$\delta(\tau_1, \tau_2)$-Continuous Functions

Chatchadaporn Prachanpol

A Thesis Submitted in Partial Fulfillment of Requirements for
degree of Master of Science in Mathematics Education
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ฟังก์ชันต่อเนื่องแบบ $\delta(\tau_1, \tau_2)$

วิทยานิพนธ์
ของ
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The examining committee has unanimously approved this Thesis, submitted by Miss Chatchadaporn Prachanpol, as a partial fulfillment of the requirements for the Master of Science in Mathematics Education at Mahasarakham University.

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บทคัดย่อ

งานวิจัยนี้ผู้วิจัยได้นำเสนอแนวคิดของฟังก์ชันต่อเนื่องแบบ $\delta(\tau_1, \tau_2)$ ฟังก์ชันเกือบต่อเนื่องแบบ $\delta(\tau_1, \tau_2)$ และฟังก์ชันต่อเนื่องอย่างอ่อนแบบ $\delta(\tau_1, \tau_2)$ นอกจากนี้ยังแสดงถึงลักษณะเฉพาะของฟังก์ชันต่อเนื่องต่อเนื่องแบบ $\delta(\tau_1, \tau_2)$ ฟังก์ชันเกือบต่อเนื่องแบบ $\delta(\tau_1, \tau_2)$ และฟังก์ชันต่อเนื่องอย่างอ่อนแบบ $\delta(\tau_1, \tau_2)$ พร้อมทั้งอภิปรายความสัมพันธ์ของฟังก์ชันต่อเนื่องแบบ $\delta(\tau_1, \tau_2)$ ฟังก์ชันเกือบต่อเนื่องแบบ $\delta(\tau_1, \tau_2)$ และฟังก์ชันต่อเนื่องอย่างอ่อนแบบ $\delta(\tau_1, \tau_2)$

คำสำคัญ : เซตเปิด $\delta(\tau_1, \tau_2)$, ฟังก์ชันต่อเนื่องแบบ $\delta(\tau_1, \tau_2)$, ฟังก์ชันเกือบต่อเนื่องแบบ $\delta(\tau_1, \tau_2)$, ฟังก์ชันต่อเนื่องอย่างอ่อนแบบ $\delta(\tau_1, \tau_2)$

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ABSTRACT

This research presents the notions of $\delta(\tau_1, \tau_2)$-continuous functions, almost $\delta(\tau_1, \tau_2)$-continuous functions, and weakly $\delta(\tau_1, \tau_2)$-continuous functions. Moreover, several characterizations of $\delta(\tau_1, \tau_2)$-continuous functions, almost $\delta(\tau_1, \tau_2)$-continuous functions, and weakly $\delta(\tau_1, \tau_2)$-continuous functions are investigated. The relationships among $\delta(\tau_1, \tau_2)$-continuous functions, almost $\delta(\tau_1, \tau_2)$-continuous functions, and weakly $\delta(\tau_1, \tau_2)$-continuous functions are also discussed.

Keywords : $\delta(\tau_1, \tau_2)$-open, $\delta(\tau_1, \tau_2)$-continuous function, almost $\delta(\tau_1, \tau_2)$-continuous function, weakly $\delta(\tau_1, \tau_2)$-continuous function

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Background

Topology is a branch of mathematics that abstracts analysis and geometry, concerned with the study of topological spaces. A topological space is a mathematical structure consisting of a set with subsets referred to as open sets. The complement of an open set is called a closed set. Both open and closed sets play important roles in topological spaces. Additionally, there are two fundamental operators: closure and interior, which correspond to closed sets and open sets, respectively. Later, mathematicians studied various generalized sets of open sets and the properties of them. In 1968, Veličko [13] introduced a class of open sets in topological spaces called δ-open sets and he investigated the concepts of δ-closed sets and δ-closure.

Continuity is a fundamental concept in general topology. This concept has been generalized using open and closed sets. In 1980, Noiri [10] introduced a new class of functions in a topological space known as δ-continuous functions. He also investigated various characterizations of δ-continuous functions and the relationship between δ-continuity and almost continuity. However, an almost-continuous function may not necessarily be δ-continuous. Rose [11] introduced the concept of weakly continuous functions and investigated the relationships between continuous and weakly continuous functions in 1984. The characterizations of weakly δ-continuous functions and the relationship between weak continuity and weak δ-continuity were studied in greater depth by Baker [2] in 1985.

Kelly [7] initiated the concept of bitopological spaces in 1963, and he investigated the separation axioms in bitopological spaces. Subsequently, many mathematicians have introduced and studied various classes of open and closed sets in bitopological spaces. In 2018, Boonpok et al. [4] introduced the notions of $\tau_1\tau_2$-open sets, $\tau_1\tau_2$-closed sets, and

$\tau_1\tau_2$-closure in bitopological spaces. Additionally, Viriyapong and Boonpok [14] introduced and investigated some properties of $(\tau_1, \tau_2)r$-open sets.

Consequently, the researchers are interested in defining and investigating various characterizations of $\delta(\tau_1, \tau_2)$-continuous functions, almost $\delta(\tau_1, \tau_2)$-continuous functions, and weakly $\delta(\tau_1, \tau_2)$-continuous functions. We further discuss the relationships among almost $\delta(\tau_1, \tau_2)$-continuous functions, weakly $\delta(\tau_1, \tau_2)$-continuous functions, and $\delta(\tau_1, \tau_2)$-continuous functions.

1.2 Objectives of the research

The purposes of the research are:

(1) To define and investigate the characterizations of $\delta(\tau_1, \tau_2)$-continuous functions, almost $\delta(\tau_1, \tau_2)$-continuous functions and weakly $\delta(\tau_1, \tau_2)$-continuous functions.

(2) To study the relationships of $\delta(\tau_1, \tau_2)$-continuous functions, almost $\delta(\tau_1, \tau_2)$-continuous functions and weakly $\delta(\tau_1, \tau_2)$-continuous functions.

1.3 Research methodology

The research procedures of this thesis consists of the following steps:

(1) To criticize possible extension of the literature review.

(2) To define and investigate the characterizations of $\delta(\tau_1, \tau_2)$-continuous functions.

(3) To define and investigate the characterizations of almost $\delta(\tau_1, \tau_2)$-continuous functions.

(4) To define and investigate the characterizations of weakly $\delta(\tau_1, \tau_2)$-continuous functions.

(5) To make the conclusions and do a complete report to offer Mahasarakham University.

1.4 Scope of the study

The scope of the study are: the characterizations and relationships of $\delta(\tau_1, \tau_2)$-continuous functions, almost $\delta(\tau_1, \tau_2)$-continuous functions and weakly $\delta(\tau_1, \tau_2)$-continuous functions.

CHAPTER 2

PRELIMINARIES

The content of this chapter consists of three sections: the first section presents the notion of topological spaces, the second section discusses the concept of continuous functions, and the third section focuses on bitopological spaces. We define some notations, deal with some preliminaries, and present some useful results that will be replicated in the next chapter.

2.1 Topological spaces

In this section, the definitions and established properties of open sets and δ-open sets in topological spaces are given. The main results are:

Definition 2.1.1. [5] Let $X \neq \emptyset$. A *topology* τ on X is a collection of subsets of X, Then, τ is called *topology* on X if and only if τ satisfies the following properties:

(1) $\emptyset,\ X \in \tau$.

(2) If $G_1, G_2 \in \tau$ then $G_1 \cap G_2 \in \tau$.

(3) If $G_i \in \tau$ for all $i \in J$ then $\cup_{i\in J} G_i \in \tau$.

The set X together with a topology τ on X is called a *topological space*, denoted by (X, τ). The elements of τ are called *open set*.

Definition 2.1.2. [1] Let (X, τ) be a topological space and $A \subseteq X$. Then, A is *closed* if $X - A$ is open.

Example 2.1.3. Let us consider the topological space (X, τ) such that $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a, b\}, \{c, d\}, X\}$. We observe that $\emptyset, \{a, b\}, \{c, d\}$ and X are open sets. Therefore, it is clear that $X, \{c, d\}, \{a, b\}$ and $\emptyset$ are closed.

Definition 2.1.4. [5] Let (X, τ) be a topological space and $A \subseteq X$. The *interior* of A is the set given by $\text{Int}(A) = \cup\{U \subseteq X : U \subseteq A \text{ and } U \text{ is open}\}$.

Example 2.1.5. Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$, and let $A = \{a, c\}$. It is easy to verify that $\text{Int}(A) = \{a, c\}$.

Theorem 2.1.6. [1] Let (X, τ) be a topological space and $A, B \subseteq X$.

(1) If U is an open set in X and $U \subseteq A$, then $U \subseteq \text{Int}(A)$.

(2) If $A \subseteq B$, then $\text{Int}(A) \subseteq \text{Int}(B)$.

(3) A is open if and only if $A = \text{Int}(A)$.

(4) $\text{Int}(A) \cup \text{Int}(B) \subseteq \text{Int}(A \cup B)$.

(5) $\text{Int}(A) \cap \text{Int}(B) = \text{Int}(A \cap B)$.

Definition 2.1.7. [5] Let (X, τ) be a topological space and $A \subseteq X$. The *closure* of A is the set given by $\text{Cl}(A) = \cap\{F \subseteq X : F \text{ is closed and } A \subseteq F\}$.

Example 2.1.8. Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$, and let $A = \{a\}$. It is easy to verify that $\{a, b\}$ and X are closed sets such that $A \subseteq \{a, b\}$ and $A \subseteq X$. Therefore, $\text{Cl}(A) = \{a, b\} \cap X = \{a, b\}$.

Theorem 2.1.9. [1] For subsets A and B in a topological space (X, τ), the following statements hold:

(1) If F is a closed set in X and $A \subseteq F$, then $\text{Cl}(A) \subseteq F$.

(2) If $A \subseteq B$, then $\text{Cl}(A) \subseteq \text{Cl}(B)$.

(3) A is closed if and only if $A = \text{Cl}(A)$.

(4) $\text{Cl}(A \cup B) = \text{Cl}(A) \cup \text{Cl}(B)$.

(5) $\text{Cl}(A \cap B) \subseteq \text{Cl}(A) \cap \text{Cl}(B)$.

(6) $\text{Int}(X - A) = X - \text{Cl}(A)$.

(7) $\text{Cl}(X - A) = X - \text{Int}(A)$.

Definition 2.1.10. [10] A subset U of a topological space (X, τ) is said to be a *regular open set* if $U = \text{Int}(\text{Cl}(U))$.

Definition 2.1.11. [10] A subset V of a topological space (X, τ) is said to be *regular closed* if $V = \text{Cl}(\text{Int}(V))$.

Example 2.1.12. Let us consider the topological space (X, τ) such that $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$. It is easy to check that $\emptyset$, $\{a\}$, $\{b, c\}$ and X are regular open sets. Moreover, the regular closed sets are $\emptyset$, $\{a, d\}$, $\{b, c, d\}$ and X.

Definition 2.1.13. [13] Let (X, τ) be a topological space and $A \subseteq X$.

(1) A point $x \in X$ is called a δ-*cluster point* of A if $A \cap U \neq \emptyset$ for every regular open set U of X containing x. The set of all δ-cluster points of A is called the δ-*closure* of A denoted by $\text{Cl}_\delta(A)$.

(2) A subset A of X is called δ-*closed* if $A = \text{Cl}_\delta(A)$. The complement of a δ-closed set ia called a δ-*open set* in X.

(3) The δ-*interior* of A is denoted by $\text{Int}_\delta(A)$, and given by

$$\text{Int}_\delta(A) = \cup\{U \subseteq X : U \text{ is a regular open set}, U \subseteq A\}.$$

Example 2.1.14. From example 2.1.12, We observe that $\emptyset, \{d\}, \{a, d\}, \{b, c, d\}$ and X are δ-closed sets. Therefore, the complements of δ-closed are δ-open.

Example 2.1.15. Let $A = \{a, c\}$. From example 2.1.12, we obtain that regular open sets are $\emptyset, \{a\}, \{b, c\}$ and X. It is easy to verify that $\emptyset \subseteq A$ and $\{a\} \subseteq A$. Therefore, $\text{Int}_\delta(A) = \emptyset \cup \{a\} = \{a\}$.

