a w-open set V such that $x \in V \subset A$.

Lemma 2.1.4. [3] If w is a WS on X, $x \in c_w(A)$ if and only if $V \cap A \neq \emptyset$ for any w-open set V containing x

Proposition 2.1.5. [3] If w is a WS on X and $A \in w$, then $A = i_w(A)$ and if A is w-closed, then $A = c_w(A)$.

Definition 2.1.6. [3] Let w be a WS on X and $A \subset X$. Then

1. $A \in \alpha(w)$ if $A \subset i_w(c_w(i_w(A)))$.
2. $A \in \pi(w)$ if $A \subset i_w(c_w(A))$.
3. $A \in \sigma(w)$ if $A \subset c_w(i_w(A))$.
4. $A \in \beta(w)$ if $A \subset c_w(i_w(c_w(A)))$.

Lemma 2.1.7. [3] Let w be a WS on X. Then $i_w(c_w(i_w(c_w(A)))) = i_w(c_w(A))$ and $c_w(i_w(c_w(i_w(A)))) = c_w(i_w(A))$ for all $A \subset X$.

Theorem 2.1.8. [3] If w is a WS on X, each of the structures $\alpha(w), \sigma(w), \pi(w), \beta(w)$ is a generalized topology.

Definition 2.1.9. [4] Let w be a WS on X and $A \subset X$. Then

1. $A \in r(w)$ if $A = i_w(c_w(A))$.
2. $A \in rc(w)$ if $A = c_w(i_w(A))$.

Theorem 2.1.10. [4] Let w be a WS on X and $A \subset X$. The $A \in r(w)$ if and only if $A \in \alpha(w)$ and $X - A \in \beta(w)$.

Theorem 2.1.11. [4] Let w be a WS on X and $A \subset X$. The $A \in r(w)$ if and only if $A \in \pi(w)$ and $X - A \in \sigma(w)$.

Theorem 2.1.12. [4] Let w be a WS on X and $A \subset X$. If $A \in \pi(w)$, then there exists a $B \in r(w)$ such that $A \subset B$ and $c_w(A) = c_w(B)$.

Theorem 2.1.13. [4] The following properties hold for a WS w on X and $A, B \subset X$:

1. $i_w(A \cap B) \subset i_w(A) \cap i_w(B)$.
2. $c_w(A) \cup c_w(B) \subset c_w(A \cup B)$.

2.2 Bi-Weak Structure Spaces

Definition 2.2.1. [9] Let X be a nonempty set and w^1, w^2 be two weak structures on X. A triple (X, w^1, w^2) is called a *bi-weak structure space* (briefly *bi-w space*).

Let (X, w^1, w^2) be a bi-weak structure space and A be a subset of X. The w-closure and w-interior of A with respect to w^j denote by $c_{w^j}(A)$ and $i_{w^j}(A)$, respectively, for $j \in \{1, 2\}$.

Definition 2.2.2. [9] A subset A of a bi-weak structure space (X, w^1, w^2) is called *closed* if $A = c_{w^1}(c_{w^2}(A))$. The complement of a closed set is called *open*.

Theorem 2.2.3. [9] Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then the following are equivalent:

1. A is closed;

2. $A = c_{w^1}(A)$ and $A = c_{w^2}(A)$;

3. $A = c_{w^2}(c_{w^1}(A))$.

Theorem 2.2.4. [9] Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then the following are equivalent:

1. A is open;

2. $A = i_{w^1}(i_{w^2}(A))$;

3. $A = i_{w^1}(A)$ and $A = i_{w^2}(A)$;

4. $A = i_{w^2}(i_{w^1}(A))$.

CHAPTER 3

GENERALIZED OPEN SETS IN BI-WEAK STRUCTURE SPACES

In this chapter, we will introduce the notions of generalized open sets in bi-weak structure spaces. Now, let $k, l \in \{1, 2\}$ be such that $k \neq l$.

3.1 $w(k,l)$-semi-open set

In this section, we will introduce the notion of $w(k,l)$-semi-open sets and investigate some of their properties.

Definition 3.1.1. Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then A is called a $w(k,l)$-*semi-open set* if $A \subset c_{w^l}(i_{w^k}(A))$. The complement of a $w(k,l)$-semi-open set is called a $w(k,l)$-*semi-closed set*.

Remark. Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then A is $w(k,l)$-semi-closed if and only if $X - A$ is $w(k,l)$-semi-open.

Example 3.1.2. Let $X = \{1, 2, 3\}$. Define WS w^1 and w^2 on X as follows:
$w^1 = \{\emptyset, \{1, 2\}, \{1, 3\}\}$ and $w^2 = \{\emptyset, \{1\}, \{2\}\}$. Then $X, \{3\}, \{2\}$ are w^1-closed and $X, \{2, 3\}, \{1, 3\}$ are w^2-closed. Consider

A	$i_{w^1}(A)$	$c_{w^2}(i_{w^1}(A))$	A is $w(1,2)$-semi-open set
$\emptyset$	$\emptyset$	$\{3\}$	✓
$\{1\}$	$\emptyset$	$\{3\}$	✗
$\{2\}$	$\emptyset$	$\{3\}$	✗
$\{3\}$	$\emptyset$	$\{3\}$	✓
$\{1,2\}$	$\{1,2\}$	X	✓
$\{1,3\}$	$\{1,3\}$	$\{1,3\}$	✓
$\{2,3\}$	$\emptyset$	$\{3\}$	✗
X	X	X	✓

and

A	$i_{w^2}(A)$	$c_{w^1}(i_{w^2}(A))$	A is $w(2,1)$-semi-open set
$\emptyset$	$\emptyset$	$\emptyset$	✓
$\{1\}$	$\{1\}$	X	✓
$\{2\}$	$\{2\}$	$\{2\}$	✓
$\{3\}$	$\emptyset$	$\emptyset$	✗
$\{1,2\}$	$\{1,2\}$	X	✓
$\{1,3\}$	$\{1\}$	X	✓
$\{2,3\}$	$\{2\}$	$\{2\}$	✗
X	$\{1,2\}$	X	✓

From the above table, we obtain that:

1. $\emptyset, \{1,2\}, \{1,3\}, X$ are $w(1,2)$-semi-open sets and $w(2,1)$-semi-open sets.

2. $\{3\}$ is $w(1,2)$-semi-open set but is not a $w(2,1)$-semi-open set.

3. $\{1\}, \{2\}$ are $w(2,1)$-semi-open sets but not a $w(1,2)$-semi-open sets.

4. $\{2,3\}$ is not $w(1,2)$-semi-open set and $w(2,1)$-semi-open set.

Theorem 3.1.3. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-semi-closed if and only if $i_{w^l}(c_{w^k}(A)) \subset A$.

Proof. ($\Longrightarrow$) Assume that A is $w(k,l)$-semi-closed. Then $X - A$ is $w(k,l)$-semi-open. Thus $X - A \subset c_{w^l}(i_{w^k}(X - A))$, and so $X - A \subset c_{w^l}(X - c_{w^k}(A))$. Hence $X - A \subset X - i_{w^l}(c_{w^k}(A))$. Then $i_{w^l}(c_{w^k}(A)) \subset A$.

($\Longleftarrow$) Assume that $i_{w^l}(c_{w^k}(A)) \subset A$. Then $X - A \subset X - i_{w^l}(c_{w^k}(A))$. Thus $X - A \subset c_{w^l}(X - c_{w^k}(A))$, and so $X - A \subset c_{w^l}(i_{w^k}(X - A))$. Therefore, $X - A$ is $w(k,l)$-semi-open. Hence A is $w(k,l)$-semi-closed. □

Proposition 3.1.4. Let (X, w^1, w^2) be a bi-w space. If A_j is $w(k,l)$-semi-open for all $j \in \Lambda$, then $\bigcup_{j\in\Lambda} A_j$ is $w(k,l)$-semi-open.

Proof. Assume that A_j is $w(k,l)$-semi-open for all $j \in \Lambda$ and let $A = \bigcup_{j\in\Lambda} A_j$. Fix $j_0 \in \Lambda$. Then $A_{j_0} \subset A$. Thus $c_{w^l}(i_{w^k}(A_{j_0})) \subset c_{w^l}(i_{w^k}(A))$. Since A_{j_0} is $w(k,l)$-semi-open, $A_{j_0} \subset c_{w^l}(i_{w^k}(A_{j_0}))$. This implies that $A_{j_0} \subset c_{w^l}(i_{w^k}(A))$. Since j_0 is an arbitrary element of Λ, we have $A_j \subset c_{w^l}(i_{w^k}(A))$ for all $j \in \Lambda$. Therefore $\bigcup_{j\in\Lambda} A_j \subset c_{w^l}(i_{w^k}(A))$. Thus $\bigcup_{j\in\Lambda} A_j$ is $w(k,l)$-semi-open. □

Remark. The intersection of two $w(k,l)$-semi-open sets is not an $w(k,l)$-semi-open set in general as can be seen from the following example.

Example 3.1.5. From Example 3.1.2, we obtain that $\{1,2\}$ and $\{1,3\}$ are $w(1,2)$-semi-open but $\{1,2\} \cap \{1,3\} = \{1\}$ is not $w(1,2)$-semi-open.

Proposition 3.1.6. Let (X, w^1, w^2) be a bi-w space. If A_j is $w(k,l)$-semi-closed for all $j \in \Lambda$, then $\bigcap_{j\in\Lambda} A_j$ is $w(k,l)$-semi-closed.

Proof. Assume that A_j is $w(k,l)$-semi-closed for all $j \in \Lambda$ and let $A = \bigcap_{j\in\Lambda} A_j$. Fix $j_0 \in \Lambda$. Then $A \subset A_{j_0}$. Thus $i_{w^l}(c_{w^k}(A)) \subset i_{w^l}(c_{w^k}(A_{j_0}))$. Since A_{j_0} is $w(k,l)$-semi-closed, $i_{w^l}(c_{w^k}(A_{j_0})) \subset A_{j_0}$. This implies that $i_{w^l}(c_{w^k}(A)) \subset A_{j_0}$. Since j_0 is an arbitrary element of Λ, we have $i_{w^l}(c_{w^k}(A)) \subset A_j$ for all $j \in \Lambda$. Therefore $i_{w^l}(c_{w^k}(A)) \subset \bigcap_{j\in\Lambda} A_j$. Thus $\bigcap_{j\in\Lambda} A_j$ is $w(k,l)$-semi-closed. □

Remark. The union of two $w(k,l)$-semi-closed sets is not an $w(k,l)$-semi-closed set in general as can be seen from the following example.

