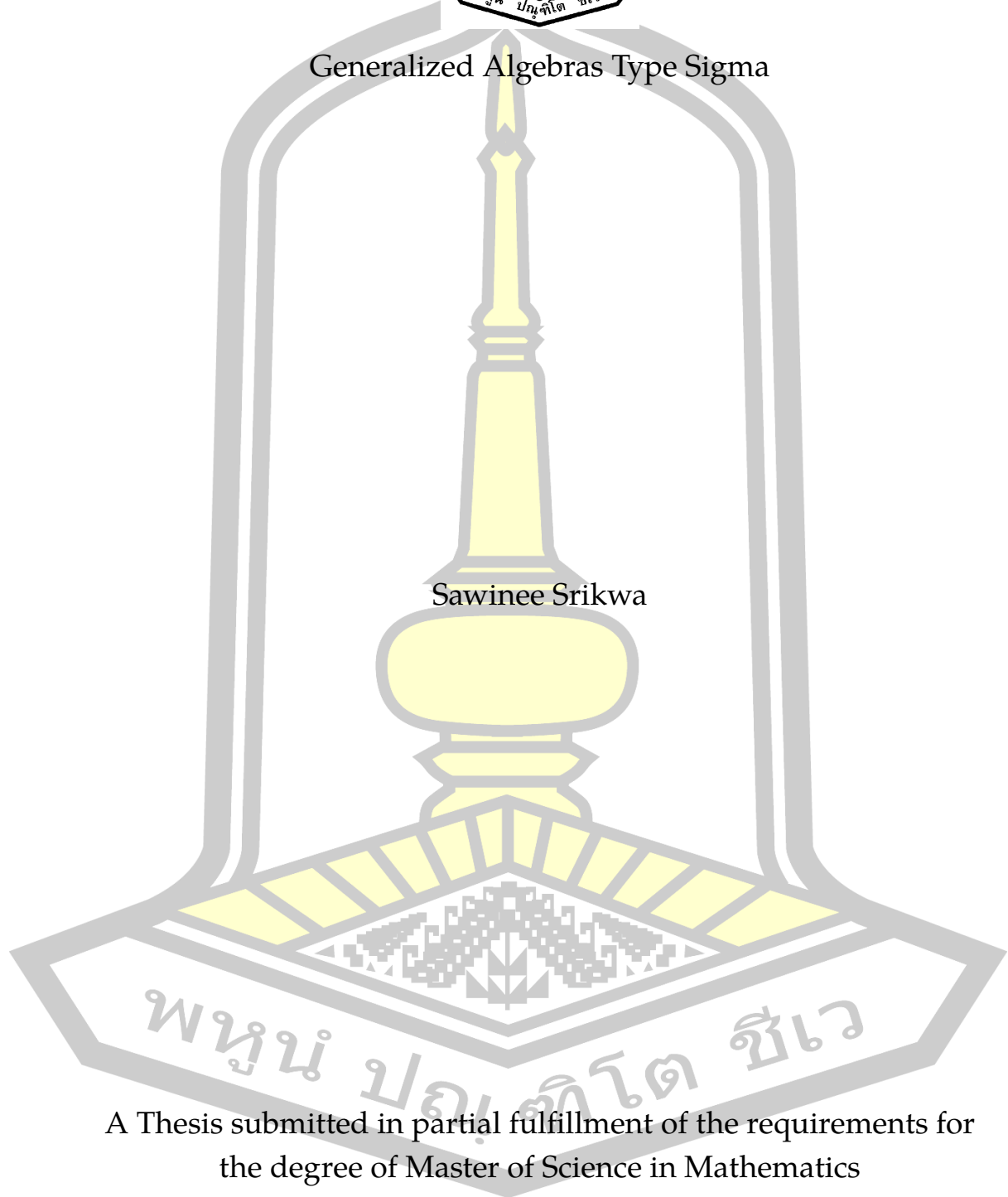


Generalized Algebras Type Sigma

Sawinee Srikwa



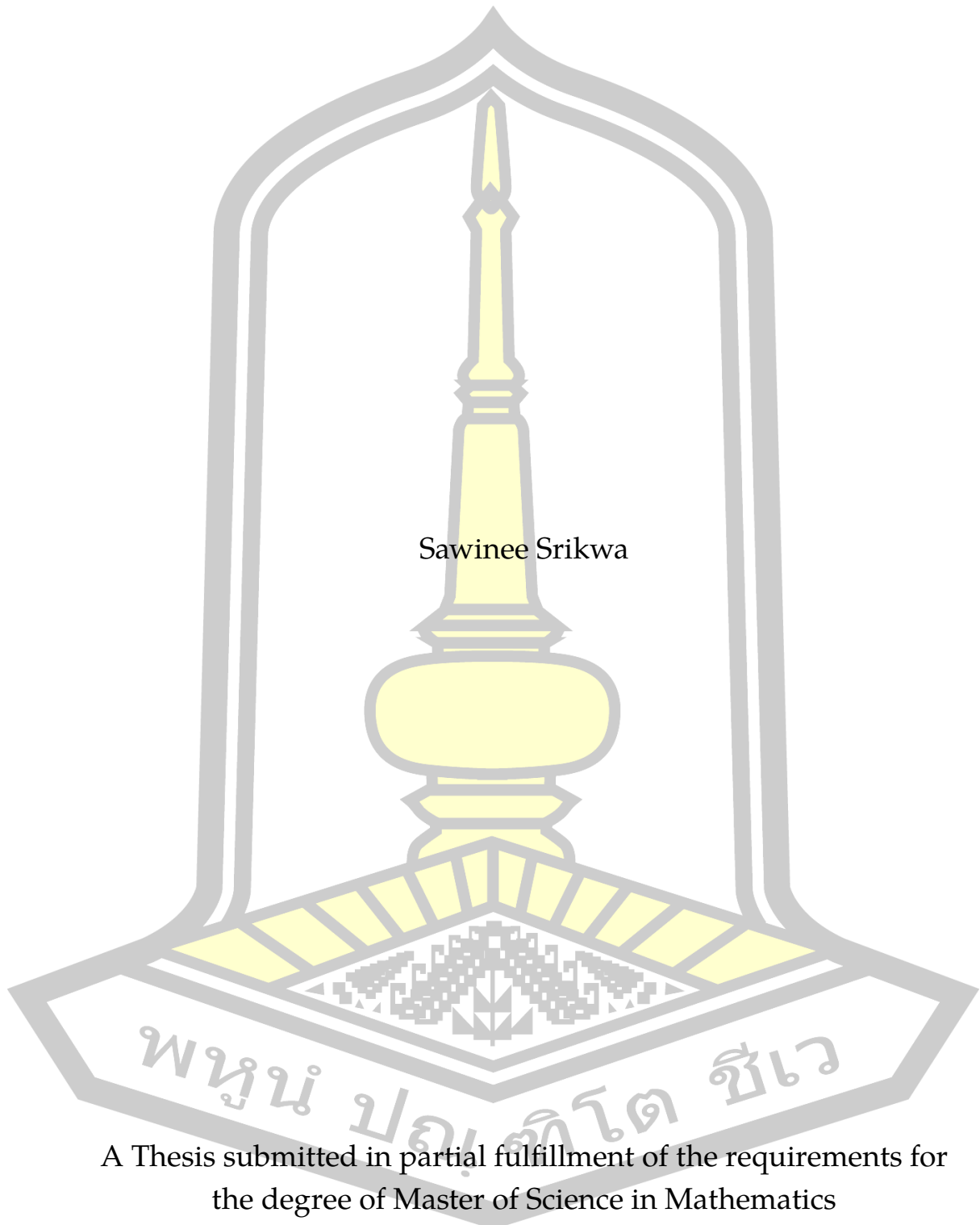
A Thesis submitted in partial fulfillment of the requirements for
the degree of Master of Science in Mathematics
at Mahasarakham University

June 2025

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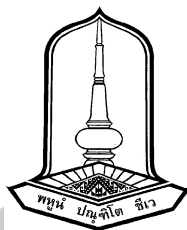
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The examining committee has unanimously approved this thesis, submitted by Miss. Sawinee Srikwa, as a partial fulfillment of the requirements for the Master of Science in Mathematics at Mahasarakham University.

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ชื่อเรื่อง การวางนัยทั่วไปของพีชคณิตซิกมา
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บทคัดย่อ

ในการวิจัยนี้ เป็นการนำเสนอแนวคิดของการวางนัยทั่วไปของพีชคณิตซิกมา (ชนิด- $\tau = (|\Sigma_j|)_{j \in \Omega}$) และได้แสดงให้เห็นสมบัติบางประการของพีชคณิตนัยทั่วไปซิกมาที่นิยามขึ้น อันประกอบด้วยแนวคิดของการเป็นพีชคณิตนัยทั่วไปซิกมาย่อย ความสัมพันธ์สมภาค สมสัณฐานและสาทิสสัณฐาน โดยศึกษาบนโครงสร้างของพีชคณิตนัยทั่วไปซิกมา รวมทั้งสมบัติบางประการของแนวคิดเหล่านี้ที่ค้นพบ