Theorem 2.1.16. [13] For subsets A and B in a topological space (X, τ), the following statements hold:

(1) $\text{Int}_\delta(A)$ is the largest δ-open set contained in A.

(2) A is δ-open if and only if $A = \text{Int}_\delta(A)$.

(3) $\text{Int}_\delta(\text{Int}_\delta(A)) = \text{Int}_\delta(A)$.

(4) $X - \text{Int}_\delta(A) = \text{Cl}_\delta(X - A)$.

(5) $X - \text{Cl}_\delta(A) = \text{Int}_\delta(X - A)$.

(6) If $A \subseteq B$, then $\text{Int}_\delta(A) \subseteq \text{Int}_\delta(B)$.

(7) $\text{Int}_\delta(A) \cup \text{Int}_\delta(B) \subseteq \text{Int}_\delta(A \cup B)$.

(8) $\text{Int}_\delta(A \cap B) = \text{Int}_\delta(A) \cap \text{Int}_\delta(B)$.

(9) If A_α is δ-open in X for each $\alpha \in \Lambda$, then $\cup_{\alpha\in\Lambda} A_\alpha$ is δ-open.

2.2 Continuous functions

The concept of continuous function is a basic to study general topology. In this section, we mention some definitions and characterizations of continuous functions, almost continuous functions, weakly continuous functions, δ-continuous functions, and weakly δ-continuous functions in topological spaces. Additionally, the relations among these functions are discussed. The main results are:

Definition 2.2.1. Let (X, τ) and (Y, σ) be topological spaces. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be:

(1) *continuous* [1] if, for each $x \in X$ and every open set V containing $f(x)$, there exists a neighborhood U of x such that $f(U) \subseteq V$.

(2) *almost continuous* [12] if, for each $x \in X$ and for each open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f(U) \subseteq \text{Int}(\text{Cl}(V))$.

Example 2.2.2. Let $X = \{a, b, c\}$ with topologies $\tau = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$. A function $f : (X, \tau) \to (X, \sigma)$ is defined as follow: $f(a) = a$, $f(b) = b$ and $f(c) = c$. Then f is continuous.

Remark. For a function $f : (X, \tau) \to (Y, \sigma)$, the following implication holds:

$$\text{continuous} \;\Rightarrow\; \text{almost continuous.}$$

The converse of the implication is not true in general. We can see from the following example.

Example 2.2.3. Let (X, τ) and (X, σ) be topological spaces such that $X = \{a, b, c, d\}$,

$\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$ and $\sigma = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}, X\}$.

A function $f : (X, \tau) \to (X, \sigma)$ is defined as follow: $f(a) = b$, $f(b) = f(d) = d$ and $f(c) = c$. Then f is almost continuous, but f is not continuous.

Definition 2.2.4. [9] Let (X, τ) and (Y, σ) be topological spaces. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *weakly continuous* at a point $x \in X$ if, for each neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subseteq \text{Cl}(V)$. A function f is said to be *weakly continuous* if f is weakly continuous at each point $x \in X$.

Remark. For a function $f : (X, \tau) \to (Y, \sigma)$, the following implication holds:

$$\text{almost continuous} \;\Rightarrow\; \text{weakly continuous.}$$

The converse of the implication is not true in general. We can see from the following example.

Example 2.2.5. Let (X, τ) and (X, σ) be topological spaces such that $X = \{a, b, c\}$,

with topologies $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$.

A function $f : (X, \tau) \to (X, \sigma)$ is defined as follow: $f(a) = f(b) = b$ and $f(c) = a$. Then f is weakly continuous, but f is not almost continuous.

Theorem 2.2.6. [1] For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:

(1) f is continuous.

(2) $f^{-1}(V)$ is open of X for every open subset V of Y.

(3) $f^{-1}(F)$ is closed of X for every closed subset F of Y.

Theorem 2.2.7. [9] A function $f : (X, \tau) \to (Y, \sigma)$ is weakly continuous if and only if $f^{-1}(V) \subseteq \text{Int}(f^{-1}(\text{Cl}(V)))$ for each open subset V of Y.

Theorem 2.2.8. [11] A function $f : (X, \tau) \to (Y, \sigma)$ is weakly continuous if and only if $\text{Cl}(f^{-1}(V)) \subseteq f^{-1}(\text{Cl}(V))$ for each open subset V of Y.

Theorem 2.2.9. [12] For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:

(1) f is almost continuous.

(2) $f^{-1}(V)$ is an open subset of X for every regular open subset V of Y.

(3) $f^{-1}(F)$ is a closed subset of X for every regular closed subset F of Y.

(4) For each point x of X and for each regular open neighborhood M of $f(x)$, there is a neighborhood N of x such that $f(N) \subseteq M$.

(5) $f^{-1}(A) \subseteq \text{Int}(f^{-1}(\text{Int}(\text{Cl}(A))))$ for every open subset A of Y.

(6) $\text{Cl}(f^{-1}(\text{Cl}(\text{Int}(B)))) \subseteq f^{-1}(B)$ for every closed subset B of Y.

Theorem 2.2.10. [11] For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:

(1) f is almost continuous.

(2) $f^{-1}(V) \subseteq$ Int(Cl($f^{-1}(V)$)) for each open subset V of Y.

(3) f(Cl(U)) $\subseteq$ Cl($f(U)$) for each open subset U of X.

Definition 2.2.11. Let (X, τ) and (Y, σ) be topological spaces. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be:

(1) *δ-continuous* [10] if, for each $x \in X$ and each open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that f(Int(Cl(U))) $\subseteq$ Int(Cl(V)).

(2) *weakly δ-continuous* [2] if, for each $x \in X$ and each open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that f(Int(Cl(U))) $\subseteq$ Cl(V).

Example 2.2.12. From example 2.2.3, we obtain that f is almost continuous. Moreover, f is weakly δ-continuous function.

Remark. For a function $f : (X, \tau_1) \to (Y, \tau_2)$, the following implication holds:

δ-continuous $\Rightarrow$ almost continuous $\Rightarrow$ weakly δ-continuous $\Rightarrow$ weakly continuous.

The converse of the implication is not true in general. We can see from the following example.

Example 2.2.13. From example 2.2.2, we obtain that f is continuous. Moreover, f is almost continuous, but f is not δ-continuous.

Example 2.2.14. Let (X, τ) and (X, σ) be topological spaces such that $X = \{a, b, c\}$,

with topologies $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$.

A function $f : (X, \tau) \to (X, \sigma)$ is defined as follow: $f(a) = f(b) = b$ and $f(c) = a$. Then f is weakly δ-continuous, but f is not almost continuous.

Theorem 2.2.15. [10] For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:

(1) f is δ-continuous.

(2) For each $x \in X$ and each regular open set V containing $f(x)$, there exists a regular open set U containing x such that $f(U) \subseteq V$.

(3) $f(\mathrm{Cl}_\delta(A)) \subseteq \mathrm{Cl}_\delta(f(A))$ for every $A \subseteq X$.

(4) $\mathrm{Cl}_\delta(f^{-1}(B)) \subseteq f^{-1}(\mathrm{Cl}_\delta(B))$ for every $B \subseteq Y$.

(5) $f^{-1}(F)$ is δ-closed in X for every regular closed set F of Y.

(6) $f^{-1}(F)$ is δ-closed in X for every δ-closed set F of Y.

(7) $f^{-1}(V)$ is δ-open in X for every δ-open set V of Y.

(8) $f^{-1}(V)$ is δ-open in X for every regular open set V of Y.

Remark. For a function $f : (X, \tau) \to (Y, \sigma)$, from the definitions 2.2.1, 2.2.4 and 2.2.11 defined above, we have the following diagram implications.

$$\begin{tikzcd}
\text{continuous} \arrow[dr] & & \text{weakly continuous} \\
& \text{almost continuous} \arrow[ur] \arrow[dr] & \\
\delta\text{-continuous} \arrow[ur] \arrow[rr] & & \text{weakly } \delta\text{-continuous} \arrow[uu]
\end{tikzcd}$$

2.3 Bitopological spaces

In this section, we present some main notion of bitopological spaces. Kelly [7] introduced the concept of bitopological spaces, then many studies have been conducted to extend topological ideas into a bitopological context.

According to Kelly [7], a bitopological space (X, τ_1, τ_2) is a set X with two topologies, τ_1 and τ_2. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$.

Definition 2.3.1. [4] Let (X, τ_1, τ_2) be a bitopological space and $A \subseteq X$. A is called a $\tau_1\tau_2$-*open set* if $A = \tau_1$-Int(τ_2-Int(A)). The complement of $\tau_1\tau_2$-open is $\tau_1\tau_2$-*closed.*

Example 2.3.2. Let $X = \{a, b, c\}$ with topologies $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b, c\}, X\}$. It is easy to check that $\{a\}$ is a $\tau_1\tau_2$-open set, but $\{a, b\}$ is not a $\tau_1\tau_2$-open set. Therefore, the complement of $\{a\}$ is $\tau_1\tau_2$-closed.

Definition 2.3.3. [4] Let (X, τ_1, τ_2) be a bitopological space and $A \subseteq X$. Then, the $\tau_1\tau_2$-*closure* of A denoted by $\tau_1\tau_2$-Cl(A) is defined as

$$\tau_1\tau_2\text{-Cl}(A) = \cap\{F \subseteq X : F \text{ is } \tau_1\tau_2\text{-closed in } X \text{ and } A \subseteq F\}.$$

Example 2.3.4. Let $A = \{b\}$. From Example 2.3.2, it is easy to verify that X, $\{b, c\}$ and $\emptyset$ are $\tau_1\tau_2$-closed. Since $A \subseteq X$ and $A \subseteq \{b, c\}$, then $\tau_1\tau_2$-Cl$(A) = X \cap \{b, c\} = \{b, c\}$.

Definition 2.3.5. [4] Let (X, τ_1, τ_2) be a bitopological space and $A \subseteq X$. Then, the $\tau_1\tau_2$-*interior* of A denoted by $\tau_1\tau_2$-Int(A) is defined as

$$\tau_1\tau_2\text{-Int}(A) = \cup\{G \subseteq X : G \text{ is } \tau_1\tau_2\text{-open in } X \text{ and } G \subseteq A\}.$$

Example 2.3.6. Let $X = \{a, b, c\}$ with topologies $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b, c\}, X\}$. Let $A = \{a, b\}$. It is easy to verify that $\emptyset$, $\{a\}$ and X are $\tau_1\tau_2$-open in X. Since $\emptyset \subseteq A$ and $\{a\} \subseteq A$, then $\tau_1\tau_2$-Int$(A) = \emptyset \cup \{a\} = \{a\}$.

Definition 2.3.7. [4] A subset N of a bitopological space (X, τ_1, τ_2) is said to be a $\tau_1\tau_2$-*neighborhood* of x if there exists a $\tau_1\tau_2$-open set V of X such that $x \in V \subseteq N$.

Example 2.3.8. Let $X = \{a, b, c\}$ with topologies $\tau_1 = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$. Let $N = \{a, b\}$. It is easy to verify that $\emptyset$, $\{a\}$, $\{c\}$ and X are $\tau_1\tau_2$-open in X. Clearly, $a \in \{a\} \subseteq N$. Then, N is $\tau_1\tau_2$-neighborhood of $a \in X$.

Lemma 2.3.9. [4] Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:

(1) $A \subseteq \tau_1\tau_2\text{-Cl}(A)$ and $\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Cl}(A)) = \tau_1\tau_2\text{-Cl}(A)$.

(2) If $A \subseteq B$, then $\tau_1\tau_2\text{-Cl}(A) \subseteq \tau_1\tau_2\text{-Cl}(B)$.

(3) $\tau_1\tau_2\text{-Cl}(A)$ is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2\text{-Cl}(A)$.

(5) $\tau_1\tau_2\text{-Cl}(X - A) = X - \tau_1\tau_2\text{-Int}(A)$.

Proposition 2.3.10. Let (X, τ_1, τ_2) be a bitopological space and $A \subseteq X$. Then $x \in \tau_1\tau_2\text{-Cl}(A)$ if and only if $V \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set V containing x.