Example 3.1.7. From Example 3.1.2, we obtain that $\{2\}$ and $\{3\}$ are $w(1,2)$-semi-closed but $\{2\} \cup \{3\} = \{2,3\}$ is not $w(1,2)$-semi-closed.

Theorem 3.1.8. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-semi-open if and only if there exists a subset U of X such that $i_{w^k}(U) = U$ and $U \subset A \subset c_{w^l}(U)$.

Proof. ($\Longrightarrow$) Assume that A is $w(k,l)$-semi-open. Then $A \subset c_{w^l}(i_{w^k}(A))$. Let $U = i_{w^k}(A)$. Then $i_{w^k}(U) = i_{w^k}(i_{w^k}(A)) = i_{w^k}(A) = U$. Since $i_{w^k}(A) \subset A \subset c_{w^l}(i_{w^k}(A))$, $U \subset A \subset c_{w^l}(U)$.

($\Longleftarrow$) Assume that there exists a subset U of X such that $i_{w^k}(U) = U$ and $U \subset A \subset c_{w^l}(U)$. Since $U \subset A$, $c_{w^l}(i_{w^k}(U)) \subset c_{w^l}(i_{w^k}(A))$. Since $i_{w^k}(U) = U$, $A \subset c_{w^l}(U) = c_{w^l}(i_{w^k}(U)) \subset c_{w^l}(i_{w^k}(A))$. Hence A is $w(k,l)$-semi-open. $\square$

Corollary 3.1.9. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If there exists $U \in w^k$ such that $U \subset A \subset c_{w^l}(U)$, then A is $w(k,l)$-semi-open.

Proof. Assume that there exists $U \in w^k$ such that $U \subset A \subset c_{w^l}(U)$. Since $U \in w^k$, $i_{w^k}(U) = U$. By Theorem 3.1.8, A is $w(k,l)$-semi-open. $\square$

Corollary 3.1.10. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If A is $w(k,l)$-semi-open and $A \subset B \subset c_{w^l}(A)$, then B is $w(k,l)$-semi-open.

Proof. Assume that A is $w(k,l)$-semi-open and $A \subset B \subset c_{w^l}(A)$. Then there exists a subset U of X such that $i_{w^k}(U) = U$ and $U \subset A \subset c_{w^l}(U)$. Thus $U \subset A \subset B$. Since $A \subset B \subset c_{w^l}(A) \subset c_{w^l}(U)$, $U \subset B \subset c_{w^l}(U)$. By Theorem 3.1.8, B is $w(k,l)$-semi-open. $\square$

Corollary 3.1.11. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If A is $w(k,l)$-semi-closed and $i_{w^l}(A) \subset B \subset A$, then B is $w(k,l)$-semi-closed.

Proof. Assume that A is $w(k,l)$-semi-closed and $i_{w^l}(A) \subset B \subset A$. Then $X - A$ is $w(k,l)$-semi-open and $X - A \subset X - B \subset X - i_{w^l}(A)$. Thus $X - A \subset X - B \subset c_{w^l}(X - A)$. Therefore $X - B$ is $w(k,l)$-semi-open. Hence B is $w(k,l)$-semi-closed. $\square$

Definition 3.1.12. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. The intersection of all $w(k,l)$-semi-closed sets containing A is called the $w(k,l)$-*semi-closure* of A and is denoted by $c_\sigma^{kl}(A)$. The union of all $w(k,l)$-semi-open sets contained in A is called the $w(k,l)$-*semi-interior* of A and is denoted by $i_\sigma^{kl}(A)$, respectively, that is,
$c_\sigma^{kl}(A) = \bigcap\{F : F \text{ is } w(k,l)\text{-semi-closed and } A \subset F\}$,
$i_\sigma^{kl}(A) = \bigcup\{G : G \text{ is } w(k,l)\text{-semi-open and } G \subset A\}$.

Remark. Form Definition 3.1.12, it is obvious that $i_\sigma^{kl}(A) \subset A \subset c_\sigma^{kl}(A)$.

Lemma 3.1.13. Let (X, w^1, w^2) be a bi-w space and $A \subset X$, $x \subset X$. Then $x \in c_\sigma^{kl}(A)$ if and only if $V \cap A \neq \emptyset$ for all $w(k,l)$-semi-open set V containing x.

Proof. Assume that there exists a $w(k,l)$-semi-open set V containing x such that $V \cap A = \emptyset$. Then $X - V$ is $w(k,l)$-semi-closed such that $x \notin X - V$ and $A \subset X - V$. Hence, $x \notin c_\sigma^{kl}(A)$. Conversely, assume that $x \notin c_\sigma^{kl}(A)$. Then there exists a $w(k,l)$-semi-closed set F such that $A \subset F$ and $x \notin F$. Set $V = X - F$. Thus V is $w(k,l)$-semi-open set containing x such that $V \cap A = \emptyset$. □

Lemma 3.1.14. Let (X, w^1, w^2) be a bi-w space and $A \subset X$, $x \subset X$. Then $x \in i_\sigma^{kl}(A)$ if and only if there exists a $w(k,l)$-semi-open set V such that $x \in V \subset A$.

Proof. Assume that $x \in i_\sigma^{kl}(A)$. Then there exists a $w(k,l)$-semi-open set V such that $x \in V \subset A$. Conversely, assume that there exists a $w(k,l)$-semi-open set V such that $x \in V \subset A$. Then $x \in i_\sigma^{kl}(A)$. □

Lemma 3.1.15. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then $i_\sigma^{kl}(X - A) = X - c_\sigma^{kl}(A)$ and $c_\sigma^{kl}(X - A) = X - i_\sigma^{kl}(A)$.

Proof. Assume that $x \in i_\sigma^{kl}(X - A)$. Then there exists a $w(k,l)$-semi-open set V such that $x \in V \subset X - A$. Thus $V \cap A = \emptyset$, and so $x \notin c_\sigma^{kl}(A)$. Hence $x \in X - c_\sigma^{kl}(A)$. On the other hand, assume that $x \in X - c_\sigma^{kl}(A)$. Then $x \notin c_\sigma^{kl}(A)$, and so there exists a $w(k,l)$-semi-open set V containing x such that $V \cap A = \emptyset$. Thus $V \subset X - A$. Hence $x \in i_\sigma^{kl}(X - A)$. This implies that $i_\sigma^{kl}(X - A) = X - c_\sigma^{kl}(A)$. The second part follows from the first part. □

Theorem 3.1.16. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-semi-closed if and only if $A = c_\sigma^{kl}(A)$.

Proof. Assume that A is $w(k,l)$-semi-closed. Then $c_\sigma^{kl}(A) \subset A$. This implies $A = c_\sigma^{kl}(A)$. Conversely, assume that $A = c_\sigma^{kl}(A)$. By Proposition 3.1.6, A is $w(k,l)$-semi-closed. □

Theorem 3.1.17. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-semi-open if and only if $A = i_\sigma^{kl}(A)$.

Proof. Assume that A is $w(k,l)$-semi-open. Then $A \subset i_\sigma^{kl}(A)$. This implies $A = i_\sigma^{kl}(A)$. Conversely, assume that $A = i_\sigma^{kl}(A)$. By Proposition 3.1.4, A is $w(k,l)$-semi-open. □

Finally, we shall introduce the concept of the continuity using $w(k,l)$-semi-open sets.

Definition 3.1.18. A function $f : (X, w^1, w^2) \to (Y, \nu^1, \nu^2)$ is called $w(k,l)$-*semi-continuous* at x if, for all ν^k-open set V containing $f(x)$, there exists a $w(k,l)$-semi-open set U containing x such that $f(U) \subset V$. Then f is called $w(k,l)$-*semi-continuous* if f is $w(k,l)$-semi-continuous at x, for all $x \in X$.

Theorem 3.1.19. Let (X, w^1, w^2) and (Y, ν^1, ν^2) be two bi-weak structure spaces and $f : (X, w^1, w^2) \to (Y, \nu^1, \nu^2)$. Then the following are equivalent:

1. f is $w(k,l)$-semi-continuous.
2. $f^{-1}(V)$ is $w(k,l)$-semi-open for all $V \in \nu^k$.
3. $f^{-1}(F)$ is $w(k,l)$-semi-closed for all ν^k-closed set F.
4. $f(c_\sigma^{kl}(A)) \subset c_{\nu^k}(f(A))$ for all $A \subset X$.
5. $c_\sigma^{kl}(f^{-1}(B)) \subset f^{-1}(c_{\nu^k}(B))$ for all $B \subset Y$.
6. $f^{-1}(i_{\nu^k}(B)) \subset i_\sigma^{kl}(f^{-1}(B))$ for all $B \subset Y$.

Proof. (1$\Longrightarrow$2) Assume that f is $w(k,l)$-semi-continuous and $V \in \nu^k$. We will show that $f^{-1}(V) = i_\sigma^{kl}(f^{-1}(V))$. Let $x \in f^{-1}(V)$. Then $f(x) \in V$. Since $V \in \nu^k$ and f is $w(k,l)$-semi-continuous, there exists a $w(k,l)$-semi-open set U containing x such that $f(U) \subset V$. Then $x \in U \subset f^{-1}(V)$. Thus $x \in i_\sigma^{kl}(f^{-1}(V))$. Therefore $f^{-1}(V) = i_\sigma^{kl}(f^{-1}(V))$. Hence $f^{-1}(V)$ is $w(k,l)$-semi-open.

(2$\Longrightarrow$3) Assume that $f^{-1}(V)$ is $w(k,l)$-semi-open for all $V \in \nu^k$. Let F be a ν^k-closed set. Then $X - F$ is ν^k-open. By 2, $f^{-1}(X - F)$ is $w(k,l)$-semi-open. Since $f^{-1}(X - F) = X - f^{-1}(F)$, $X - f^{-1}(F)$ is $w(k,l)$-semi-open. Thus $f^{-1}(F)$ is $w(k,l)$-semi-closed.