คำสำคัญ: พีชคณิตนัยทั่วไปซิกมา (ชนิด $\tau = (|\Sigma_j|)_{j \in \Omega}$), พีชคณิตนัยนัยทั่วไปซิกมาย่อย, ความสัมพันธ์สมภาค, สมสัณฐาน, สาทิสสัณฐาน

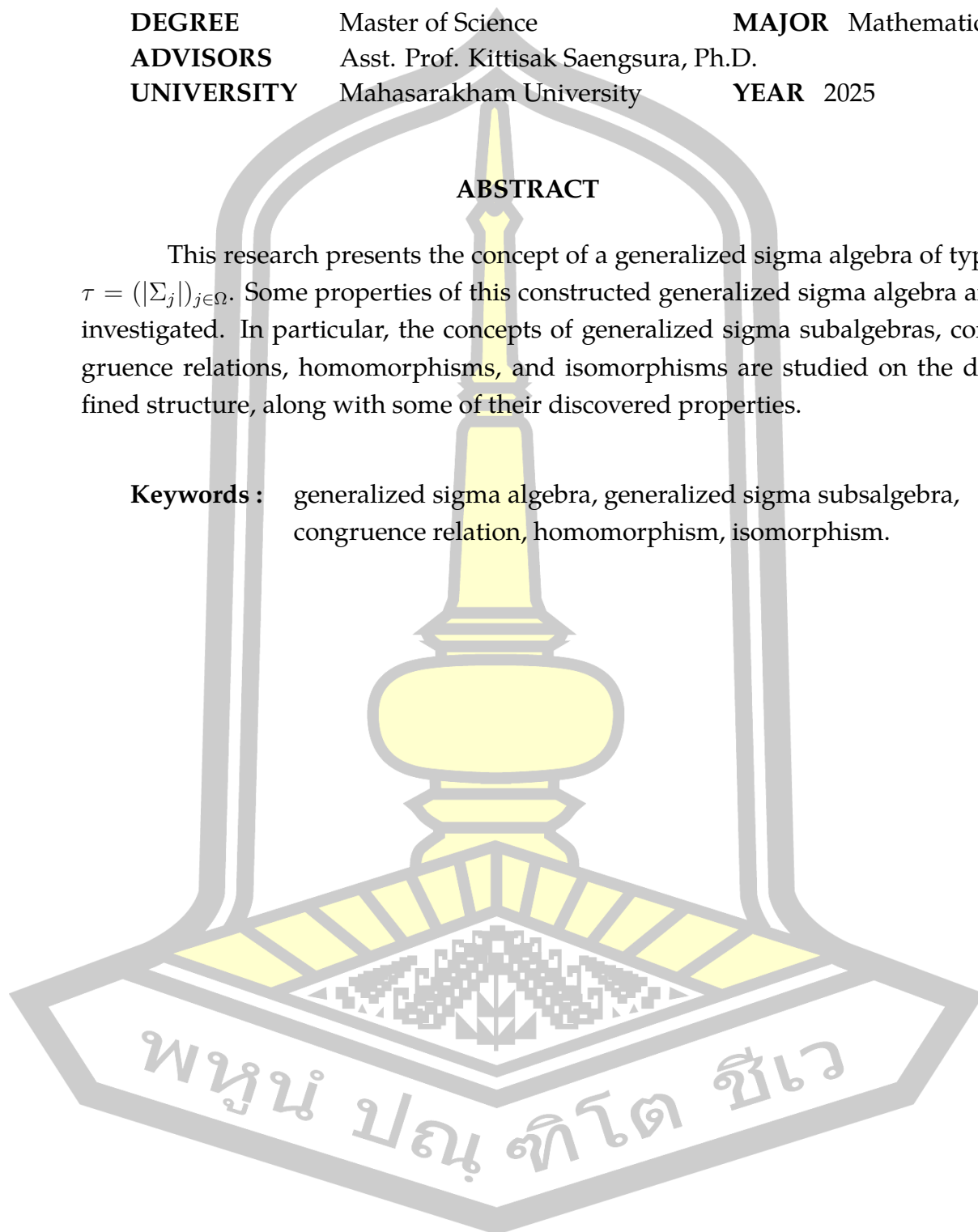
พจนัน ปณ จิตโต ชีเว

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ABSTRACT

This research presents the concept of a generalized sigma algebra of type $\tau = (|\Sigma_j|)_{j \in \Omega}$. Some properties of this constructed generalized sigma algebra are investigated. In particular, the concepts of generalized sigma subalgebras, congruence relations, homomorphisms, and isomorphisms are studied on the defined structure, along with some of their discovered properties.