Proof. Let $x \in \tau_1\tau_2\text{-Cl}(A)$ and V be any $\tau_1\tau_2$-open set containing x.
Suppose that $V \cap A = \emptyset$ for some $\tau_1\tau_2$-open set V containing x, so $X - V$ is $\tau_1\tau_2$-closed such that $A \subseteq X - V$. Hence, $x \in \tau_1\tau_2\text{-Cl}(A) \subseteq \tau_1\tau_2\text{-Cl}(X - V) = X - V$. Thus, $x \in X - V$ which is a contradiction that $x \in V$. Therefore, $V \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set V containing x. Conversely, assume that $V \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set V containing x. Suppose that $x \notin \tau_1\tau_2\text{-Cl}(A)$, so there exists a $\tau_1\tau_2$-closed set F such that $A \subseteq F$ and $x \notin F$. This implies that $X - F$ is a $\tau_1\tau_2$-open set containing x and $(X - F) \cap A = \emptyset$, which is a contradiction that $V \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set V containing x. Thus, $x \in \tau_1\tau_2\text{-Cl}(A)$. $\square$

Definition 2.3.11. [3] A subset A of a bitopological space (X, τ_1, τ_2) is said to be a $(\tau_1, \tau_2)s$-*open* if $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$. The complement of a $(\tau_1, \tau_2)s$-open set is said to be $(\tau_1, \tau_2)s$-*closed*.

Example 2.3.12. Let $X = \{a, b, c\}$ with topologies $\tau_1 = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$. It is easy to verify that $\emptyset$, $\{a\}$, $\{c\}$, $\{a, c\}$ and X are $\tau_1\tau_2$-open in X. Furthermore, $\emptyset$, $\{b\}$, $\{a, b\}$, $\{b, c\}$ and X are $\tau_1\tau_2$-closed in X. Let $A = \{a, c\}$, we obtain that $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)) = X$. It can be seen that $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$, and hence $\{a, c\}$ is a $(\tau_1, \tau_2)s$-open set in X. Therefore, the complement of $\{a, c\}$ is a $(\tau_1, \tau_2)s$-closed set.

Definition 2.3.13. [14] A subset A of a bitopological space (X, τ_1, τ_2) is said to be:

(1) $(\tau_1, \tau_2)r$-*open* if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$.

(2) $(\tau_1, \tau_2)r$-*closed* if $A = \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$.

Definition 2.3.14. [3] A subset A of a bitopological space (X, τ_1, τ_2) is said to be:

(1) $(\tau_1, \tau_2)p$-*open* if $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$.

(2) $(\tau_1, \tau_2)\beta$-*open* if $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A)))$.

Example 2.3.15. From Example 2.3.12, we have $\emptyset$, $\{a\}$, $\{c\}$, $\{a, c\}$ and X are $\tau_1\tau_2$-open in X. Furthermore, and the $\emptyset$, $\{b\}$, $\{a, b\}$, $\{b, c\}$ and X are $\tau_1\tau_2$-closed

(1) Let $A = \{c\}$. Then, we have $\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(\{c\})) = \{c\}$. Therefore, $\{c\}$ is a $(\tau_1, \tau_2)r$-open set.

(2) Let $B = \{a, b\}$. Then, we have $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\{a, b\})) = \{a, b\}$. Thus, $\{a, b\}$ is a $(\tau_1, \tau_2)r$-closed set.

(3) Let $C = \{a, c\}$. Then, we have $\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(\{a, c\})) = X$. It can be seen that $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$. Thus, $\{a, c\}$ is a $(\tau_1, \tau_2)p$-open set.

(4) Let $D = \{a\}$. Then, we have $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(\{a\}))) = \{a, b\}$. It can be seen that $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A)))$. Thus, $\{a\}$ is a $(\tau_1, \tau_2)\beta$-open set.

Remark. For a bitopological space, the properties are holds:

(1) Every $(\tau_1, \tau_2)r$-closed set is $(\tau_1, \tau_2)s$-open.

(2) Every $(\tau_1, \tau_2)r$-open set is $(\tau_1, \tau_2)p$-open.

(3) Every $(\tau_1, \tau_2)s$-open set is $(\tau_1, \tau_2)\beta$-open.

(4) Every $(\tau_1, \tau_2)p$-open set is $(\tau_1, \tau_2)\beta$-open.

The following examples show that the converse of the implication is not true in general.

Example 2.3.16. Let $X = \{a, b, c, d\}$ with topologies $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, X\}$. It is easy to verify that $\emptyset, \{a\}, \{b\}$ and X are $(\tau_1, \tau_2)r$-open sets in X.

(1) Let $A = \{a, b\}$. Then, we have $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2)$-Cl$(A) = X$. Therefore, A is a $(\tau_1, \tau_2)p$-open set, but A is not $(\tau_1, \tau_2)r$-open.

(2) Let $B = \{b, d\}$. Then, we have $B \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2)$-Int$(B) = \{b, c, d\}$. Therefore, B is a $(\tau_1, \tau_2)s$-open set, but B is not $(\tau_1, \tau_2)r$-closed.

(3) Let $C = \{a, c\}$. Then, $\{a, c\}$ is a $(\tau_1, \tau_2)\beta$-open set. Since

$$\{a, c\} \nsubseteq \{a\} = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\{a, c\})),$$

we obtain that $\{a, c\}$ is not $(\tau_1, \tau_2)p$-open.

Example 2.3.17. Let (X, τ_1, τ_2) be a bitopological space such that $X = \{a, b, c\}$ with topologies $\tau_1 = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b, c\}, X\}$. Then, $\{b\}$ is a $(\tau_1, \tau_2)\beta$-open set. Since $\{b\} \nsubseteq \emptyset = \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\{b\}))$, we obtain that $\{b\}$ is not $(\tau_1, \tau_2)s$-open.

CHAPTER 3

ON $\delta(\tau_1, \tau_2)$-CONTINUOUS FUNCTIONS

In this chapter, we introduce the concept and investigate some characterizations of $\delta(\tau_1, \tau_2)$-continuous functions. The chapter is organized as follows: Section 3.1 defines and investigates the properties of $\delta(\tau_1, \tau_2)$-open sets. In section 3.2, we introduce and discuss several characterizations of $\delta(\tau_1, \tau_2)$-continuous functions. Finally, the concept of almost $\delta(\tau_1, \tau_2)$-continuous functions and the relations between $\delta(\tau_1, \tau_2)$-continuous functions and almost $\delta(\tau_1, \tau_2)$-continuous functions are presented in Section 3.3.

3.1 $\delta(\tau_1, \tau_2)$-open sets

In this section, we define and investigate the properties of $\delta(\tau_1, \tau_2)$-open sets. Additionally, we discuss the notions of $\delta(\tau_1, \tau_2)$-open sets and $\delta(\tau_1, \tau_2)$-closed sets, as well as the $\delta(\tau_1, \tau_2)$-closure and the $\delta(\tau_1, \tau_2)$-interior.

Definition 3.1.1. Let A be a subset of a bitopological space (X, τ_1, τ_2). A point x of X is called a $\delta(\tau_1, \tau_2)$-*cluster point* of A if $V \cap A \neq \emptyset$ for every $(\tau_1, \tau_2)r$-open set V containing x. The set of all $\delta(\tau_1, \tau_2)$-cluster points of A is called the $\delta(\tau_1, \tau_2)$-*closure* of A and is denoted by $\delta(\tau_1, \tau_2)$-Cl(A).

Definition 3.1.2. Let A be a subset of a bitopological space (X, τ_1, τ_2). Then, A is called $\delta(\tau_1, \tau_2)$-*closed* if $A = \delta(\tau_1, \tau_2)$-Cl$(A)$. The complement of a $\delta(\tau_1, \tau_2)$-closed set is called $\delta(\tau_1, \tau_2)$-*open*.

The family of all $\delta(\tau_1, \tau_2)$-open (resp. $\delta(\tau_1, \tau_2)$-closed) sets of a bitopological space (X, τ_1, τ_2) is denoted by $\delta(\tau_1, \tau_2)O(X)$ (resp. $\delta(\tau_1, \tau_2)C(X)$).

Example 3.1.3. Let $X = \{a, b, c\}$ with topologies $\tau_1 = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$. It is easy to check that $\{a\}, \{c\}, \emptyset$ and X are $(\tau_1, \tau_2)r$-open sets in X. Let $A = \{a, b\}$. The set of all δ-cluster points of A is $\{a, b\}$. Then

$\delta(\tau_1, \tau_2)$-Cl$(A) = \{a, b\} = A$. Therefore, $\{a, b\}$ is a $\delta(\tau_1, \tau_2)$-closed set and $\{c\}$ is a $\delta(\tau_1, \tau_2)$-open set.

Lemma 3.1.4. Every $(\tau_1, \tau_2)r$-open set is $\delta(\tau_1, \tau_2)$-open.

Proof. Let A be any $(\tau_1, \tau_2)r$-open set of X. We shall shows that $X - A$ is $\delta(\tau_1, \tau_2)$-closed. Assume that $x \notin X - A$. Then $x \in A$ and $A \cap (X - A) = \emptyset$. It follows from Definition 3.1.1 that $x \notin \delta(\tau_1, \tau_2)$-Cl$(X - A)$. Thus $\delta(\tau_1, \tau_2)$-Cl$(X - A) \subseteq X - A$. On the other hand, assume that $x \notin \delta(\tau_1, \tau_2)$-Cl$(X - A)$. Then, there exists a $(\tau_1, \tau_2)r$-open set V containing x such that $V \cap (X - A) = \emptyset$ and hence $x \in V \subseteq A$. Thus $x \notin X - A$. Then $X - A \subseteq \delta(\tau_1, \tau_2)$-Cl$(X - A)$. This implies that $X - A = \delta(\tau_1, \tau_2)$-Cl$(X - A)$. By Definition 3.1.2, we obtain that $X - A$ is a $\delta(\tau_1, \tau_2)$-closed set, and hence A is $\delta(\tau_1, \tau_2)$-open. □

Remark. The converse of Lemma 3.1.4 is not true. We can be seen from the following example.

Example 3.1.5. Let $X = \{a, b, c\}$ with topologies $\tau_1 = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$. We can be checked that $\emptyset, \{a\}, \{c\}, \{a, c\}$ and X are the $\delta(\tau_1, \tau_2)$-open sets. Let $A = \{a, c\}$. Since $\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(\{a, c\})) = X \neq \{a, c\}$, $\{a, c\}$ is not a $(\tau_1, \tau_2)r$-open set.

Proposition 3.1.6. For subsets A, B and A_γ $(\gamma \in \Gamma)$ of a bitopological space (X, τ_1, τ_2), the following properties hold:

(1) $A \subseteq \delta(\tau_1, \tau_2)$-Cl$(A)$ and $\tau_1\tau_2$-Cl$(A) \subseteq \delta(\tau_1, \tau_2)$-Cl$(A)$.

(2) If $A \subseteq B$, then $\delta(\tau_1, \tau_2)$-Cl$(A) \subseteq \delta(\tau_1, \tau_2)$-Cl$(B)$.

(3) $\delta(\tau_1, \tau_2)$-Cl$(\cap_{\gamma \in \Gamma} A_\gamma) \subseteq \cap_{\gamma \in \Gamma}(\delta(\tau_1, \tau_2)$-Cl$(A_\gamma))$.

(4) If A_γ is a $\delta(\tau_1, \tau_2)$-closed set for each $\gamma \in \Gamma$, then $\cap_{\gamma \in \Gamma} A_\gamma$ is $\delta(\tau_1, \tau_2)$-closed.

(5) $x \in \delta(\tau_1, \tau_2)$-Cl$(A)$ if and only if $V \cap A \neq \emptyset$ for every $V \in \delta(\tau_1, \tau_2)O(X)$ containing x.

(6) $\delta(\tau_1, \tau_2)$-Cl$(A) = \cap\{F \in \delta(\tau_1, \tau_2)C(X) | A \subseteq F\}$.