(3$\Longrightarrow$4) Assume that $f^{-1}(F)$ is $w(k,l)$-semi-closed for all ν^k-closed set F. Let $A \subset X$ and $\mathscr{F} = \{F : F \text{ is } \nu^k\text{-closed and } f(A) \subset F\}$. Then $A \subset f^{-1}(f(A))$. Thus $A \subset f^{-1}(c_{\nu^k}(f(A))) = f^{-1}(\bigcap_{F \in \mathscr{F}} F) = \bigcap_{F \in \mathscr{F}} f^{-1}(F)$. By 3, $\bigcap_{F \in \mathscr{F}} f^{-1}(F)$ is $w(k,l)$-semi-closed. Moreover, $A \subset \bigcap_{F \in \mathscr{F}} f^{-1}(F)$. Then $c_\sigma^{kl}(A) \subset \bigcap_{F \in \mathscr{F}} f^{-1}(F) = f^{-1}(c_{\nu^k}(f(A)))$. Thus $f(c_\sigma^{kl}(A)) \subset c_{\nu^k}(f(A))$.

(4⟹5) Assume that $f(c_\sigma^{kl}(A)) \subset c_{\nu^k}(f(A))$ for all $A \subset X$. Let $B \subset Y$. Then $f^{-1}(B) \subset X$. By 4, $f(c_\sigma^{kl}(f^{-1}(B))) \subset c_{\nu^k}(f(f^{-1}(B))) \subset c_{\nu^k}(B)$. Thus $c_\sigma^{kl}(f^{-1}(B)) \subset f^{-1}(f(c_\sigma^{kl}(f^{-1}(B))) \subset f^{-1}(c_{\nu^k}(B))$.

(5⟹6) Assume that $c_\sigma^{kl}(f^{-1}(B)) \subset f^{-1}(c_{\nu^k}(B))$ for all $B \subset Y$. Let $B \subset Y$. Then $Y - B \subset Y$. By 5, $c_\sigma^{kl}(f^{-1}(Y-B)) \subset f^{-1}(c_{\nu^k}(Y-B))$. Since $c_\sigma^{kl}(f^{-1}(Y-B)) = c_\sigma^{kl}(X - f^{-1}(B)) = X - i_\sigma^{kl}(f^{-1}(B))$ and $f^{-1}(c_{\nu^k}(Y-B)) = f^{-1}(Y - c_{\nu^k}(B)) = X - f^{-1}(i_{\nu^k}(B))$, $X - i_\sigma^{kl}(f^{-1}(B)) \subset X - f^{-1}(i_{\nu^k}(B))$. Hence $f^{-1}(i_{\nu^k}(B)) \subset i_\sigma^{kl}(f^{-1}(B))$.

(6⟹1) Assume that $f^{-1}(i_{\nu^k}(B)) \subset i_\sigma^{kl}(f^{-1}(B))$ for all $B \subset Y$. We show that f is $w(k,l)$-semi-continuous. Let $x \in X$ and V is ν^k-open containing $f(x)$. Then $V = i_{\nu^k}(V)$. By 6, $f^{-1}(V) = f^{-1}(i_{\nu^k}(V)) \subset i_\sigma^{kl}(f^{-1}(V))$. Then $x \in i_\sigma^{kl}(f^{-1}(V))$. Thus there exists a $w(k,l)$-semi-open set G such that $x \in G \subset f^{-1}(V)$. Then $f(G) \subset V$. Thus f is $w(k,l)$-semi-continuous at x. Hence f is $w(k,l)$-semi-continuous. □

3.2 $w(k,l)$-α-open set, $w(k,l)$-preopen set and $w(k,l)$-β-open set

In this section, we will introduce the notion of $w(k,l)$-α-open set and investigate some of their properties

Definition 3.2.1. Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then A is called a $w(k,l)$-α-*open set* if $A \subset i_{w^k}(c_{w^l}(i_{w^k}(A)))$. The complement of a $w(k,l)$-α-open set is called a $w(k,l)$-α-*closed set*.

Remark. Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then A is $w(k,l)$-α-closed if and only if $X - A$ is $w(k,l)$-α-open.

Example 3.2.2. Let $X = \{1,2,3\}$. Define WS w^1 and w^2 on X as follows:

$w^1 = \{\emptyset, \{1,2\}, \{1,3\}\}$ and $w^2 = \{\emptyset, \{1\}, \{2\}\}$. Then $X, \{3\}, \{2\}$ are w^1-closed and $X, \{2,3\}, \{1,3\}$ are w^2-closed. Consider

A	$i_{w^1}(A)$	$c_{w^2}(i_{w^1}(A))$	$i_{w^1}(c_{w^2}(i_{w^1}(A)))$	A is $w(1,2)$-α-open set
$\emptyset$	$\emptyset$	$\{3\}$	$\emptyset$	✓
$\{1\}$	$\emptyset$	$\{3\}$	$\emptyset$	✗
$\{2\}$	$\emptyset$	$\{3\}$	$\emptyset$	✗
$\{3\}$	$\emptyset$	$\{3\}$	$\emptyset$	✗
$\{1,2\}$	$\{1,2\}$	X	X	✓
$\{1,3\}$	$\{1,3\}$	$\{1,3\}$	$\{1,3\}$	✓
$\{2,3\}$	$\emptyset$	$\{3\}$	$\emptyset$	✗
X	X	X	X	✓

and

A	$i_{w^2}(A)$	$c_{w^1}(i_{w^2}(A))$	$i_{w^2}(c_{w^1}(i_{w^2}(A)))$	A is $w(2,1)$-α-open set
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	✓
$\{1\}$	$\{1\}$	X	$\{1,2\}$	✓
$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	✓
$\{3\}$	$\emptyset$	$\emptyset$	$\emptyset$	✗
$\{1,2\}$	$\{1,2\}$	X	$\{1,2\}$	✓
$\{1,3\}$	$\{1\}$	X	$\{1,2\}$	✗
$\{2,3\}$	$\{2\}$	$\{2\}$	$\{2\}$	✗
X	$\{1,2\}$	X	$\{1,2\}$	✗

From the above table, we obtain that:

1. $\emptyset, \{1,2\}$ are $w(1,2)$-α-open sets and $w(2,1)$-α-open sets.

2. $\{1,3\}, X$ are $w(1,2)$-α-open sets but not a $w(2,1)$-α-open sets.

3. $\{1\}, \{2\}$ are $w(2,1)$-α-open sets but not a $w(1,2)$-α-open sets.

4. $\{3\}, \{2,3\}$ are not $w(1,2)$-α-open sets and $w(2,1)$-α-open sets.

Theorem 3.2.3. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-α-closed if and only if $c_{w^k}(i_{w^l}(c_{w^k}(A))) \subset A$.

Proof. ($\Longrightarrow$) Assume that A is $w(k,l)$-α-closed. Then $X - A$ is $w(k,l)$-α-open. Thus $X - A \subset i_{w^k}(c_{w^l}(i_{w^k}(X - A)))$, and so $X - A \subset X - c_{w^k}(i_{w^l}(c_{w^k}(A)))$. Hence $c_{w^k}(i_{w^l}(c_{w^k}(A))) \subset A$.

($\Longleftarrow$) Assume that $c_{w^k}(i_{w^l}(c_{w^k}(A))) \subset A$. Then $X - A \subset X - c_{w^k}(i_{w^l}(c_{w^k}(A)))$. Thus $X - A \subset i_{w^k}(c_{w^l}(i_{w^k}(X - A)))$. Therefore, $X - A$ is $w(k,l)$-α-open. Hence A is $w(k,l)$-α-closed. □

Proposition 3.2.4. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If A_j is $w(k,l)$-α-open for all $j \in \Lambda$, then $\bigcup_{j \in \Lambda} A_j$ is $w(k,l)$-α-open.

Proof. Assume that A_j is $w(k,l)$-α-open for all $j \in \Lambda$ and let $A = \bigcup_{j \in \Lambda} A_j$. Fix $j_0 \in \Lambda$. Then $A_{j_0} \subset A$. Thus $i_{w^k}(c_{w^l}(i_{w^k}(A_{j_0}))) \subset i_{w^k}(c_{w^l}(i_{w^k}(A)))$. Since A_{j_0} is $w(k,l)$-α-open, $A_{j_0} \subset i_{w^k}(c_{w^l}(i_{w^k}(A_{j_0})))$. This implies that $A_{j_0} \subset i_{w^k}(c_{w^l}(i_{w^k}(A)))$. Since j_0 is an arbitrary element of Λ, we have $A_j \subset i_{w^k}(c_{w^l}(i_{w^k}(A)))$ for all $j \in \Lambda$. Therefore $\bigcup_{j \in \Lambda} A_j \subset i_{w^k}(c_{w^l}(i_{w^k}(A)))$. Thus $\bigcup_{j \in \Lambda} A_j$ is $w(k,l)$-α-open. □

Remark. The intersection of two $w(k,l)$-α-open sets is not an $w(k,l)$-α-open set in general as can be seen from the following example.

Example 3.2.5. From Example 3.2.2, we obtain that $\{1,2\}$ and $\{1,3\}$ are $w(1,2)$-α-open but $\{1,2\} \cap \{1,3\} = \{1\}$ is not $w(1,2)$-α-open.

Proposition 3.2.6. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If A_j is $w(k,l)$-α-closed for all $j \in \Lambda$, then $\bigcap_{j \in \Lambda} A_j$ is $w(k,l)$-α-closed.

Proof. Assume that A_j is $w(k,l)$-α-closed for all $j \in \Lambda$. Then $X - A_j$ is $w(k,l)$-α-open for all $j \in \Lambda$. By Proposition 3.2.4, $\bigcup_{j \in \Lambda}(X - A_j)$ is $w(k,l)$-α-open. Since $\bigcup_{j \in \Lambda}(X - A_j) = X - \bigcap_{j \in \Lambda} A_j$, $X - \bigcap_{j \in \Lambda} A_j$ is $w(k,l)$-α-open. Thus $\bigcap_{j \in \Lambda} A_j$ is $w(k,l)$-α-closed. □

Remark. The union of two $w(k,l)$-α-closed sets is not an $w(k,l)$-α-closed set in general as can be seen from the following example.