(7) $\delta(\tau_1,\tau_2)$-Cl(A) is $\delta(\tau_1,\tau_2)$-closed, that is

$$\delta(\tau_1,\tau_2)\text{-Cl}(A) = \delta(\tau_1,\tau_2)\text{-Cl}(\delta(\tau_1,\tau_2)\text{-Cl}(A)).$$

Proof. (1) : Let $x \in X$ and $A \subseteq X$ such that $x \notin \delta(\tau_1,\tau_2)$-Cl$(A)$. Then, there exists a $(\tau_1,\tau_2)r$-open set V containing x such that $V \cap A = \emptyset$. Thus, $x \in V \subseteq X - A$. This implies that $x \notin A$. Therefore, $A \subseteq \delta(\tau_1,\tau_2)$-Cl$(A)$.

Let $x \notin \delta(\tau_1,\tau_2)$-Cl$(A)$. Then, there exists a $(\tau_1,\tau_2)r$-open set V containing x such that $V \cap A = \emptyset$. Since every $(\tau_1,\tau_2)r$-open set is $\tau_1\tau_2$-open, V is a $\tau_1\tau_2$-open set. Hence, $x \notin \tau_1\tau_2$-Cl(A). Therefore, $\tau_1\tau_2$-Cl$(A) \subseteq \delta(\tau_1,\tau_2)$-Cl$(A)$.

(2) : Let $A \subseteq B$. Assume that $x \notin \delta(\tau_1,\tau_2)$-Cl$(B)$. Then, there exists a $(\tau_1,\tau_2)r$-open set V containing x such that $V \cap B = \emptyset$. Hence $B \subseteq X - V$. Since $A \subseteq B$, we have that $A \subseteq X - V$. Thus $V \cap A = \emptyset$. This implies that $x \notin \delta(\tau_1,\tau_2)$-Cl$(A)$. Therefore, $\delta(\tau_1,\tau_2)$-Cl$(A) \subseteq \delta(\tau_1,\tau_2)$-Cl$(B)$.

(3) : Let $\{A_\gamma | \gamma \in \Gamma\}$ be a family of subsets of X. Note that $\cap_{\gamma\in\Gamma}A_\gamma \subseteq A_\gamma$ for all $\gamma \in \Gamma$. By (2), $\delta(\tau_1,\tau_2)$-Cl$(\cap_{\gamma\in\Gamma}A_\gamma) \subseteq \delta(\tau_1,\tau_2)$-Cl$(A_\gamma)$ and hence

$$\delta(\tau_1,\tau_2)\text{-Cl}(\cap_{\gamma\in\Gamma}A_\gamma) \subseteq \cap_{\gamma\in\Gamma}(\delta(\tau_1,\tau_2)\text{-Cl}(A_\gamma)).$$

(4) : Let A_γ be any $\delta(\tau_1,\tau_2)$-closed set for each $\gamma \in \Gamma$. Then, we obtain that $A_\gamma = \delta(\tau_1,\tau_2)$-Cl$(A_\gamma)$. It follows from (3) that

$$\delta(\tau_1,\tau_2)\text{-Cl}(\cap_{\gamma\in\Gamma}A_\gamma) \subseteq \cap_{\gamma\in\Gamma}\delta(\tau_1,\tau_2)\text{-Cl}(A_\gamma) = \cap_{\gamma\in\Gamma}A_\gamma.$$

By (1), $\cap_{\gamma\in\Gamma}A_\gamma \subseteq \delta(\tau_1,\tau_2)$-Cl$(\cap_{\gamma\in\Gamma}A_\gamma)$. Hence, $\delta(\tau_1,\tau_2)$-Cl$(\cap_{\gamma\in\Gamma}A_\gamma) = \cap_{\gamma\in\Gamma}A_\gamma$. This implies that $\cap_{\gamma\in\Gamma}A_\gamma$ is $\delta(\tau_1,\tau_2)$-closed.

(5) : $(\Rightarrow)$ Let $x \in \delta(\tau_1,\tau_2)$-Cl$(A)$. We shall show that $V \cap A \neq \emptyset$ for every $\delta(\tau_1,\tau_2)$-open set V of X containing x. Suppose that $V \cap A = \emptyset$ for some $\delta(\tau_1,\tau_2)$-open set V containing x, so $X - V$ is $\delta(\tau_1,\tau_2)$-closed and $A \subseteq X - V$. Hence,

$$x \in \delta(\tau_1,\tau_2)\text{-Cl}(A) \subseteq \delta(\tau_1,\tau_2)\text{-Cl}(X - V) = X - V.$$

Thus $x \in X - V$, which is a contradiction that $x \in V$. Therefore, $V \cap A \neq \emptyset$ for every $\delta(\tau_1, \tau_2)$-open set V of X containing x.

($\Leftarrow$) Assume that $V \cap A \neq \emptyset$ for every $\delta(\tau_1, \tau_2)$-open set V containing x. We shall show that $x \in \delta(\tau_1, \tau_2)$-Cl$(A)$. Suppose that $x \notin \delta(\tau_1, \tau_2)$-Cl$(A)$. There exists a $(\tau_1, \tau_2)r$-open set V containing x such that $V \cap A = \emptyset$. By Lemma 3.1.4, V is a $\delta(\tau_1, \tau_2)$-open set of X such that $V \cap A = \emptyset$, which is a contradiction that $V \cap A \neq \emptyset$. Thus, $x \in \delta(\tau_1, \tau_2)$-Cl$(A)$.

(6) : Let $x \in \delta(\tau_1, \tau_2)$-Cl$(A)$. Suppose that $x \notin \cap\{F \in \delta(\tau_1, \tau_2)C(X) | A \subseteq F\}$. Then, there exists a $\delta(\tau_1, \tau_2)$-closed set F_0 such that $A \subseteq F_0$ and $x \notin F_0$. Thus, $X - F_0$ is a $\delta(\tau_1, \tau_2)$-open set containing x, and hence $(X - F_0) \cap A = \emptyset$. By (5), we obtain that $x \notin \delta(\tau_1, \tau_2)$-Cl$(A)$, which is a contradiction that $x \in \delta(\tau_1, \tau_2)$-Cl$(A)$. Thus,
$x \in \cap\{F \in \delta(\tau_1, \tau_2)C(X) | A \subseteq F\}$ and hence

$$\delta(\tau_1, \tau_2)\text{-Cl}(A) \subseteq \cap\{F \in \delta(\tau_1, \tau_2)C(X) | A \subseteq F\}.$$

Conversely, assume that $x \notin \delta(\tau_1, \tau_2)$-Cl$(A)$. There exists a $\delta(\tau_1, \tau_2)$-open set V of X containing x such that $V \cap A = \emptyset$. Then, $X - V$ is a $\delta(\tau_1, \tau_2)$-closed set such that $A \subseteq X - V$ and $x \notin X - V$. Thus, $x \notin \cap\{F \in \delta(\tau_1, \tau_2)C(X) | A \subseteq F\}$ and hence $\cap\{F \in \delta(\tau_1, \tau_2)C(X) | A \subseteq F\} \subseteq \delta(\tau_1, \tau_2)$-Cl$(A)$. Consequently,

$$\delta(\tau_1, \tau_2)\text{-Cl}(A) = \cap\{F \in \delta(\tau_1, \tau_2)C(X) | A \subseteq F\}.$$

(7) : By (4) and (6), we have $\delta(\tau_1, \tau_2)$-Cl(A) is $\delta(\tau_1, \tau_2)$-closed. It follows from Definition 3.1.2 that $\delta(\tau_1, \tau_2)$-Cl$(\delta(\tau_1, \tau_2)$-Cl$(A)) = \delta(\tau_1, \tau_2)$-Cl$(A)$. $\square$

Remark. The converse of Proposition 3.1.6 (3) is not true. We can see from the following example.

Example 3.1.7. Let $X = \{a, b, c\}$ with topologies $\tau_1 = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$. Let $A = \{a\}$ and $B = \{b\}$. It is easy to verify that $\delta(\tau_1, \tau_2)$-Cl$(A) = \{a, b\}$, $\delta(\tau_1, \tau_2)$-Cl$(B) = \{b\}$ and $\delta(\tau_1, \tau_2)$-Cl$(A \cap B) = \emptyset$. Then $\delta(\tau_1, \tau_2)$-Cl$(A) \cap \delta(\tau_1, \tau_2)$-Cl$(B) = \{a, b\} \cap \{b\} = \{b\} \nsubseteq \emptyset = \delta(\tau_1, \tau_2)$-Cl$(A \cap B)$.

Definition 3.1.8. Let A be a subset of a bitopological space (X, τ_1, τ_2). The $\delta(\tau_1, \tau_2)$-interior of A denoted by $\delta(\tau_1, \tau_2)$-Int(A) is defined as follows:

$$\delta(\tau_1, \tau_2)\text{-Int}(A) = \cup\{G \subseteq X \mid G \text{ is a } \delta(\tau_1, \tau_2)\text{-open set in } X \text{ and } G \subseteq A\}.$$

Example 3.1.9. Let $A = \{a, b\}$. From Example 3.1.5, we obtain that $\emptyset$ and $\{a\}$ are the $\delta(\tau_1, \tau_2)$-open sets such that $\emptyset \subseteq A$ and $\{a\} \subseteq A$. Then, $\delta(\tau_1, \tau_2)$-Int$(A) = \emptyset \cup \{a\} = \{a\}$.

Lemma 3.1.10. Let (X, τ_1, τ_2) be a bitopological space. The $\delta(\tau_1, \tau_2)$-open set is the union of $(\tau_1, \tau_2)r$-open sets in X.

Proof. Let A be any $\delta(\tau_1, \tau_2)$-open of X and $x \in A$. Then $X - A$ is $\delta(\tau_1, \tau_2)$-closed, and hence $x \notin X - A = \delta(\tau_1, \tau_2)$-Cl$(X - A)$. There exists a $(\tau_1, \tau_2)r$-open set V_x containing x such that $V_x \cap (X - A) = \emptyset$. Thus, $x \in V_x \subseteq A$ and hence $\cup_{x \in A}\{x\} \subseteq \cup_{x \in A} V_x \subseteq A$. Since $A \subseteq \cup_{x \in A} V_x$. Therefore, $A = \cup_{x \in A} V_x$. This shows that A is the union of $(\tau_1, \tau_2)r$-open sets in X.

Conversely, let V_γ be any $(\tau_1, \tau_2)r$-open sets for each $\gamma \in \Gamma$. By Lemma 3.1.4, V_γ is $\delta(\tau_1, \tau_2)$-open for every $\gamma \in \Gamma$. Then, we have $X - V_\gamma$ is $\delta(\tau_1, \tau_2)$-closed for every $\gamma \in \Gamma$. By Proposition 3.1.6 (4), $X - \cup_{\gamma \in \Gamma} V_\gamma = \cap_{\gamma \in \Gamma}(X - V_\gamma)$ is $\delta(\tau_1, \tau_2)$-closed. Thus, $\cup_{\gamma \in \Gamma} V_\gamma$ is $\delta(\tau_1, \tau_2)$-open. □

Proposition 3.1.11. For a subset A of a bitopological space (X, τ_1, τ_2), the following properties holds:

(1) $\delta(\tau_1, \tau_2)$-Int$(X - A) = X - \delta(\tau_1, \tau_2)$-Cl$(A)$.

(2) $\delta(\tau_1, \tau_2)$-Cl$(X - A) = X - \delta(\tau_1, \tau_2)$-Int$(A)$.

Proof. (1) : Let $x \in X - \delta(\tau_1, \tau_2)$-Cl$(A)$. Then $x \notin \delta(\tau_1, \tau_2)$-Cl$(A)$. By Proposition 3.1.6 (5), there exists a $\delta(\tau_1, \tau_2)$-open set V of X containing x such that $V \cap A = \emptyset$. Therefore, $x \in V \subseteq X - A$. Hence, $x \in \delta(\tau_1, \tau_2)$-Int$(X - A)$. This implies that

$$X - \delta(\tau_1, \tau_2)\text{-Cl}(A) \subseteq \delta(\tau_1, \tau_2)\text{-Int}(X - A).$$

On the other hand, suppose that $x \in \delta(\tau_1, \tau_2)$-Int$(X - A)$. There exists a $\delta(\tau_1, \tau_2)$-open set V of X containing x such that $V \subseteq X - A$, and hence $V \cap A = \emptyset$. Therefore, $x \notin \delta(\tau_1, \tau_2)$-Cl$(A)$. Then $x \in X - \delta(\tau_1, \tau_2)$-Cl$(A)$. Hence,

$$\delta(\tau_1, \tau_2)\text{-Int}(X - A) \subseteq X - \delta(\tau_1, \tau_2)\text{-Cl}(A).$$

Consequently, $\delta(\tau_1, \tau_2)$-Int$(X - A) = X - \delta(\tau_1, \tau_2)$-Cl$(A)$.