Example 3.2.7. From Example 3.2.2, we obtain that $\{2\}$ and $\{3\}$ are $w(1,2)$-α-closed but $\{2\} \cup \{3\} = \{2,3\}$ is not $w(1,2)$-α-closed.

Theorem 3.2.8. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-α-open if and only if there exists a set U such that $U = i_{w^k}(U)$ and $U \subset A \subset i_{w^k}(c_{w^l}(U))$.

Proof. ($\Longrightarrow$) Assume that A is $w(k,l)$-α-open. Then $A \subset i_{w^k}(c_{w^l}(i_{w^k}(A)))$. Let $U = i_{w^k}(A)$. Then $i_{w^k}(U) = i_{w^k}(i_{w^k}(A))$. Thus $i_{w^k}(U) = i_{w^k}(A) = U$. Since $i_{w^k}(A) \subset A$, $U \subset A$. Since $i_{w^k}(A) = U$, $i_{w^k}(c_{w^l}(i_{w^k}(A))) = i_{w^k}(c_{w^l}(U))$. Since A is $w(k,l)$-α-open, $A \subset i_{w^k}(c_{w^l}(i_{w^k}(A))) = i_{w^k}(c_{w^l}(U))$. Hence $U \subset A \subset i_{w^k}(c_{w^l}(U))$.

($\Longleftarrow$) Assume that there exists a set U such that $U = i_{w^k}(U)$ and $U \subset A \subset i_{w^k}(c_{w^l}(U))$. Since $U \subset A$, $i_{w^k}(c_{w^l}(i_{w^k}(U))) \subset i_{w^k}(c_{w^l}(i_{w^k}(A)))$. Since $U = i_{w^k}(U)$, $A \subset i_{w^k}(c_{w^l}(U)) \subset i_{w^k}(c_{w^l}(i_{w^k}(A)))$. Hence A is $w(k,l)$-α-open. $\square$

Corollary 3.2.9. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If there exists $U \in w^k$ such that $U \subset A \subset i_{w^k}(c_{w^l}(U))$, then A is $w(k,l)$-α-open.

Proof. Assume that there exists $U \in w^k$ such that $U \subset A \subset i_{w^k}(c_{w^l}(U))$. Since $U \in w^k$, $i_{w^k}(U) = U$. By Theorem 3.2.8, A is $w(k,l)$-α-open. $\square$

Corollary 3.2.10. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If A is $w(k,l)$-α-open and $A \subset B \subset i_{w^k}(c_{w^l}(A))$, then B is $w(k,l)$-α-open.

Proof. Assume that A is $w(k,l)$-α-open and $A \subset B \subset i_{w^k}(c_{w^l}(A))$. Since A is $w(k,l)$-α-open, By Theorem 3.2.8, there exists a set U such that $i_{w^k}(U) = U$ and $U \subset A \subset i_{w^k}(c_{w^l}(U))$. Since $A \subset B \subset i_{w^k}(c_{w^l}(A))$ and $i_{w^k}(c_{w^l}(i_{w^k}(c_{w^l}(U)))) = i_{w^k}(c_{w^l}(U))$, $U \subset B \subset i_{w^k}(c_{w^l}(U))$. By Theorem 3.2.8, B is $w(k,l)$-α-open. $\square$

Corollary 3.2.11. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If A is $w(k,l)$-α-closed and $c_{w^k}(i_{w^l}(A)) \subset B \subset A$, then B is $w(k,l)$-α-closed.

Proof. Assume that A is $w(k,l)$-α-closed and $c_{w^k}(i_{w^l}(A)) \subset B \subset A$. Then $X - A$ is $w(k,l)$-α-open and $X - A \subset X - B \subset i_{w^k}(c_{w^l}(X - A))$. By Corollary 3.2.10, $X - B$ is $w(k,l)$-α-open. Hence B is $w(k,l)$-α-closed. $\square$

Next, we will introduce the notion of $w(k,l)$-preopen sets and investigate some of their properties.

Definition 3.2.12. Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then A is called a *$w(k,l)$-preopen set* if $A \subset i_{w^k}(c_{w^l}(A))$. The complement of a $w(k,l)$-preopen set is called a *$w(k,l)$-preclosed set*.

Remark. Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then A is $w(k,l)$-preclosed if and only if $X - A$ is $w(k,l)$-preopen.

Example 3.2.13. Let $X = \{1,2,3\}$. Define WS w^1 and w^2 on X as follows:

$w^1 = \{\emptyset, \{1,3\}, \{2,3\}$ and $w^2 = \{\emptyset, \{1\}, \{1,2\}\}$. Then $X, \{1\}, \{2\}$ are w^1-closed and $X, \{3\}, \{2,3\}$ are w^2-closed. Consider

A	$c_{w^2}(A)$	$i_{w^1}(c_{w^2}(A))$	A is $w(1,2)$-preopen set
$\emptyset$	$\{3\}$	$\emptyset$	✓
$\{1\}$	X	X	✓
$\{2\}$	$\{2,3\}$	$\{2,3\}$	✓
$\{3\}$	$\{3\}$	$\emptyset$	✗
$\{1,2\}$	X	X	✓
$\{1,3\}$	X	X	✓
$\{2,3\}$	$\{2,3\}$	$\{2,3\}$	✓
X	X	X	✓

and

A	$c_{w^1}(A)$	$i_{w^2}(c_{w^1}(A))$	A is $w(2,1)$-preopen set
$\emptyset$	$\emptyset$	$\emptyset$	✓
$\{1\}$	$\{1\}$	$\{1\}$	✓
$\{2\}$	$\{2\}$	$\emptyset$	✗
$\{3\}$	X	$\{1,2\}$	✗
$\{1,2\}$	X	$\{1,2\}$	✓
$\{1,3\}$	X	$\{1,2\}$	✗
$\{2,3\}$	X	$\{1,2\}$	✗
X	X	$\{1,2\}$	✗

From the above table, we obtain that:

1. $\emptyset, \{1\}, \{1,2\}$ are $w(1,2)$-preopen sets and $w(2,1)$-preopen sets.

2. $\{2\}, \{1,3\}, \{2,3\}, X$ are $w(1,2)$-preopen sets but not a $w(2,1)$-preopen sets.

3. $\{3\}$ is not $w(1,2)$-preopen set and $w(2,1)$-preopen set.

Theorem 3.2.14. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-preclosed if and only if $c_{w^k}(i_{w^l}(A)) \subset A$.

Proof. ($\Longrightarrow$) Assume that A is $w(k,l)$-preclosed. Then $X - A$ is $w(k,l)$-preopen. Thus $X - A \subset i_{w^k}(c_{w^l}(X - A))$, and so $X - A \subset X - c_{w^k}(i_{w^l}(A))$. Hence $c_{w^k}(i_{w^l}(A)) \subset A$.

($\Longleftarrow$) Assume that $c_{w^k}(i_{w^l}(A)) \subset A$. Then $X - A \subset X - c_{w^k}(i_{w^l}(A))$. Thus $X - A \subset i_{w^k}(c_{w^l}(X - A))$. Therefore, $X - A$ is $w(k,l)$-preopen. Hence A is $w(k,l)$-preclosed. □

Proposition 3.2.15. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If A_j is $w(k,l)$-preopen, for all $j \in \Lambda$, then $\bigcup_{j \in \Lambda} A_j$ is $w(k,l)$-preopen.

Proof. Assume that A_j is $w(k,l)$-preopen for all $j \in \Lambda$ and let $A = \bigcup_{j \in \Lambda} A_j$. Fix $j_0 \in \Lambda$. Then $A_{j_0} \subset A$. Thus $i_{w^k}(c_{w^l}(A_{j_0})) \subset i_{w^k}(c_{w^l}(A))$. Since A_{j_0} is $w(k,l)$-preopen, $A_{j_0} \subset i_{w^k}(c_{w^l}(A_{j_0}))$. This implies that $A_{j_0} \subset i_{w^k}(c_{w^l}(A))$. Since j_0 is an arbitrary element of Λ, we have $A_j \subset i_{w^k}(c_{w^l}(A))$ for all $j \in \Lambda$. Therefore $\bigcup_{j \in \Lambda} A_j \subset i_{w^k}(c_{w^l}(A))$. Thus $\bigcup_{j \in \Lambda} A_j$ is $w(k,l)$-preopen. □

Remark. The intersection of two $w(k,l)$-preopen sets is not an $w(k,l)$-preopen set in general as can be seen from the following example.

Example 3.2.16. From Example 3.2.13, we obtain that $\{1,2\}$ and $\{1,3\}$ are $w(1,2)$-preopen but $\{1,3\} \cap \{2,3\} = \{3\}$ is not $w(1,2)$-preopen.

Proposition 3.2.17. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If A_j is $w(k,l)$-preclosed, for all $j \in \Lambda$ then $\bigcap_{j \in \Lambda} A_j$ is $w(k,l)$-preclosed.

Proof. Assume that A_j is $w(k,l)$-preclosed for all $j \in \Lambda$. Then $X - A_j$ is $w(k,l)$-preopen for all $j \in \Lambda$. By Proposition 3.2.26, $\bigcup_{j \in \Lambda}(X - A_j)$ is $w(k,l)$-preopen. Since $\bigcup_{j \in \Lambda}(X - A_j) = X - \bigcap_{j \in \Lambda} A_j$, $X - \bigcap_{j \in \Lambda} A_j$ is $w(k,l)$-preopen. Thus $\bigcap_{j \in \Lambda} A_j$ is $w(k,l)$-preclosed. □

Remark. The union of two $w(k,l)$-preclosed sets is not an $w(k,l)$-preclosed set in general as can be seen from the following example.

Example 3.2.18. From Example 3.2.13, we obtain that $\{1\}$ and $\{2\}$ are $w(1,2)$-preclosed but $\{1\} \cup \{2\} = \{1,2\}$ is not $w(1,2)$-preclosed.

Theorem 3.2.19. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-preopen if and only if there exists a set U such that $U = i_{w^k}(U)$ and $A \subset U \subset c_{w^l}(A)$.

Proof. ($\Longrightarrow$) Assume that A is $w(k,l)$-preopen. Then $A \subset i_{w^k}(c_{w^l}(A))$. Let $U = i_{w^k}(c_{w^l}(A))$. Then $i_{w^k}(U) = i_{w^k}(i_{w^k}(c_{w^l}(A))) = i_{w^k}(c_{w^l}(A)) = U$. Since $A \subset i_{w^k}(c_{w^l}(A))$, $A \subset U$. Since $U = i_{w^k}(c_{w^l}(A)) \subset c_{w^l}(A)$, $U \subset c_{w^l}(A)$. Hence $A \subset U \subset c_{w^l}(A)$.

($\Longleftarrow$) Assume that there exists a set U such that $U = i_{w^k}(U)$ and $A \subset U \subset c_{w^l}(A)$. Since $U \subset c_{w^l}(A)$, $i_{w^k}(U) \subset i_{w^k}(c_{w^l}(A))$. Since $U = i_{w^k}(U)$ and $A \subset U$, $A \subset i_{w^k}(c_{w^l}(A))$. Hence A is $w(k,l)$-preopen. $\square$

Corollary 3.2.20. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If there exists $U \in w^k$ such that $A \subset U \subset c_{w^l}(A)$ then A is $w(k,l)$-preopen.

Proof. Assume that there exists $U \in w^k$ such that $A \subset U \subset c_{w^l}(A)$. Since $U \in w^k$, $i_{w^k}(U) = U$. By Theorem 3.2.19, A is $w(k,l)$-preopen. $\square$

Corollary 3.2.21. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If A is $w(k,l)$-preopen and $B \subset A \subset c_{w^l}(B)$, then B is $w(k,l)$-preopen.

Proof. Assume that A is $w(k,l)$-preopen and $B \subset A \subset c_{w^l}(A)$. Since A is $w(k,l)$-preopen, by Theorem 3.2.19, there exists a set U such that $i_{w^k}(U) = U$ and $A \subset U \subset c_{w^l}(A)$. Since $B \subset A \subset c_{w^l}(A)$, $B \subset A \subset U \subset c_{w^l}(A) \subset c_{w^l}(B)$. By Theorem 3.2.19, B is $w(k,l)$-preopen. $\square$

Corollary 3.2.22. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If A is $w(k,l)$-preclosed and $i_{w^l}(B) \subset A \subset B$, then B is $w(k,l)$-preclosed.

Proof. Assume that A is $w(k,l)$-preclosed and $i_{w^l}(B) \subset A \subset B$. Then $X - A$ is $w(k,l)$-preopen and $X - B \subset X - A \subset c_{w^l}(X - B)$. By Corollary 3.2.21, $X - B$ is $w(k,l)$-preopen. Hence B is $w(k,l)$-preclosed. $\square$

Finally, we shall introduce the notions of $w(k,l)$-β-open sets and investigate some of their properties.

Definition 3.2.23. Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then A is called a $w(k,l)$-β-*open set* if $A \subset c_{w^l}(i_{w^k}(c_{w^l}(A)))$. The complement of a $w(k,l)$-β-open set is called a $w(k,l)$-β-*closed set*.

Remark. Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then A is $w(k,l)$-β-closed if and only if $X - A$ is $w(k,l)$-β-open.

Example 3.2.24. Let $X = \{1,2,3\}$. Define WS w^1 and w^2 on X as follows:

$w^1 = \{\emptyset, \{1,2\}, \{1,3\}\}$ and $w^2 = \{\emptyset, \{1\}, \{2\}\}$. Then $X, \{3\}, \{2\}$ are w^1-closed and $X, \{2,3\}, \{1,3\}$ are w^2-closed. Consider

A	$c_{w^2}(A)$	$i_{w^1}(c_{w^2}(A))$	$c_{w^2}(i_{w^1}(c_{w^2}(A)))$	A is $w(1,2)$-β-open set
$\emptyset$	$\{3\}$	$\emptyset$	$\{3\}$	✓
$\{1\}$	$\{1,3\}$	$\{1,3\}$	$\{1,3\}$	✓
$\{2\}$	$\{2,3\}$	$\emptyset$	$\{3\}$	✗
$\{3\}$	$\{3\}$	$\emptyset$	$\{3\}$	✓
$\{1,2\}$	X	X	X	✓
$\{1,3\}$	$\{1,3\}$	$\{1,3\}$	$\{1,3\}$	✓
$\{2,3\}$	$\{2,3\}$	$\emptyset$	$\{3\}$	✗
X	X	X	X	✓

and

A	$c_{w^1}(A)$	$i_{w^2}(c_{w^1}(A))$	$c_{w^1}(i_{w^2}(c_{w^1}(A)))$	A is $w(2,1)$-β-open set
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	✓
$\{1\}$	X	$\{1,2\}$	X	✓
$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	✓
$\{3\}$	$\{3\}$	$\emptyset$	$\emptyset$	✗
$\{1,2\}$	X	$\{1,2\}$	X	✓
$\{1,3\}$	X	$\{1,2\}$	X	✓
$\{2,3\}$	X	$\{1,2\}$	X	✓
X	X	$\{1,2\}$	X	✓

From the above table, we obtain that:

1. $\emptyset, \{1\}, \{1,2\}, \{1,3\}, X$ are $w(1,2)$-β-open sets and $w(2,1)$-β-open sets.

2. $\{3\}$ is $w(1,2)$-β-open set but is not a $w(2,1)$-β-open set.

3. $\{2\}, \{2,3\}$ are $w(2,1)$-β-open sets but not a $w(1,2)$-β-open sets.

Theorem 3.2.25. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-β-closed if and only if $i_{w^l}(c_{w^k}(i_{w^l}(A))) \subset A$.

Proof. ($\Longrightarrow$) Assume that A is $w(k,l)$-β-closed. Then $X - A$ is $w(k,l)$-β-open. Thus $X - A \subset c_{w^l}(i_{w^k}(c_{w^l}(X - A)))$, and so $X - A \subset X - i_{w^l}(c_{w^k}(i_{w^l}(A)))$. Hence $i_{w^l}(c_{w^k}(i_{w^l}(A))) \subset A$.

($\Longleftarrow$) Assume that $i_{w^l}(c_{w^k}(i_{w^l}(A))) \subset A$. Then $X - A \subset X - i_{w^l}(c_{w^k}(i_{w^l}(A)))$. Thus $X - A \subset c_{w^l}(i_{w^k}(c_{w^l}(X - A)))$. Therefore, $X - A$ is $w(k,l)$-β-open. Hence A is $w(k,l)$-β-closed. □

Proposition 3.2.26. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If A_j is $w(k,l)$-β-open for all $j \in \Lambda$, then $\bigcup_{j\in\Lambda} A_j$ is $w(k,l)$-β-open.

Proof. Assume that A_j is $w(k,l)$-β-open for all $j \in \Lambda$ and let $A = \bigcup_{j\in\Lambda} A_j$. Fix $j_0 \in \Lambda$. Then $A_{j_0} \subset A$. Thus $c_{w^l}(i_{w^k}(c_{w^l}(A_{j_0}))) \subset c_{w^l}(i_{w^k}(c_{w^l}(A)))$. Since A_{j_0} is $w(k,l)$-β-open, $A_{j_0} \subset c_{w^l}(i_{w^k}(c_{w^l}(A_{j_0})))$. This implies that $A_{j_0} \subset c_{w^l}(i_{w^k}(c_{w^l}(A)))$. Since j_0

is an arbitrary element of Λ, we have $A_j \subset c_{w^l}(i_{w^k}(c_{w^l}(A)))$ for all $j \in \Lambda$. Therefore $\bigcup_{j\in\Lambda} A_j \subset c_{w^l}(i_{w^k}(c_{w^l}(A)))$. Thus $\bigcup_{j\in\Lambda} A_j$ is $w(k,l)$-β-open. □

Remark. The intersection of two $w(k,l)$-β-open sets is not an $w(k,l)$-β-open set in general as can be seen from the following example.

Example 3.2.27. From Example 3.2.24, we obtain that $\{1,3\}$ and $\{2,3\}$ are $w(2,1)$-β-open but $\{1,3\} \cap \{2,3\} = \{3\}$ is not $w(2,1)$-β-open.

Proposition 3.2.28. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If A_j is $w(k,l)$-β-closed for all $j \in \Lambda$, then $\bigcap_{j\in\Lambda} A_j$ is $w(k,l)$-β-closed.

Proof. Assume that A_j is $w(k,l)$-β-closed for all $j \in \Lambda$. Then $X - A_j$ is $w(k,l)$-β-open for all $j \in \Lambda$. By Proposition 3.2.26, $\bigcup_{j\in\Lambda}(X - A_j)$ is $w(k,l)$-β-open. Since $\bigcup_{j\in\Lambda}(X - A_j) = X - \bigcap_{j\in\Lambda} A_j$, $X - \bigcap_{j\in\Lambda} A_j$ is $w(k,l)$-β-open. Thus $\bigcap_{j\in\Lambda} A_j$ is $w(k,l)$-β-closed. □

Remark. The union of two $w(k,l)$-β-closed sets is not an $w(k,l)$-β-closed set in general as can be seen from the following example.

Example 3.2.29. From Example 3.2.24, we obtain that $\{1\}$ and $\{2\}$ are $w(2,1)$-β-closed but $\{1\} \cup \{2\} = \{1,2\}$ is not $w(2,1)$-β-closed.

Theorem 3.2.30. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-β-open if and only if there exists a subset U of X such that $U = i_{w^k}(U)$ and $A \subset c_{w^l}(U) \subset c_{w^l}(A)$.

Proof. ($\Longrightarrow$) Assume that A is $w(k,l)$-β-open. Then $A \subset c_{w^l}(i_{w^k}(c_{w^l}(A)))$. Let $U = i_{w^k}(c_{w^l}(A))$. Then $i_{w^k}(U) = i_{w^k}(i_{w^k}(c_{w^l}(A))) = i_{w^k}(c_{w^l}(A)) = U$. Since $U = i_{w^k}(c_{w^l}(A))$ and $A \subset c_{w^l}(i_{w^k}(c_{w^l}(A)))$, $A \subset c_{w^l}(U) = c_{w^l}(i_{w^k}(c_{w^l}(A))) \subset c_{w^l}(A)$.