(2) : Let $A \subseteq X$. Then $X - A \subseteq X$. By (1), we have

$$\delta(\tau_1, \tau_2)\text{-Cl}(X - A) = X - \delta(\tau_1, \tau_2)\text{-Int}(A).$$

□

Proposition 3.1.12. For a subset A of a bitopological space (X, τ_1, τ_2), the following properties holds:

(1) $\delta(\tau_1, \tau_2)$-Int$(A) \subseteq \tau_1\tau_2$-Int$(A)$.

(2) If $A \subseteq B$, then $\delta(\tau_1, \tau_2)$-Int$(A) \subseteq \delta(\tau_1, \tau_2)$-Int$(B)$.

(3) If A_γ is a $\delta(\tau_1, \tau_2)$-open set for each $\gamma \in \Gamma$, then $\cup_{\gamma\in\Gamma} A_\gamma$ is $\delta(\tau_1, \tau_2)$-open.

(4) $\delta(\tau_1, \tau_2)$-Int(A) is $\delta(\tau_1, \tau_2)$-open, that is

$$\delta(\tau_1, \tau_2)\text{-Int}(\delta(\tau_1, \tau_2)\text{-Int}(A)) = \delta(\tau_1, \tau_2)\text{-Int}(A).$$

(5) $\cup_{\gamma\in\Gamma}(\delta(\tau_1, \tau_2)\text{-Int}(A_\gamma)) \subseteq \delta(\tau_1, \tau_2)\text{-Int}(\cup_{\gamma\in\Gamma}(A_\gamma))$.

(6) A is $\delta(\tau_1, \tau_2)$-open if and only if $\delta(\tau_1, \tau_2)$-Int$(A) = A$.

Proof. (1) : Let $x \in \delta(\tau_1, \tau_2)$-Int$(A)$. By Lemma 3.1.10, there exists a $(\tau_1, \tau_2)r$-open set V containing x such that $V \subseteq A$. Since every $(\tau_1, \tau_2)r$-open set is $\tau_1\tau_2$-open, V is a $\tau_1\tau_2$-open set such that $x \in V \subseteq A$. Then, $x \in \tau_1\tau_2$-Int(A). Hence $\delta(\tau_1, \tau_2)$-Int$(A) \subseteq \tau_1\tau_2$-Int$(A)$.

(2) : Let $A \subseteq B$. Assume that $x \in \delta(\tau_1, \tau_2)$-Int$(A)$, so there exists a $\delta(\tau_1, \tau_2)$-open set V containing x such that $V \subseteq A$. Since $A \subseteq B$, we have that $V \subseteq B$. Hence, $x \in \delta(\tau_1, \tau_2)$-Int$(B)$. Therefore, $\delta(\tau_1, \tau_2)$-Int$(A) \subseteq \delta(\tau_1, \tau_2)$-Int$(B)$.

(3) : Let A_γ be any $\delta(\tau_1,\tau_2)$-open set for each $\gamma \in \Gamma$. Then, we have $X - A_\gamma$ is $\delta(\tau_1,\tau_2)$-closed for every $\gamma \in \Gamma$. By Proposition 3.1.6 (4), $X - \cup_{\gamma\in\Gamma} A_\gamma = \cap_{\gamma\in\Gamma}(X - A_\gamma)$ is $\delta(\tau_1,\tau_2)$-closed. Thus $\cup_{\gamma\in\Gamma} A_\gamma$ is $\delta(\tau_1,\tau_2)$-open.

(4) : By (3), $\delta(\tau_1,\tau_2)$-Int(A) is $\delta(\tau_1,\tau_2)$-open. Let $A \subseteq X$. It follows from Proposition 3.1.11 (2) and Proposition 3.1.6 (7) that

$$\begin{aligned} X - \delta(\tau_1,\tau_2)\text{-Int}(A) &= \delta(\tau_1,\tau_2)\text{-Cl}(X - A) \\ &= \delta(\tau_1,\tau_2)\text{-Cl}(\delta(\tau_1,\tau_2)\text{-Cl}(X - A)) \\ &= X - \delta(\tau_1,\tau_2)\text{-Int}(\delta(\tau_1,\tau_2)\text{-Int}(A)). \end{aligned}$$

Hence, $\delta(\tau_1,\tau_2)$-Int$(\delta(\tau_1,\tau_2)$-Int$(A)) = \delta(\tau_1,\tau_2)$-Int$(A)$.

(5) : Let $\{A_\gamma | \gamma \in \Gamma\}$ be a family of subsets of X. By Proposition 3.1.11 (2) and Proposition 3.1.6 (3),

$$\begin{aligned} X - \delta(\tau_1,\tau_2)\text{-Int}(\cup_{\gamma\in\Gamma} A_\gamma) &= \delta(\tau_1,\tau_2)\text{-Cl}(X - \cup_{\gamma\in\Gamma} A_\gamma) \\ &= \delta(\tau_1,\tau_2)\text{-Cl}(\cap_{\gamma\in\Gamma}(X - A_\gamma)) \\ &\subseteq \cap_{\gamma\in\Gamma}(\delta(\tau_1,\tau_2)\text{-Cl}(X - A_\gamma)) \\ &= \cap_{\gamma\in\Gamma}(X - \delta(\tau_1,\tau_2)\text{-Int}(A_\gamma)) \\ &= X - \cup_{\gamma\in\Gamma}(\delta(\tau_1,\tau_2)\text{-Int}(A_\gamma)). \end{aligned}$$

Therefore, $\cup_{\gamma\in\Gamma}(\delta(\tau_1,\tau_2)\text{-Int}(A_\gamma)) \subseteq \delta(\tau_1,\tau_2)\text{-Int}(\cup_{\gamma\in\Gamma} A_\gamma)$.

(6) : ($\Rightarrow$) Let A be any $\delta(\tau_1,\tau_2)$-open set, so $X - A$ is $\delta(\tau_1,\tau_2)$-closed. By Proposition 3.1.11 (2), we have $X - A = \delta(\tau_1,\tau_2)$-Cl$(X - A) = X - \delta(\tau_1,\tau_2)$-Int$(A)$. Therefore, $\delta(\tau_1,\tau_2)$-Int$(A) = A$.

($\Leftarrow$) Let $\delta(\tau_1,\tau_2)$-Int$(A) = A$. Then,

$$X - A = X - \delta(\tau_1,\tau_2)\text{-Int}(A) = \delta(\tau_1,\tau_2)\text{-Cl}(X - A)$$

By Proposition 3.1.6 (7), $X - A$ is a $\delta(\tau_1,\tau_2)$-closed set. Therefore, A is a $\delta(\tau_1,\tau_2)$-open set. $\square$

Remark. The converse of Proposition 3.1.12 (5) is not true. We can be seen from the following example.

Example 3.1.13. Let $X = \{a, b, c\}$ with topologies $\tau_1 = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$. Now, consider the sets $A = \{a\}$ and $B = \{b, c\}$. We obtain that $\delta(\tau_1, \tau_2)$-Int$(A) = \{a\}$, $\delta(\tau_1, \tau_2)$-Int$(B) = \{c\}$ and $\delta(\tau_1, \tau_2)$-Int$(A \cup B) = X$. It can be seen that $X \nsubseteq \{a, c\} = \delta(\tau_1, \tau_2)$-Int$(A) \cup \delta(\tau_1, \tau_2)$-Int$(B)$.

3.2 $\delta(\tau_1, \tau_2)$-continuous functions

In this section, we introduce the concept of $\delta(\tau_1, \tau_2)$-continuous functions using $\delta(\tau_1, \tau_2)$-open sets. We also discuss several characterizations of $\delta(\tau_1, \tau_2)$-continuous functions.

Definition 3.2.1. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called $\delta(\tau_1, \tau_2)$-*continuous* at $x \in X$ if, for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\delta(\tau_1, \tau_2)$-open set U of X containing x such that $f(U) \subseteq V$. A function f is said to be $\delta(\tau_1, \tau_2)$-*continuous* if f is $\delta(\tau_1, \tau_2)$-continuous at each point of X.

Example 3.2.2. Let $X = \{a, b, c\}$ with topologies $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. Let $Y = \{1, 2, 3\}$ with topologies $\sigma_1 = \{\emptyset, \{1\}, \{3\}, \{1, 3\}, Y\}$ and $\sigma_2 = \{\emptyset, \{1\}, \{3\}, \{1, 2\}, \{1, 3\}, Y\}$. Define a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ as follow: $f(a) = 2$, $f(b) = 3$ and $f(c) = 2$. Then f is $\delta(\tau_1, \tau_2)$-continuous.

Theorem 3.2.3. For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) f is $\delta(\tau_1, \tau_2)$-continuous at x.

(2) $x \in \delta(\tau_1\tau_2)$-Int$(f^{-1}(V))$ for every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$.

(3) $x \in f^{-1}(\sigma_1\sigma_2\text{-Cl}(f(A)))$ for every $A \subseteq X$ such that $x \in \delta(\tau_1, \tau_2)\text{-Cl}(A)$.

(4) $x \in f^{-1}(\sigma_1\sigma_2\text{-Cl}(B)))$ for every $B \subseteq Y$ such that $x \in \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(B))$.

(5) $x \in \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(B))$ for every $B \subseteq Y$ such that $x \in f^{-1}(\sigma_1\sigma_2\text{-Int}(B))$.

(6) $x \in f^{-1}(F)$ for every $\sigma_1\sigma_2$-closed set F of Y such that

$$x \in \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(F)).$$

Proof. (1) $\Rightarrow$ (2) : Let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By (1), there exists a $\delta(\tau_1, \tau_2)$-open set U of X containing x such that $f(U) \subseteq V$. Hence, $x \in U \subseteq f^{-1}(V)$. Therefore, $x \in \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(V))$.

(2) $\Rightarrow$ (3) : Let $A \subseteq X$ such that $x \in \delta(\tau_1, \tau_2)\text{-Cl}(A)$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By (2), $x \in \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(V))$. Then, there exists a $\delta(\tau_1, \tau_2)$-open set U of X containing x such that $U \subseteq f^{-1}(V)$. Since $x \in \delta(\tau_1, \tau_2)\text{-Cl}(A)$ and by Proposition 3.1.6 (5), $U \cap A \neq \emptyset$. Thus, $\emptyset \neq f(U \cap A) \subseteq f(U) \cap f(A) \subseteq V \cap f(A)$. This implies that $V \cap f(A) \neq \emptyset$ for each $\sigma_1\sigma_2$-open set V containing $f(x)$. Hence, $f(x) \in \sigma_1\sigma_2\text{-Cl}(f(A))$. Therefore, $x \in f^{-1}(\sigma_1\sigma_2\text{-Cl}(f(A)))$.

(3) $\Rightarrow$ (4) : Let $B \subseteq Y$ and $x \in \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(B))$. By (3), we have

$$x \in f^{-1}(\sigma_1\sigma_2\text{-Cl}(f(f^{-1}(B)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(B)).$$

(4) $\Rightarrow$ (5) : Let $B \subseteq Y$ and $x \in f^{-1}(\sigma_1\sigma_2\text{-Int}(B))$.
Suppose that $x \notin \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(B))$. Then $x \in X - \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(B))$. Since

$$\begin{aligned} X - \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(B)) &= \delta(\tau_1, \tau_2)\text{-Cl}(X - f^{-1}(B)) \\ &= \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(Y - B)) \end{aligned}$$

and by (4), we obtain that
$$\begin{aligned} x \in f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - B)) &= f^{-1}(Y - \sigma_1\sigma_2\text{-Int}(B)) \\ &= X - f^{-1}(\sigma_1\sigma_2\text{-Int}(B)). \end{aligned}$$

Therefore, $x \notin f^{-1}(\sigma_1\sigma_2\text{-Int}(B))$, which is a contradiction that $x \in f^{-1}(\sigma_1\sigma_2\text{-Int}(B))$. Hence, $x \in \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(B))$.

(5) $\Rightarrow$ (6) : Let F be any $\sigma_1\sigma_2$-closed set of Y and $x \in \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(F))$. Then $Y - F$ is a $\sigma_1\sigma_2$-open set in Y. Suppose that $x \notin f^{-1}(F)$, so

$$x \in X - f^{-1}(F) = f^{-1}(Y - F) = f^{-1}(\delta(\sigma_1, \sigma_2)\text{-Int}(Y - F)).$$

By (6) and Proposition 3.1.12 (3), we have

$$\begin{aligned} x \in \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(Y - F)) &= \delta(\tau_1, \tau_2)\text{-Int}(X - f^{-1}(F)) \\ &= X - \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(F)). \end{aligned}$$

Then, $x \notin \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(F))$. This is a contradiction. Therefore, $x \in f^{-1}(F)$.

(6) $\Rightarrow$ (2) : Let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Then $x \in f^{-1}(V)$. Suppose that $x \notin \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(V))$. By Proposition 3.1.12 (4), we obtain that

$$\begin{aligned} x \in X - \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(V)) &= \delta(\tau_1, \tau_2)\text{-Cl}(X - f^{-1}(V)) \\ &= \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(Y - V)). \end{aligned}$$

Since $Y - V$ is a $\sigma_1\sigma_2$-closed set. By (6), we have $x \in f^{-1}(Y - V) = X - f^{-1}(V)$. This implies that $x \notin f^{-1}(V)$, which is a contradiction. Thus, $x \in \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(V))$.

(2) $\Rightarrow$ (1) : Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By (2), we have $x \in \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(V))$. Then, there exists a $\delta(\tau_1, \tau_2)$-open set U such that $x \in U \subseteq f^{-1}(V)$. Thus, $f(U) \subseteq V$. Therefore, f is $\delta(\tau_1, \tau_2)$-continuous at $x \in X$. $\square$

Theorem 3.2.4. For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) f is $\delta(\tau_1, \tau_2)$-continuous.

(2) $f^{-1}(V)$ is $\delta(\tau_1, \tau_2)$-open in X for every $\sigma_1\sigma_2$-open set V of Y.

(3) $f(\delta(\tau_1, \tau_2)\text{-Cl}(A)) \subseteq \sigma_1\sigma_2\text{-Cl}(f(A))$ for every $A \subseteq X$.

(4) $\delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(B))$ for every $B \subseteq Y$.

(5) $f^{-1}(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(B))$ for every $B \subseteq Y$.

(6) $f^{-1}(F)$ is $\delta(\tau_1, \tau_2)$-closed in X for every $\sigma_1\sigma_2$-closed set F of Y.

Proof. (1) $\Rightarrow$ (2) : Let V be any $\sigma_1\sigma_2$-open of Y and $x \in f^{-1}(V)$. By (1), there exists a $\delta(\tau_1, \tau_2)$-open set U of X containing x such that $f(U) \subseteq V$. Thus, $x \in U \subseteq f^{-1}(V)$ and hence $x \in \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(V))$. This implies that $f^{-1}(V) \subseteq \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is $\delta(\tau_1, \tau_2)$-open in X.

(2) $\Rightarrow$ (3) : Let $A \subseteq X$ and $x \in \delta(\tau_1, \tau_2)\text{-Cl}(A)$. Then $f(x) \in f(\delta(\tau_1, \tau_2)\text{-Cl}(A))$. Let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By (2), we have $f^{-1}(V)$ is $\delta(\tau_1, \tau_2)$-open, and hence $x \in f^{-1}(V) = \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(V))$. Then, there exists a $\delta(\tau_1, \tau_2)$-open set U of X containing x such that $U \subseteq f^{-1}(V)$. Since $x \in \delta(\tau_1, \tau_2)\text{-Cl}(A)$, then $U \cap A \neq \emptyset$. Hence, $\emptyset \neq f(U \cap A) \subseteq f(U) \cap f(A) \subseteq V \cap f(A)$. This implies that, $V \cap f(A) \neq \emptyset$ for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$. Then, $f(x) \in \sigma_1\sigma_2\text{-Cl}(f(A))$. Consequently, $f(\delta(\tau_1, \tau_2)\text{-Cl}(A)) \subseteq \sigma_1\sigma_2\text{-Cl}(f(A))$.

(3) $\Rightarrow$ (4) : Let $B \subseteq Y$. By (3), we have that

$$\begin{aligned} f(\delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(B))) &\subseteq \sigma_1\sigma_2\text{-Cl}(f(f^{-1}(B))) \\ &\subseteq \sigma_1\sigma_2\text{-Cl}(B). \end{aligned}$$

Therefore, $\delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(B))$.

(4) ⇒ (5) : Let $B \subseteq Y$. By (4) and Proposition 3.1.12 (4), we obtain that

$$\begin{aligned} X - \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(B)) &= \delta(\tau_1, \tau_2)\text{-Cl}(X - f^{-1}(B)) \\ &= \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(Y - B)) \\ &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - B)) \\ &= f^{-1}(Y - \sigma_1\sigma_2\text{-Int}(B)) \\ &= X - f^{-1}(\sigma_1\sigma_2\text{-Int}(B)). \end{aligned}$$

Hence, $f^{-1}(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(B))$.

(5) ⇒ (6) : Let F be any $\sigma_1\sigma_2$-closed set of Y. Then, $Y - F$ is $\sigma_1\sigma_2$-open in Y. By (5) and Proposition 3.1.12 (3), we obtain that

$$\begin{aligned} X - f^{-1}(F) &= f^{-1}(Y - F) \\ &= f^{-1}(\sigma_1\sigma_2\text{-Int}(Y - F)) \\ &\subseteq \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(Y - F)) \\ &= \delta(\tau_1, \tau_2)\text{-Int}(X - (f^{-1}(F))) \\ &= X - \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(F)). \end{aligned}$$

Therefore, $\delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(F)) \subseteq f^{-1}(F)$. This implies that $f^{-1}(F)$ is $\delta(\tau_1, \tau_2)$-closed in X.

(6) ⇒ (2) : Let V be any $\sigma_1\sigma_2$-open set of Y. Then, $Y - V$ is $\sigma_1\sigma_2$-closed in Y. By (6), we have $X - f^{-1}(V) = f^{-1}(Y - V)$ is $\delta(\tau_1, \tau_2)$-closed in X. Thus, $f^{-1}(V)$ is $\delta(\tau_1, \tau_2)$-open.

(2) ⇒ (1) : Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By (2), we obtain $f^{-1}(V)$ is $\delta(\tau_1, \tau_2)$-open. Hence, $x \in f^{-1}(V) = \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(V))$. Then, there exists a $\delta(\tau_1, \tau_2)$-open set U such that $x \in U \subseteq f^{-1}(V)$. Therefore, $f(U) \subseteq V$. Thus, f is $\delta(\tau_1, \tau_2)$-continuous at x. This shows that f is $\delta(\tau_1, \tau_2)$-continuous. □

3.3 Almost $\delta(\tau_1, \tau_2)$-continuous functions

In this section, we introduce the notion of almost $\delta(\tau_1, \tau_2)$-continuous functions and investigate some characterizations of almost $\delta(\tau_1, \tau_2)$-continuous functions. Moreover, the relationships between $\delta(\tau_1, \tau_2)$-continuous functions and almost $\delta(\tau_1, \tau_2)$-continuous functions are established.

Definition 3.3.1. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *almost* $\delta(\tau_1, \tau_2)$*-continuous* at $x \in X$ if, for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\delta(\tau_1, \tau_2)$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Int($\sigma_1\sigma_2$-Cl($V$)). A function f is called *almost* $\delta(\tau_1, \tau_2)$*-continuous* if f is almost $\delta(\tau_1, \tau_2)$-continuous at each point of X.

Remark. For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following implication holds:

$$\delta(\tau_1, \tau_2)\text{-continuous} \;\Rightarrow\; \text{almost } \delta(\tau_1, \tau_2)\text{-continuous.}$$

The converse of the implication is not true in general. We give an example for the implication as follows.

Example 3.3.2. Let $X = \{1, 2, 3, 4\}$ with topologies $\tau_1 = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, X\}$ and $\tau_2 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, X\}$. Let $Y = \{a, b, c\}$ with topologies $\sigma_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$ and $\sigma_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, Y\}$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is defined as follow: $f(1) = a$, $f(2) = b$ and $f(3) = c$ and $f(4) = c$. Then f is almost $\delta(\tau_1, \tau_2)$-continuous, but f is not $\delta(\tau_1, \tau_2)$-continuous.

Theorem 3.3.3. For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) f is almost $\delta(\tau_1, \tau_2)$-continuous at x.

(2) $x \in \delta(\tau_1, \tau_2)$-Int($f^{-1}$($\sigma_1\sigma_2$-Int($\sigma_1\sigma_2$-Cl($V$)))) for every $\sigma_1\sigma_2$-open set V containing $f(x)$.

(3) $x \in \delta(\tau_1, \tau_2)$-Int($f^{-1}(V)$) for every $(\sigma_1, \sigma_2)r$-open set V containing $f(x)$.

(4) For each $x \in X$ and each $(\sigma_1, \sigma_2)r$-open set V of Y containing $f(x)$, there exists a $\delta(\tau_1, \tau_2)$-open set U of X containing x such that $f(U) \subseteq V$.

Proof. (1) $\Rightarrow$ (2) : Let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Since f is almost $\delta(\tau_1, \tau_2)$-continuous at $x \in X$. Then, there exists a $\delta(\tau_1, \tau_2)$-open set U of X containing

x such that $f(U) \subseteq \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$. Thus, $x \in U \subseteq f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))$. Therefore, $x \in \delta(\tau_1, \tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))$.

(2) $\Rightarrow$ (3) : Let V be any $(\sigma_1, \sigma_2)r$-open set of Y containing $f(x)$. By (2), we obtain that $x \in \delta(\tau_1, \tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) = \delta(\tau_1, \tau_2)$-Int$(f^{-1}(V))$. Therefore,

$$x \in \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(V)).$$

(3) $\Rightarrow$ (4) : Let V be any $(\sigma_1, \sigma_2)r$-open set of Y containing $f(x)$. By (3), we have $x \in \delta(\tau_1, \tau_2)$-Int$(f^{-1}(V))$. Then, there exists a $\delta(\tau_1, \tau_2)$-open set U of X such that

$$x \in U \subseteq f^{-1}(V).$$

Thus, $f(U) \subseteq V$.

(4) $\Rightarrow$ (1) : Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Since $\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$ is a $(\sigma_1, \sigma_2)r$-open set and by (4), there exists a $\delta(\tau_1, \tau_2)$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$. Consequently, f is almost $\delta(\tau_1, \tau_2)$-continuous at x. $\square$

Theorem 3.3.4. For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) f is almost $\delta(\tau_1, \tau_2)$-continuous.

(2) $f^{-1}(V) \subseteq \delta(\tau_1, \tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))))$ for every $\sigma_1\sigma_2$-open set V of Y.

(3) $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(F)))) \subseteq f^{-1}(F)$ for every $\sigma_1\sigma_2$-closed set F of Y.

(4) $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(B))))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(B))$ for every $B \subseteq Y$.

(5) $f^{-1}(\sigma_1\sigma_2$-Int$(B)) \subseteq \delta(\tau_1, \tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(B)))))$ for every $B \subseteq Y$.

(6) $f^{-1}(V)$ is $\delta(\tau_1,\tau_2)$-open in X for every $(\sigma_1,\sigma_2)r$-open set V of Y.

(7) $f^{-1}(F)$ is $\delta(\tau_1,\tau_2)$-closed in X for every $(\sigma_1,\sigma_2)r$-closed set F of Y.

Proof. (1) $\Rightarrow$ (2) : Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in f^{-1}(V)$. Since f is almost $\delta(\tau_1,\tau_2)$-continuous at $x \in X$, there exists a $\delta(\tau_1,\tau_2)$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$. Then $x \in U \subseteq f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$, and hence $x \in \delta(\tau_1,\tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$. This implies that

$$f^{-1}(V) \subseteq \delta(\tau_1,\tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))).$$

(2) $\Rightarrow$ (3) : Let F be any $\sigma_1\sigma_2$-closed set of Y. Then, $Y - F$ is $\sigma_1\sigma_2$-open. By (2),

$$\begin{aligned} X - f^{-1}(F) &= f^{-1}(Y - F) \\ &\subseteq \delta(\tau_1,\tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - F)))) \\ &= \delta(\tau_1,\tau_2)\text{-Int}(f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(F)))) \\ &= X - \delta(\tau_1,\tau_2)\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(F)))). \end{aligned}$$

Therefore, $\delta(\tau_1,\tau_2)\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(F)))) \subseteq f^{-1}(F)$.