($\Longleftarrow$) Assume that there exists a subset U of X such that $U = i_{w^k}(U)$ and $A \subset c_{w^l}(U) \subset c_{w^l}(A)$. Thus $A \subset c_{w^l}(U) \subset c_{w^l}(i_{w^k}(U)) = c_{w^l}(i_{w^k}(c_{w^l}(i_{w^k}(U)))) = c_{w^l}(i_{w^k}(c_{w^l}(U))) \subset c_{w^l}(i_{w^k}(c_{w^l}(A)))$. Hence A is $w(k,l)$-β-open. □

Corollary 3.2.31. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If there exists $U \in w^k$ such that $A \subset c_{w^l}(U) \subset c_{w^l}(A)$, then A is $w(k,l)$-β-open.

Proof. Assume that there exists $U \in w^k$ such that $A \subset c_{w^l}(U) \subset c_{w^l}(A)$. Since $U \in w^k$, $i_{w^k}(U) = U$. By Theorem 3.2.30, A is $w(k,l)$-β-open. □

Corollary 3.2.32. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If A is $w(k,l)$-β-open and $B \subset A \subset c_{w^l}(B)$, then B is $w(k,l)$-β-open.

Proof. Assume that A is $w(k,l)$-β-open and $B \subset A \subset c_{w^l}(B)$. Since A is $w(k,l)$-β-open, By Theorem 3.2.30, there exists a subset U of X such that $i_{w^k}(U) = U$ and $A \subset c_{w^l}(U) \subset c_{w^l}(A)$. Since $B \subset A \subset c_{w^l}(B)$, $B \subset A \subset c_{w^l}(U) \subset c_{w^l}(A) \subset c_{w^l}(B)$. By Theorem 3.2.30, B is $w(k,l)$-β-open. □

Corollary 3.2.33. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If A is $w(k,l)$-β-closed and $i_{w^l}(B) \subset A \subset B$, then B is $w(k,l)$-β-closed.

Proof. Assume that A is $w(k,l)$-β-closed and $i_{w^l}(B) \subset A \subset B$. Then $X - A$ is $w(k,l)$-β-open and $X - A \subset X - B \subset c_{w^l}(X - B)$. By Corollary 3.2.32, $X - B$ is $w(k,l)$-β-open. Hence B is $w(k,l)$-β-closed. □

CHAPTER 4

RELATIONSHIPS OF $W(K,L)$-SEMI-OPEN SETS, $W(K,L)$-α-OPEN SETS, $W(K,L)$-PREOPEN SETS AND $W(K,L)$-β-OPEN SETS

In this chapter, we will discuss some relations of $w(k,l)$-semi-open sets, $w(k,l)$-α-open sets, $w(k,l)$-preopen sets and $w(k,l)$-β-open sets.

4.1 Relation of $w(k,l)$-semi-open set, $w(k,l)$-α-open set, $w(k,l)$-preopen set and $w(k,l)$-β-open set

In this section, it begins with a review of the definitions of open sets in spaces. Let (X, w^1, w^2) be a bi-w space and $A \subset X$.

A is called $w(k,l)$-semi-open if and only if $A \subset c_{w^l}(i_{w^k}(A))$.

A is called $w(k,l)$-α-open if and only if $A \subset i_{w^k}(c_{w^l}(i_{w^k}(A)))$.

A is called $w(k,l)$-preopen if and only if $A \subset i_{w^k}(c_{w^l}(A))$.

A is called $w(k,l)$-β-open if and only if $A \subset c_{w^l}(i_{w^k}(c_{w^l}(A)))$.

Theorem 4.1.1. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Every open is $w(k,l)$-α-open.

Proof. Assume that A is open. Then $A = i_{w^k}(A)$ and $A = i_{w^l}(A)$. Thus $A \subset c_{w^l}(A) = c_{w^l}(i_{w^k}(A))$ and so $A = i_{w^k}(A) \subset i_{w^k}(c_{w^l}(i_{w^k}(A)))$. Hence A is $w(k,l)$-α-open. □

Theorem 4.1.2. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Every $w(k,l)$-α-open is $w(k,l)$-semi-open.

Proof. Assume that A is $w(k,l)$-α-open. Then $A \subset i_{w^k}(c_{w^l}(i_{w^k}(A)))$. Thus $A \subset i_{w^k}(c_{w^l}(i_{w^k}(A))) \subset c_{w^l}(i_{w^k}(A))$. Hence A is $w(k,l)$-semi-open. □

Theorem 4.1.3. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Every $w(k,l)$-α-open is $w(k,l)$-preopen.

Proof. Assume that A is $w(k,l)$-α-open. Then $A \subset i_{w^k}(c_{w^l}(i_{w^k}(A)))$. Thus $A \subset i_{w^k}(c_{w^l}(i_{w^k}(A))) \subset i_{w^k}(c_{w^l}(A))$. Hence A is $w(k,l)$-preopen. $\square$

Remark. In general, the inverse of Theorem 4.1.2 and Theorem 4.1.3 are not true. Moreover, $w(k,l)$-semi-open sets and $w(k,l)$-preopen sets have no relationship as shown in the following example.

Example 4.1.4. Let $X = \{1,2,3\}$. Define WS w^1 and w^2 on X as follows:

$w^1 = \{\emptyset, \{1,2\}, \{1,3\}\}$ and $w^2 = \{\emptyset, \{1\}, \{2\}\}$. Then $X, \{3\}, \{2\}$ are w^1-closed and $X, \{2,3\}, \{1,3\}$ are w^2-closed. Consider

A	$i_{w^1}(A)$	$c_{w^2}(i_{w^1}(A))$	$i_{w^1}(c_{w^2}(i_{w^1}(A)))$	A is $w(1,2)$-α-open set
$\emptyset$	$\emptyset$	$\{3\}$	$\emptyset$	✓
$\{1\}$	$\emptyset$	$\{3\}$	$\emptyset$	✗
$\{2\}$	$\emptyset$	$\{3\}$	$\emptyset$	✗
$\{3\}$	$\emptyset$	$\{3\}$	$\emptyset$	✗
$\{1,2\}$	$\{1,2\}$	X	X	✓
$\{1,3\}$	$\{1,3\}$	$\{1,3\}$	$\{1,3\}$	✓
$\{2,3\}$	$\emptyset$	$\{3\}$	$\emptyset$	✗
X	X	X	X	✓

A	$i_{w^1}(A)$	$c_{w^2}(i_{w^1}(A))$	A is $w(1,2)$-semi-open set
$\emptyset$	$\emptyset$	$\{3\}$	✓
$\{1\}$	$\emptyset$	$\{3\}$	✗
$\{2\}$	$\emptyset$	$\{3\}$	✗
$\{3\}$	$\emptyset$	$\{3\}$	✓
$\{1,2\}$	$\{1,2\}$	X	✓
$\{1,3\}$	$\{1,3\}$	$\{1,3\}$	✓
$\{2,3\}$	$\emptyset$	$\{3\}$	✗
X	X	X	✓

and

A	$c_{w^2}(A)$	$i_{w^1}(c_{w^2}(A))$	A is $w(1,2)$-preopen set
$\emptyset$	$\{3\}$	$\emptyset$	✓
$\{1\}$	$\{1,3\}$	$\{1,3\}$	✓
$\{2\}$	$\{2,3\}$	$\emptyset$	✗
$\{3\}$	$\{3\}$	$\emptyset$	✗
$\{1,2\}$	X	X	✓
$\{1,3\}$	$\{1,3\}$	$\{1,3\}$	✓
$\{2,3\}$	$\{2,3\}$	$\emptyset$	✗
X	X	X	✓

From the table above, it can be seen that:

1. $\{3\}$ is a $w(1,2)$-semi-open set but $\{3\}$ is not a $w(2,1)$-α-open set and $w(1,2)$-preopen set.
2. $\{1\}$ is a $w(1,2)$-preopen set but $\{1\}$ is not a $w(2,1)$-α-open set and $w(1,2)$-semi-open set.

Theorem 4.1.5. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Every $w(k,l)$-semi-open is $w(k,l)$-β-open.

Proof. Assume that A is $w(k,l)$-semi-open. Then $A \subset c_{w^l}(i_{w^k}(A))$. Thus $A \subset c_{w^l}(i_{w^k}(A)) \subset c_{w^l}(i_{w^k}(c_{w^l}(A)))$. Hence A is $w(k,l)$-β-open. □

Theorem 4.1.6. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Every $w(k,l)$-preopen is $w(k,l)$-β-open.

Proof. Assume that A is $w(k,l)$-preopen. Then $A \subset i_{w^k}(c_{w^l}(A))$. Thus $A \subset c_{w^l}(A) \subset c_{w^l}(i_{w^k}(c_{w^l}(A)))$. Hence A is $w(k,l)$-β-open. □

Remark. In general, the inverse of Theorem 4.1.5 and Theorem 4.1.6 are not true, as in the following example.

Example 4.1.7. Let $X = \{1,2,3\}$. Define WS w^1 and w^2 on X as follows:

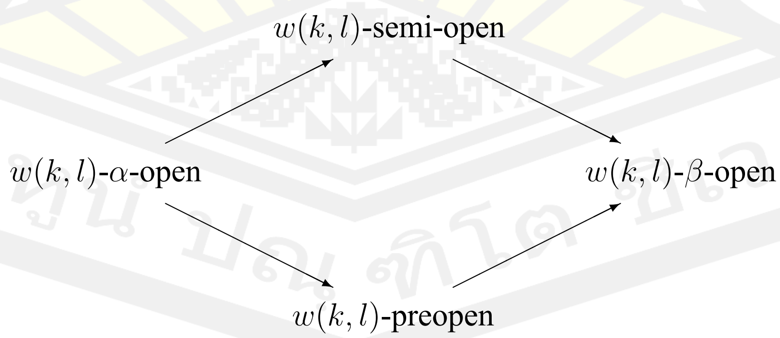
$w^1 = \{\emptyset, \{1,2\}, \{1,3\}\}$ and $w^2 = \{\emptyset, \{1\}, \{2\}\}$. Then $X, \{3\}, \{2\}$ are w^1-closed and $X, \{2,3\}, \{1,3\}$ are w^2-closed. Consider

A	$c_{w^2}(A)$	$i_{w^1}(c_{w^2}(A))$	$c_{w^2}(i_{w^1}(c_{w^2}(A)))$	A is $w(1,2)$-β-open set
$\emptyset$	$\{3\}$	$\emptyset$	$\{3\}$	✓
$\{1\}$	$\{1,3\}$	$\{1,3\}$	$\{1,3\}$	✓
$\{2\}$	$\{2,3\}$	$\emptyset$	$\{3\}$	✗
$\{3\}$	$\{3\}$	$\emptyset$	$\{3\}$	✓
$\{1,2\}$	X	X	X	✓
$\{1,3\}$	$\{1,3\}$	$\{1,3\}$	$\{1,3\}$	✓
$\{2,3\}$	$\{2,3\}$	$\emptyset$	$\{3\}$	✗
X	X	X	X	✓

From the table above and Example 4.1.4, it can be seen that

1. $\{1\}$ is a $w(1,2)$-β-open set but $\{1\}$ is not a $w(1,2)$-semi-open set,
2. $\{3\}$ is a $w(1,2)$-β-open set but $\{3\}$ is not a $w(1,2)$-preopen set.

Remark. From the theorems and and examples above in this topic. We can conclude that the relationship between $w(k,l)$-semi-open set $w(k,l)$-α-open set $w(k,l)$-preopen set as the following diagram $w(k,l)$-β-open set



4.2 $w(k,l)$-regular open set

Definition 4.2.1. Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then A is called a $w(k,l)$-*regular open set* if $A = i_{w^k}(c_{w^l}(A))$. The complement of a $w(k,l)$-regular open set is called a $w(k,l)$*-regular closed set*.

Remark. Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then A is $w(k,l)$-regular closed if and only if $X - A$ is $w(k,l)$- regular open.

Theorem 4.2.2. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-regular closed if and only if $c_{w^k}(i_{w^l}(A)) = A$.

Proof. ($\Longrightarrow$) Assume that A is $w(k,l)$-regular closed. Then $X - A$ is $w(k,l)$-regular open. Thus $X - A = i_{w^k}(c_{w^l}(X - A))$, and so $X - A = X - c_{w^k}(i_{w^l}(A))$. Hence $c_{w^k}(i_{w^l}(A)) = A$.

($\Longleftarrow$) Assume that $c_{w^k}(i_{w^l}(A)) = A$. Then $X - A = X - c_{w^k}(i_{w^l}(A))$. Thus $X - A = i_{w^k}(c_{w^l}(X - A))$. Therefore, $X - A$ is $w(k,l)$-regular open. Hence A is $w(k,l)$-regular closed. □

Theorem 4.2.3. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-regular open if and only if $A = i_{w^k}(A)$ and A is $w(l,k)$-semi-closed.

Proof. ($\Longrightarrow$) Assume that A is $w(k,l)$-regular open. Then $A = i_{w^k}(c_{w^l}(A))$. Thus $i_{w^k}(c_{w^l}(A)) \subset A$. Hence A is $w(l,k)$-semi-closed. Since $A = i_{w^k}(c_{w^l}(A))$, Then $i_{w^k}(A) = i_{w^k}(i_{w^k}(c_{w^l}(A))) = i_{w^k}(c_{w^l}(A)) = A$.

($\Longleftarrow$) Assume that $A = i_{w^k}(A)$ and A is $w(l,k)$-semi-closed. Then $i_{w^k}(c_{w^l}(A)) \subset A$. Since $A \subset c_{w^l}(A)$, Then $A = i_{w^k}(A) \subset i_{w^k}(c_{w^l}(A))$. Such that $A = i_{w^k}(c_{w^l}(A))$. Hence, A is $w(k,l)$-regular open. □

Theorem 4.2.4. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-regular open if and only if A is $w(k,l)$-α-open and $w(l,k)$-semi-closed.

Proof. ($\Longrightarrow$) Assume that A is $w(k,l)$-regular open. Then $A = i_{w^k}(c_{w^l}(A))$. Thus $i_{w^k}(c_{w^l}(A)) \subset A$. Hence A is $w(l,k)$-semi-closed. Since $A \subset i_{w^k}(c_{w^l}(A))$ and $A = i_{w^k}(A)$. Then $A \subset i_{w^k}(c_{w^l}(i_{w^k}(A)))$. Hence, A is $w(k,l)$-α-open.

($\Longleftarrow$) Assume that A $w(k,l)$-α-open and $w(l,k)$-semi-closed. Then $A \subset i_{w^k}(c_{w^l}(i_{w^k}(A)))$ and $i_{w^k}(c_{w^l}(A)) \subset A$. Since $i_{w^k}(A) \subset A$, $A \subset i_{w^k}(c_{w^l}(i_{w^k}(A))) \subset i_{w^k}(c_{w^l}(A))$. Such that $A = i_{w^k}(c_{w^l}(A))$. Hence A is $w(k,l)$-regular open. □

Theorem 4.2.5. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-regular open if and only if A is $w(k,l)$-preopen and $w(l,k)$-semi-closed.

Proof. ($\Longrightarrow$) Assume that A is $w(k,l)$-regular open. By Theorem 4.2.4, A is $w(l,k)$-semi-closed and $w(k,l)$-α-open. By Theorem 4.1.3, A is $w(k,l)$-preopen.

($\Longleftarrow$) Assume that A $w(k,l)$-preopen and $w(l,k)$-semi-closed. Then $A \subset i_{w^k}(c_{w^l}(A))$ and $i_{w^k}(c_{w^l}(A)) \subset A$. Thus $A = i_{w^k}(c_{w^l}(A))$. Hence A is $w(k,l)$-regular open. □

Theorem 4.2.6. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-regular open if and only if $A = i_{w^k}(A)$ and $w(l,k)$-β-closed.

Proof. ($\Longrightarrow$) Assume that A is $w(k,l)$-regular open. Then $A = i_{w^k}(c_{w^l}(A))$. Thus $i_{w^k}(A) = i_{w^k}(i_{w^k}(c_{w^l}(A)))$, and so $i_{w^k}(A) = i_{w^k}(c_{w^l}(A)) = A$. Since $i_{w^k}(c_{w^l}(A)) \subset A$, $i_{w^k}(c_{w^l}(i_{w^k}(A))) \subset A$. Hence A is $w(l,k)$-β-closed.

($\Longleftarrow$) Assume that $A = i_{w^k}(A)$ and $w(l,k)$-β-closed. Then $i_{w^k}(c_{w^l}(i_{w^k}(A))) \subset A$, and so $i_{w^k}(c_{w^l}(A)) \subset A$. Since $A \subset c_{w^l}(A)$, $A = i_{w^k}(A) \subset i_{w^k}(c_{w^l}(A))$. Then $A = i_{w^k}(c_{w^l}(A))$. Hence A is $w(k,l)$-regular open. □

Theorem 4.2.7. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-regular open if and only if $w(k,l)$-α-open and $w(l,k)$-β-closed.

Proof. ($\Longrightarrow$) Assume that A is $w(k,l)$-regular open. Then $A = i_{w^k}(c_{w^l}(A))$. Since $A = i_{w^k}(A)$, $A \subset i_{w^k}(c_{w^l}(i_{w^k}(A)))$. Then $w(k,l)$-α-open. Since $i_{w^k}(c_{w^l}(A)) \subset A$ and $A = i_{w^k}(A)$, $i_{w^k}(c_{w^l}(i_{w^k}(A))) \subset A$. Hence A is $w(l,k)$-β-closed.

($\Longleftarrow$) Assume that $w(k,l)$-α-open and $w(l,k)$-β-closed. Then $A \subset i_{w^k}(c_{w^l}(i_{w^k}(A)))$ and $i_{w^k}(c_{w^l}(i_{w^k}(A))) \subset A$. Thus $A = i_{w^k}(c_{w^l}(i_{w^k}(A)))$, and so $i_{w^k}(A) = i_{w^k}(c_{w^l}(i_{w^k}(A))) = A$. By Theorem 4.2.6, A is $w(k,l)$-regular open. □

CHAPTER 5

CONCLUSIONS

5.1 Conclusions

The aim of this thesis is to study the $w(k,l)$-semi-open sets in a bi-weak structure space, and introduce. Some properties of $w(k,l)$-semi-open sets are obtained. Furthermore, the notion of $w(k,l)$-semi-continuity using $w(k,l)$-semi-open sets are introduced. Moreover, we introduce $w(k,l)$-preopen sets $w(k,l)$-α-open sets and $w(k,l)$-β-open sets in bi-weak structure space and study some of their properties and relationships of those sets. The results are follows :

1) Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then A is called a $w(k,l)$-*semi-open set* if $A \subset c_{w^l}(i_{w^k}(A))$. The complement of a $w(k,l)$-semi-open set is called a $w(k,l)$-*semi-closed set*.

From the above definition, the following theorems are derived:

1.1) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-semi-closed if and only if $i_{w^l}(c_{w^k}(A)) \subset A$.

1.2) Let (X, w^1, w^2) be a bi-w space. If A_j is $w(k,l)$-semi-open for all $j \in \Lambda$, then $\bigcup_{j\in\Lambda} A_j$ is $w(k,l)$-semi-open.

1.3) Let (X, w^1, w^2) be a bi-w space. If A_j is $w(k,l)$-semi-closed for all $j \in \Lambda$, then $\bigcap_{j\in\Lambda} A_j$ is $w(k,l)$-semi-closed.

1.4) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-semi-open if and only if there exists a subset U of X such that $i_{w^k}(U) = U$ and $U \subset A \subset c_{w^l}(U)$.

1.5) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If there exists $U \in w^k$ such that $U \subset A \subset c_{w^l}(U)$ then A is $w(k,l)$-semi-open.