(3) $\Rightarrow$ (4) : Let $B \subseteq Y$. Then $\sigma_1\sigma_2\text{-Cl}(B)$ is $\sigma_1\sigma_2$-closed in Y. By (3), we have

$$\delta(\tau_1,\tau_2)\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B))))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(B)).$$

(4) $\Rightarrow$ (5) : Let $B \subseteq Y$. By (4), we have

$$\begin{aligned} f^{-1}(\sigma_1\sigma_2\text{-Int}(B)) &= X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - B)) \\ &\subseteq X - \delta(\tau_1,\tau_2)\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - B))))) \\ &= \delta(\tau_1,\tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B))))). \end{aligned}$$

Therefore, $f^{-1}(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \delta(\tau_1,\tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))))$.

(5) $\Rightarrow$ (6) : Let V be any $(\sigma_1,\sigma_2)r$-open set of Y. Then V is a $\sigma_1\sigma_2$-open set in Y and $V = \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(V)))$. By (5), we obtain that

$$f^{-1}(V) = f^{-1}(\sigma_1\sigma_2\text{-Int}(V))$$

$$\subseteq \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(V)))) \\ = \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(V)).$$

Since $\delta(\tau_1, \tau_2)$-Int$(f^{-1}(V)) \subseteq f^{-1}(V)$. Hence $\delta(\tau_1, \tau_2)$-Int$(f^{-1}(V)) = f^{-1}(V)$. This implies that $f^{-1}(V)$ is $\delta(\tau_1, \tau_2)$-open in X.

(6) $\Rightarrow$ (7) : Let F be any $(\sigma_1, \sigma_2)r$-closed set of Y. Then $X - F$ is a $(\sigma_1, \sigma_2)r$-open set in Y and by (6), we obtain that

$$\begin{aligned} X - f^{-1}(F) &= f^{-1}(Y - F) \\ &= \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(Y - F)) \\ &= X - \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(F)). \end{aligned}$$

Thus, $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(F)) = f^{-1}(F)$. Hence, $f^{-1}(F)$ is $\delta(\tau_1, \tau_2)$-closed in X.

(7) $\Rightarrow$ (1) : Let $x \in X$ and V be any $(\sigma_1, \sigma_2)r$-open set of Y containing $f(x)$. Then $Y - V$ is $(\sigma_1, \sigma_2)r$-closed in Y. By (7), we have

$$\begin{aligned} X - f^{-1}(V) &= f^{-1}(Y - V) \\ &= \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(Y - V)) \\ &= X - \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(V)). \end{aligned}$$

Thus, $x \in f^{-1}(V) = \delta(\tau_1, \tau_2)$-Int$(f^{-1}(V))$. Then, there exists a $\delta(\tau_1, \tau_2)$-open set U of X containing x such that $U \subseteq f^{-1}(V)$. Hence $f(U) \subseteq V$. By Theorem 3.3.3 (4), we obtain that f is almost $\delta(\tau_1, \tau_2)$-continuous at x. Consequently, f is almost $\delta(\tau_1, \tau_2)$-continuous. □

Theorem 3.3.5. For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) f is almost $\delta(\tau_1, \tau_2)$-continuous.

(2) $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y.

(3) $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y.

(4) $f^{-1}(V) \subseteq \delta(\tau_1, \tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.

Proof. (1) $\Rightarrow$ (2) : Let V be any $(\sigma_1, \sigma_2)\beta$-open set of Y.
Then $V \subseteq \sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))$, and hence

$$\begin{aligned}\sigma_1\sigma_2\text{-Cl}(V) &\subseteq \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))\\ &= \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \subseteq \sigma_1\sigma_2\text{-Cl}(V).\end{aligned}$$

Therefore, $\sigma_1\sigma_2$-Cl(V) is a $(\sigma_1, \sigma_2)r$-closed set in Y. By Theorem 3.3.4 (7), we obtain that $f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ is $\delta(\tau_1, \tau_2)$-closed. Hence,

$$\begin{aligned}\delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(V)) &\subseteq \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))\\ &= f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)).\end{aligned}$$

(2) $\Rightarrow$ (3) : The proof is obvious since every $(\sigma_1, \sigma_2)s$-open set is $(\sigma_1, \sigma_2)\beta$-open.

(3) $\Rightarrow$ (1) : Let F be any $(\sigma_1, \sigma_2)r$-closed set of Y. Then F is $(\sigma_1, \sigma_2)s$-open in Y. By (3), we have that

$$\begin{aligned}\delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(F)) &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(F))\\ &= f^{-1}(F)\\ &\subseteq \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(F)).\end{aligned}$$

Hence, $f^{-1}(F)$ is $\delta(\tau_1, \tau_2)$-closed in X. By Theorem 3.3.4 (7), f is almost $\delta(\tau_1, \tau_2)$-continuous.

(1) $\Rightarrow$ (4) : Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Then $V \subseteq \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$ and $\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$ is $(\sigma_1, \sigma_2)r$-open in Y. By Theorem 3.3.4 (6), we obtain that $f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))$ is $\delta(\tau_1, \tau_2)$-open in X. Then,

$$f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) = \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))).$$

Thus, $$f^{-1}(V) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$$

$$= \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))).$$

(4) $\Rightarrow$ (1) : Let V be any $(\sigma_1, \sigma_2)r$-open set of Y. Then $V = \sigma_1\sigma_2$-Int($\sigma_1\sigma_2$-Cl(V)) and V is a $(\sigma_1, \sigma_2)p$-open set in Y. By (4), we have

$$\begin{aligned} f^{-1}(V) &\subseteq \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \\ &= \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(V)). \end{aligned}$$

Hence, $f^{-1}(V)$ is a $\delta(\tau_1, \tau_2)$-open set in X. By Theorem 3.3.4 (6), f is almost $\delta(\tau_1, \tau_2)$-continuous. $\square$

CHAPTER 4

ON WEAKLY $\delta(\tau_1, \tau_2)$-CONTINUOUS FUNCTIONS

The concepts of weakly $\delta(\tau_1, \tau_2)$-continuous functions are presented and examined in this chapter. The following is how this chapter is organized: Weakly $\delta(\tau_1, \tau_2)$-continuous functions are defined in Section 4.1, and Section 4.2 provides some characterizations of these functions. Additionally, correlations between almost $\delta(\tau_1, \tau_2)$-continuous functions, weakly $\delta(\tau_1, \tau_2)$-continuous functions, and $\delta(\tau_1, \tau_2)$-continuous functions are studied.

4.1 Weakly $\delta(\tau_1, \tau_2)$-continuous functions

In this section, we introduce the concept of weakly $\delta(\tau_1, \tau_2)$-continuous functions using $\delta(\tau_1, \tau_2)$-open sets and investigate some characteristics of weakly $\delta(\tau_1, \tau_2)$-continuous functions. Moreover, we discuss the relationships between almost $\delta(\tau_1, \tau_2)$-continuous functions and weakly $\delta(\tau_1, \tau_2)$-continuous functions.

Definition 4.1.1. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *weakly $\delta(\tau_1, \tau_2)$-continuous* at $x \in X$ if, for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\delta(\tau_1, \tau_2)$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Cl$(V)$. A function f is called *weakly $\delta(\tau_1, \tau_2)$-continuous* if f is weakly $\delta(\tau_1, \tau_2)$-continuous at each point of X.

Remark. For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following implication holds:

$$\text{almost } \delta(\tau_1, \tau_2)\text{-continuous} \Rightarrow \text{weakly } \delta(\tau_1, \tau_2)\text{-continuous.}$$

The converse of the implication is not true in general. We give an example for the implication as follows.

Example 4.1.2. Let $X = \{a, b, c\}$ with topologies $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Let $Y = \{1, 2, 3\}$ with topologies

$\sigma_1 = \{\emptyset, \{1\}, \{3\}, \{1,3\}, Y\}$ and $\sigma_2 = \{\emptyset, \{1\}, \{3\}, \{1,2\}, \{1,3\}, Y\}$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is defined as follow: $f(a) = f(b) = 2$ and $f(c) = 1$. Then f is weakly $\delta(\tau_1, \tau_2)$-continuous, but f is not almost $\delta(\tau_1, \tau_2)$-continuous.

Theorem 4.1.3. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is weakly $\delta(\tau_1, \tau_2)$-continuous at $x \in X$ if and only if $x \in \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$ for every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$.

Proof. $(\Rightarrow)$: Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Since f is weakly $\delta(\tau_1, \tau_2)$-continuous at x, so there exists a $\delta(\tau_1, \tau_2)$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. Therefore, $x \in U \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$. This implies that $x \in \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$.

$(\Leftarrow)$: Let V be any $\sigma_1\sigma_2$-open of Y containing $f(x)$. By assumption, we obtain that

$$x \in \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))).$$

Then, there exists a $\delta(\tau_1, \tau_2)$-open set U of X containing x such that
$U \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$. Hence, $f(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. Therefore, f is weakly $\delta(\tau_1, \tau_2)$-continuous at x. □

Theorem 4.1.4. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is weakly $\delta(\tau_1, \tau_2)$-continuous if and only if $f^{-1}(V) \subseteq \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$ for every $\sigma_1\sigma_2$-open set V of Y.

Proof. $(\Rightarrow)$: Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in f^{-1}(V)$. Then, $f(x) \in V$. Since f is weakly $\delta(\tau_1, \tau_2)$-continuous at $x \in X$. By Theorem 4.1.3,

$$x \in \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))).$$

Therefore, $f^{-1}(V) \subseteq \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$.

$(\Leftarrow)$: Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By assumption, we have

$$x \in f^{-1}(V) \subseteq \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))).$$

By Theorem 4.1.3, f is weakly $\delta(\tau_1, \tau_2)$-continuous at $x \in X$. This shows that f is weakly $\delta(\tau_1, \tau_2)$-continuous. □

4.2 Characterizations of weakly $\delta(\tau_1, \tau_2)$-continuous functions

We investigate several characterizations of weakly $\delta(\tau_1, \tau_2)$-continuous functions in this section.

Theorem 4.2.1. For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) f is weakly $\delta(\tau_1, \tau_2)$-continuous.

(2) $f^{-1}(V) \subseteq \delta(\tau_1, \tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2$-Cl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y.

(3) $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(F))) \subseteq f^{-1}(F)$ for every $\sigma_1\sigma_2$-closed F of Y.

(4) $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(B)))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(B))$ for every $B \subseteq Y$.

(5) $f^{-1}(\sigma_1\sigma_2$-Int$(B)) \subseteq \delta(\tau_1, \tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(B))))$ for every $B \subseteq Y$.

(6) $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $\sigma_1\sigma_2$-open set V of Y.

(7) $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(F))) \subseteq f^{-1}(F)$ for every $(\sigma_1, \sigma_2)r$-closed set F of Y.

(8) $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y.

(9) $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y.

(10) $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.

(11) $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.

(12) $f^{-1}(V) \subseteq \delta(\tau_1, \tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2$-Cl$(V)))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.

Proof. (1) $\Rightarrow$ (2) : Let V be any $\sigma_1\sigma_2$-open set of Y. It follows from Theorem 4.1.4, we obtain that $f^{-1}(V) \subseteq \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$.

(2) $\Rightarrow$ (3) : Let F be any $\sigma_1\sigma_2$-closed set of Y. Then $Y - F$ is $\sigma_1\sigma_2$-open in Y. By (2), we have

$$\begin{aligned} X - f^{-1}(F) &= f^{-1}(Y - F) \\ &\subseteq \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - F))) \\ &= \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(Y - \sigma_1\sigma_2\text{-Int}(F))) \\ &= \delta(\tau_1, \tau_2)\text{-Int}(X - f^{-1}(\sigma_1\sigma_2\text{-Int}(F))) \\ &= X - \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(F))). \end{aligned}$$

Therefore, $\delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(F))) \subseteq f^{-1}(F)$.

(3) $\Rightarrow$ (4) : Let $B \subseteq Y$. Since $\sigma_1\sigma_2\text{-Cl}(B)$ is $\sigma_1\sigma_2$-closed in Y and by (3),

$$\delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(B)).$$

(4) $\Rightarrow$ (5) : Let $B \subseteq Y$. By (4), we obtain that

$$\begin{aligned} f^{-1}(\sigma_1\sigma_2\text{-Int}(B)) &= X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - B)) \\ &\subseteq X - \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - B)))) \\ &= \delta(\tau_1, \tau_2)\text{-Int}(X - f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - B)))) \\ &= \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))). \end{aligned}$$

Thus, $f^{-1}(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B))))$.

(5) $\Rightarrow$ (6) : Let V be any $\sigma_1\sigma_2$-open set of Y and $x \notin f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$. Then $f(x) \notin \sigma_1\sigma_2\text{-Cl}(V)$. There exists a $\sigma_1\sigma_2$-open set U of Y containing $f(x)$ such that $U \cap V = \emptyset$. Hence, $\sigma_1\sigma_2\text{-Cl}(U) \cap V = \emptyset$. By (5),

$$\begin{aligned} x \in f^{-1}(U) &= f^{-1}(\sigma_1\sigma_2\text{-Int}(U)) \\ &\subseteq \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(U)))) \\ &= \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(U))). \end{aligned}$$

Then, there exists a $\delta(\tau_1, \tau_2)$-open set G of X such that $x \in G \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(U))$. Thus,

$$\begin{aligned} f^{-1}(V) \cap G &\subseteq f^{-1}(V) \cap f^{-1}(\sigma_1\sigma_2\text{-Cl}(U)) \\ &= f^{-1}(V \cap \sigma_1\sigma_2\text{-Cl}(U)) \\ &= \emptyset. \end{aligned}$$

Hence, $x \notin \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(V))$. Therefore, $\delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$.

(6) $\Rightarrow$ (7) : Let F be any $(\sigma_1, \sigma_2)r$-closed set of Y. Then $\sigma_1\sigma_2\text{-Int}(F)$ is $\sigma_1\sigma_2$-open in Y. By (6),

$$\delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(F))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(F))) = f^{-1}(F).$$

(7) $\Rightarrow$ (8) : Let V be any $(\sigma_1, \sigma_2)\beta$-open set of Y.
Then $V \subseteq \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$, and hence

$$\begin{aligned} \sigma_1\sigma_2\text{-Cl}(V) &\subseteq \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \\ &= \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq \sigma_1\sigma_2\text{-Cl}(V). \end{aligned}$$

Thus, $\sigma_1\sigma_2\text{-Cl}(V)$ is $(\sigma_1, \sigma_2)r$-closed. By (7),

$$\delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)).$$

(8) $\Rightarrow$ (9) : The proof is obvious since every $(\sigma_1, \sigma_2)s$-open set is $(\sigma_1, \sigma_2)\beta$-open.

(9) $\Rightarrow$ (10) : Let V be $(\sigma_1, \sigma_2)p$-open of Y. Then $V \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$, and hence $\sigma_1\sigma_2\text{-Cl}(V) \subseteq \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$. Therefore, $\sigma_1\sigma_2\text{-Cl}(V)$ is $(\sigma_1, \sigma_2)s$-open in Y. By (9),

$$\begin{aligned} &\delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \\ &\quad = \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Cl}(V))))) \\ &\quad \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Cl}(V))) \\ &\quad = f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)). \end{aligned}$$

(10) $\Rightarrow$ (11) : Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. By (10),

$$\delta(\tau_1,\tau_2)\text{-Cl}(f^{-1}(V)) \subseteq \delta(\tau_1,\tau_2)\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)).$$

(11) $\Rightarrow$ (12) : Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. By (11),

$$\begin{aligned} f^{-1}(V) &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \\ &= X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq X - \delta(\tau_1,\tau_2)\text{-Cl}(f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &= \delta(\tau_1,\tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))). \end{aligned}$$

(12) $\Rightarrow$ (1) : Let V be any $\sigma_1\sigma_2$-open set of Y. Then V is $(\sigma_1, \sigma_2)p$-open in Y. By (12), $f^{-1}(V) \subseteq \delta(\tau_1,\tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$. It follows from Theorem 4.1.4 that f is weakly $\delta(\tau_1,\tau_2)$-continuous. $\square$

CHAPTER 5

CONCLUSIONS

5.1 Conclusions

The purposes of this thesis are to present the concepts of $\delta(\tau_1, \tau_2)$-continuous functions, almost $\delta(\tau_1, \tau_2)$-continuous functions, and weakly $\delta(\tau_1, \tau_2)$-continuous functions. Furthermore, we obtain several characterizations of these functions. The results are as follows:

1. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces.

A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called $\delta(\tau_1, \tau_2)$-*continuous* at $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\delta(\tau_1, \tau_2)$-open set U of X containing x such that $f(U) \subseteq V$. A function f is said to be $\delta(\tau_1, \tau_2)$-*continuous* if f is $\delta(\tau_1, \tau_2)$-continuous at each point of X.

From the above definition, the following theorem are derived:

1.1 For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) f is $\delta(\tau_1, \tau_2)$-continuous at x.

(2) $x \in \delta(\tau_1\tau_2)$-Int$(f^{-1}(V))$ for every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$.

(3) $x \in f^{-1}(\sigma_1\sigma_2$-Cl$(f(A)))$ for every $A \subseteq X$ such that $x \in \delta(\tau_1, \tau_2)$-Cl$(A)$.

(4) $x \in f^{-1}(\sigma_1\sigma_2$-Cl$(B)))$ for every $B \subseteq Y$ such that $x \in \delta(\tau_1, \tau_2)$-Cl$(f^{-1}(B))$.

(5) $x \in \delta(\tau_1, \tau_2)$-Int$(f^{-1}(B))$ for every $B \subseteq Y$ such that $x \in f^{-1}(\sigma_1\sigma_2$-Int$(B))$.

(6) $x \in f^{-1}(F)$ for every $\sigma_1\sigma_2$-closed set F of Y such that

$$x \in \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(F)).$$

1.2 For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) f is $\delta(\tau_1, \tau_2)$-continuous.

(2) $f^{-1}(V)$ is $\delta(\tau_1, \tau_2)$-open in X for every $\sigma_1\sigma_2$-open set V of Y.

(3) $f(\delta(\tau_1, \tau_2)\text{-Cl}(A)) \subseteq \sigma_1\sigma_2\text{-Cl}(f(A))$ for every $A \subseteq X$.

(4) $\delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(B))$ for every $B \subseteq Y$.

(5) $f^{-1}(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(B))$ for every $B \subseteq Y$.

(6) $f^{-1}(F)$ is $\delta(\tau_1, \tau_2)$-closed in X for every $\sigma_1\sigma_2$-closed set F of Y.

2. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces.

A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *almost* $\delta(\tau_1, \tau_2)$-*continuous* at $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\delta(\tau_1, \tau_2)$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$. A function f is called *almost* $\delta(\tau_1, \tau_2)$-*continuous* if f is almost $\delta(\tau_1, \tau_2)$-continuous at each point of X.

From the above definition, the following theorem are derived:

2.1 For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) f is almost $\delta(\tau_1, \tau_2)$-continuous at x.

(2) $x \in \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$ for every $\sigma_1\sigma_2$-open set V containing $f(x)$.

(3) $x \in \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(V))$ for every $(\sigma_1, \sigma_2)r$-open set V containing $f(x)$.

(4) For each $x \in X$ and each $(\sigma_1, \sigma_2)r$-open set V of Y containing $f(x)$, there exists a $\delta(\tau_1, \tau_2)$-open set U of X containing x such that $f(U) \subseteq V$.

2.2 For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) f is almost $\delta(\tau_1, \tau_2)$-continuous.

(2) $f^{-1}(V) \subseteq \delta(\tau_1, \tau_2)$-Int($f^{-1}(\sigma_1\sigma_2$-Int($\sigma_1\sigma_2$-Cl$(V)$))) for every $\sigma_1\sigma_2$-open set V of Y.

(3) $\delta(\tau_1, \tau_2)$-Cl($f^{-1}(\sigma_1\sigma_2$-Cl($\sigma_1\sigma_2$-Int(F)))) $\subseteq f^{-1}(F)$ for every $\sigma_1\sigma_2$-closed set F of Y.

(4) $\delta(\tau_1, \tau_2)$-Cl($f^{-1}(\sigma_1\sigma_2$-Cl($\sigma_1\sigma_2$-Int($\sigma_1\sigma_2$-Cl(B))))) $\subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(B))$ for every $B \subseteq Y$.

(5) $f^{-1}(\sigma_1\sigma_2$-Int$(B)) \subseteq \delta(\tau_1, \tau_2)$-Int($f^{-1}(\sigma_1\sigma_2$-Int($\sigma_1\sigma_2$-Cl($\sigma_1\sigma_2$-Int$(B)$)))) for every $B \subseteq Y$.

(6) $f^{-1}(V)$ is $\delta(\tau_1, \tau_2)$-open in X for every $(\sigma_1, \sigma_2)r$-open set V of Y.

(7) $f^{-1}(F)$ is $\delta(\tau_1, \tau_2)$-closed in X for every $(\sigma_1, \sigma_2)r$-closed set F of Y.

2.3 For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) f is almost $\delta(\tau_1, \tau_2)$-continuous.

(2) $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y.

(3) $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y.

(4) $f^{-1}(V) \subseteq \delta(\tau_1, \tau_2)$-Int($f^{-1}(\sigma_1\sigma_2$-Int($\sigma_1\sigma_2$-Cl$(V)$))) for every $(\sigma_1, \sigma_2)p$-open set V of Y.

3. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces.

A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *weakly $\delta(\tau_1, \tau_2)$-continuous* at $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\delta(\tau_1, \tau_2)$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Cl$(V)$. A function f is called *weakly $\delta(\tau_1, \tau_2)$-continuous* if f is weakly $\delta(\tau_1, \tau_2)$-continuous at each point of X.

From the above definition, the following theorem are derived:

3.1 A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is weakly $\delta(\tau_1, \tau_2)$-continuous at $x \in X$ if and only if $x \in \delta(\tau_1, \tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2$-Cl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$.

3.2 A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is weakly $\delta(\tau_1, \tau_2)$-continuous if and only if $f^{-1}(V) \subseteq \delta(\tau_1, \tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2$-Cl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y.

3.3 For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) f is weakly $\delta(\tau_1, \tau_2)$-continuous.

(2) $f^{-1}(V) \subseteq \delta(\tau_1, \tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2$-Cl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y.

(3) $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(F))) \subseteq f^{-1}(F)$ for every $\sigma_1\sigma_2$-closed F of Y.

(4) $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(B)))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(B))$ for every $B \subseteq Y$.

(5) $f^{-1}(\sigma_1\sigma_2$-Int$(B)) \subseteq \delta(\tau_1, \tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(B))))$ for every $B \subseteq Y$.

(6) $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $\sigma_1\sigma_2$-open set V of Y.

(7) $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(F))) \subseteq f^{-1}(F)$ for every $(\sigma_1, \sigma_2)r$-closed set F of Y.

(8) $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y.

(9) $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y.

(10) $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.

(11) $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.

(12) $f^{-1}(V) \subseteq \delta(\tau_1, \tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2$-Cl$(V)))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.

3.4 For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, from the definitions defined above, the following implication holds:

$$\begin{gathered}\delta(\tau_1, \tau_2)\text{-continuity} \\ \Downarrow \\ \text{almost } \delta(\tau_1, \tau_2)\text{-continuity} \\ \Downarrow \\ \text{weak } \delta(\tau_1, \tau_2)\text{-continuity.}\end{gathered}$$

The converse of the implications are not true in general.

5.2 Recommendations

To this end, although we have presented several characterizations in this research, there is another definition of continuity that is defined by using $\delta(\tau_1, \tau_2)$-open sets interests us for further investigation. Furthermore, the relationships among these continuous functions will be elucidated in subsequent studies.

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