1.6) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If A is $w(k,l)$-semi-open and $A \subset B \subset c_{w^l}(A)$, then B is $w(k,l)$-semi-open.

1.7) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If A is $w(k,l)$-semi-closed and $i_{w^l}(A) \subset B \subset A$, then B is $w(k,l)$-semi-closed.

2) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. The intersection of all $w(k,l)$-semi-closed

sets containing A is called the $w(k,l)$-*semi-closure* of A and is denoted by $c_\sigma^{kl}(A)$. The union of all $w(k,l)$-semi-open sets contained in A is called the $w(k,l)$-*semi-interior* of A and is denoted by $i_\sigma^{kl}(A)$, respectively, that is,
$c_\sigma^{kl}(A) = \bigcap\{F : F \text{ is } w(k,l)\text{-semi-closed and } A \subset F\}$,
$i_\sigma^{kl}(A) = \bigcup\{G : G \text{ is } w(k,l)\text{-semi-open and } G \subset A\}$.

From the above definition, the following theorems are derived:

2.1) Let (X, w^1, w^2) be a bi-w space and $A \subset X$, $x \subset X$. Then $x \in c_\sigma^{kl}(A)$ if and only if $V \cap A \neq \emptyset$ for all $w(k,l)$-semi-open set V containing x.

2.2) Let (X, w^1, w^2) be a bi-w space and $A \subset X$, $x \subset X$. Then $x \in i_\sigma^{kl}(A)$ if and only if there exists a $w(k,l)$-semi-open set V such that $x \in V \subset A$.

2.3) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then $i_\sigma^{kl}(X - A) = X - c_\sigma^{kl}(A)$ and $c_\sigma^{kl}(X - A) = X - i_\sigma^{kl}(A)$.

2.4) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-semi-closed if and only if $A = c_\sigma^{kl}(A)$.

2.5) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-semi-open if and only if $A = i_\sigma^{kl}(A)$.

3) A function $f : (X, w^1, w^2) \to (Y, \nu^1, \nu^2)$ is called $w(k,l)$-*semi-continuous* at x if, for all ν^k-open set V containing $f(x)$, there exists a $w(k,l)$-semi-open set U containing x such that $f(U) \subset V$. Then f is called $w(k,l)$-*semi-continuous* if f is $w(k,l)$-semi-continuous at x, for all $x \in X$.

From the above definition, the following theorems are derived:

3.1) Let (X, w^1, w^2) and (Y, ν^1, ν^2) be two bi weak structure spaces and $f : (X, w^1, w^2) \to (Y, \nu^1, \nu^2)$. Then the following are equivalent :

1. f is $w(k,l)$-semi-continuous.
2. $f^{-1}(V)$ is $w(k,l)$-semi-open for all $V \in \nu^k$.
3. $f^{-1}(F)$ is $w(k,l)$-semi-closed for all ν^k-closed set F.
4. $f(c_\sigma^{kl}(A)) \subset c_{\nu^k}(f(A))$ for all $A \subset X$.
5. $c_\sigma^{kl}(f^{-1}(B)) \subset f^{-1}(c_{\nu^k}(B))$ for all $B \subset Y$.
6. $f^{-1}(i_{\nu^k}(B)) \subset i_\sigma^{kl}(f^{-1}(B))$ for all $B \subset Y$.

4) Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then A is called $w(k,l)$-*α-open set* if $A \subset i_{w^k}(c_{w^l}(i_{w^k}(A)))$. The complement of a $w(k,l)$-α-open set is called $w(k,l)$-*α-closed set*.

From the above definition, the following theorems are derived:

4.1) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-α-closed if and only if $c_{w^k}(i_{w^l}(c_{w^k}(A))) \subset A$.

4.2) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If A_j is $w(k,l)$-α-open for all $j \in \Lambda$, then $\bigcup_{j\in\Lambda} A_j$ is $w(k,l)$-α-open.

4.3) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If A_j is $w(k,l)$-α-closed for all $j \in \Lambda$, then $\bigcap_{j\in\Lambda} A_j$ is $w(k,l)$-α-closed.

4.4) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-α-open if and only if there exists a set U such that $U = i_{w^k}(U)$ and $U \subset A \subset i_{w^k}(c_{w^l}(U))$.

4.5) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If there exists $U \in w^k$ such that $U \subset A \subset i_{w^k}(c_{w^l}(U))$ then A is $w(k,l)$-α-open.

4.6) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If A is $w(k,l)$-α-open and $A \subset B \subset i_{w^k}(c_{w^l}(A))$, then B is $w(k,l)$-α-open.

4.7) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If A is $w(k,l)$-α-closed and $c_{w^k}(i_{w^l}(A)) \subset B \subset A$, then B is $w(k,l)$-α-closed.

5) Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then A is called $w(k,l)$-*preopen set* if $A \subset i_{w^k}(c_{w^l}(A))$. The complement of a $w(k,l)$-preopen set is called $w(k,l)$-*preclosed set*.

From the above definition, the following theorems are derived:

5.1) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-preclosed if and only if $c_{w^k}(i_{w^l}(A)) \subset A$.

5.2) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If A_j is $w(k,l)$-preopen for all $j \in \Lambda$, then $\bigcup_{j\in\Lambda} A_j$ is $w(k,l)$-preopen.

5.3) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If A_j is $w(k,l)$-preclosed for all $j \in \Lambda$, then $\bigcap_{j\in\Lambda} A_j$ is $w(k,l)$-preclosed.

5.4) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-preopen if and only if there exists a set U such that $U = i_{w^k}(U)$ and $A \subset U \subset c_{w^l}(A)$.

5.5) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If there exists $U \in w^k$ such that $A \subset U \subset c_{w^l}(A)$ then A is $w(k,l)$-preopen.

5.6) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If A is $w(k,l)$-preopen and $B \subset A \subset c_{w^l}(B)$, then B is $w(k,l)$-preopen.

5.7) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If A is $w(k,l)$-preclosed and $i_{w^l}(B) \subset A \subset B$, then B is $w(k,l)$-preclosed.

6) Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then A is called $w(k,l)$-*β-open set* if $A \subset c_{w^l}(i_{w^k}(c_{w^l}(A)))$. The complement of a $w(k,l)$-β-open set is called $w(k,l)$-*β-closed set*.

From the above definition, the following theorems are derived:

6.1) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-β-closed if and only if $i_{w^l}(c_{w^k}(i_{w^l}(A))) \subset A$.

6.2) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If A_j is $w(k,l)$-β-open for all $j \in \Lambda$, then $\bigcup\limits_{j\in\Lambda} A_j$ is $w(k,l)$-β-open.

6.3) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If A_j is $w(k,l)$-β-closed for all $j \in \Lambda$, then $\bigcap\limits_{j\in\Lambda} A_j$ is $w(k,l)$-β-closed.

6.4) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-β-open if and only if there exists a subset U of X such that $U = i_{w^k}(U)$ and $A \subset c_{w^l}(U) \subset c_{w^l}(A)$.

6.5) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If there exists $U \in w^k$ such that $A \subset c_{w^l}(U) \subset c_{w^l}(A)$ then A is $w(k,l)$-β-open.

6.6) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If A is $w(k,l)$-β-open and $B \subset A \subset c_{w^l}(B)$, then B is $w(k,l)$-β-open.

6.7) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If A is $w(k,l)$-β-closed and $i_{w^l}(B) \subset A \subset B$, then B is $w(k,l)$-β-closed.

7) Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then A is called $w(k,l)$-*regular open set* if $A = i_{w^k}(c_{w^l}(A))$. The complement of a $w(k,l)$-regular open set is called $w(k,l)$-*regular closed set*.

From the above definition, the following theorems are derived:

7.1) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-regular closed if and only if $c_{w^k}(i_{w^l}(A)) = A$.

7.2) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-regular open if and only if $A = i_{w^k}(A)$ and A is $w(l,k)$-semi-closed.

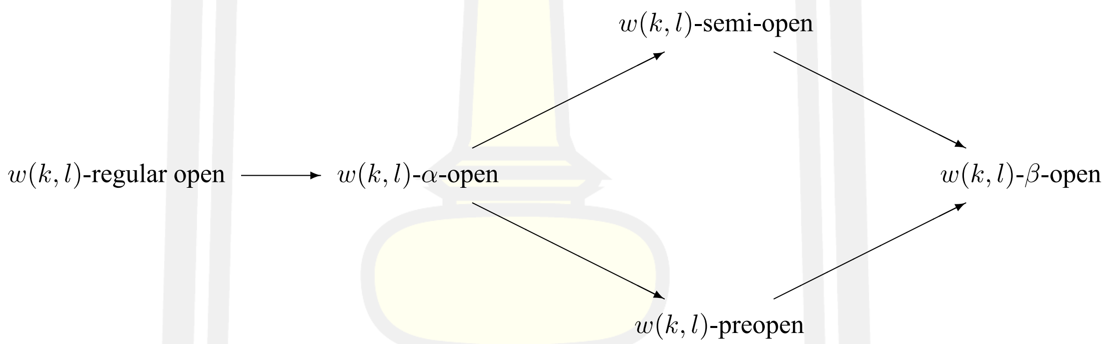
7.3) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-regular open if and only if A is $w(k,l)$-α-open and $w(l,k)$-semi-closed.

7.4) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-regular open if and only if A is $w(k,l)$-preopen and $w(l,k)$-semi-closed.

7.5) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-regular open if and only if $A = i_{w^k}(A)$ and $w(l,k)$-β-closed.

7.6) Let (X, w^1, w^2) be a bi-w space and $A \subset X$. Then A is $w(k,l)$-regular open if and only if $w(k,l)$-α-open and $w(l,k)$-β-closed.

The following diagram, showing relationships of $w(k,l)$-semi-open sets, $w(k,l)$-preopen sets, $w(k,l)$-α-open sets, $w(k,l)$-β-open sets and $w(k,l)$-regular open sets



5.2 Recommendations

Even though, I have found several properties as of the sets and space presented in this thesis, there are several questions yet to be answered and it may be worth investigating in future studies. I formulate the questions as follows :

1. Do study the notion of continuity using the $w(k,l)$-preopen sets, $w(k,l)$-α-open sets and $w(k,l)$-β-open sets ?

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