Keywords : generalized sigma algebra, generalized sigma subsalgebra, congruence relation, homomorphism, isomorphism.



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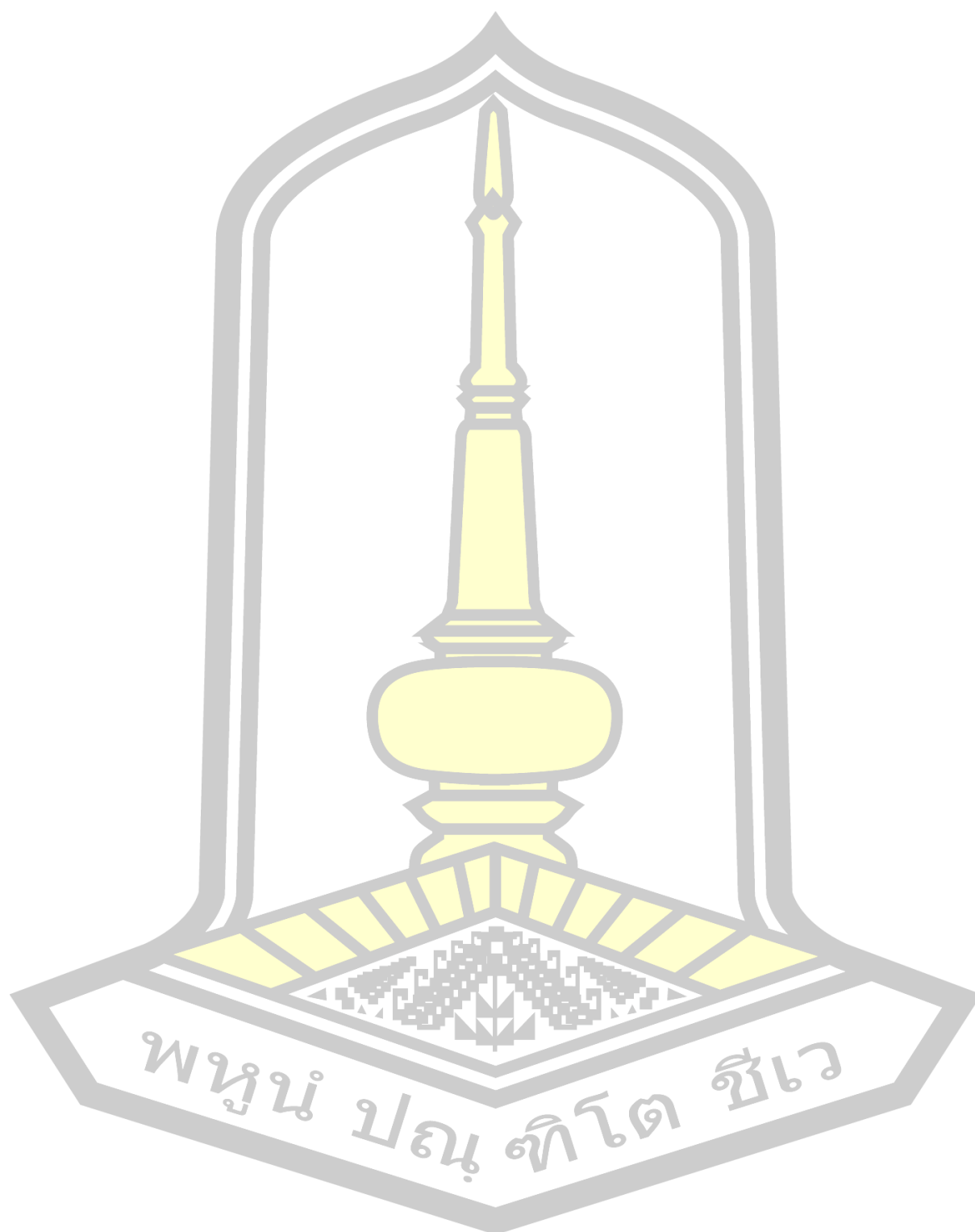
I would like to extend my sincere gratitude to Prof. Dr. Somsak Lekkoksung, Assoc. Prof. Dr. Nareupanat Lekkoksung, and my father, Asst. Prof. Dr. Yongyuth Suyoi, for their insightful comments, academic guidance, and thorough review, which have significantly contributed to the refinement and successful completion of this thesis. I wish to express my deepest appreciation to my parents for their unconditional love, constant encouragement, and endless support in all aspects of my life. Their belief in me has been a cornerstone of my academic success.

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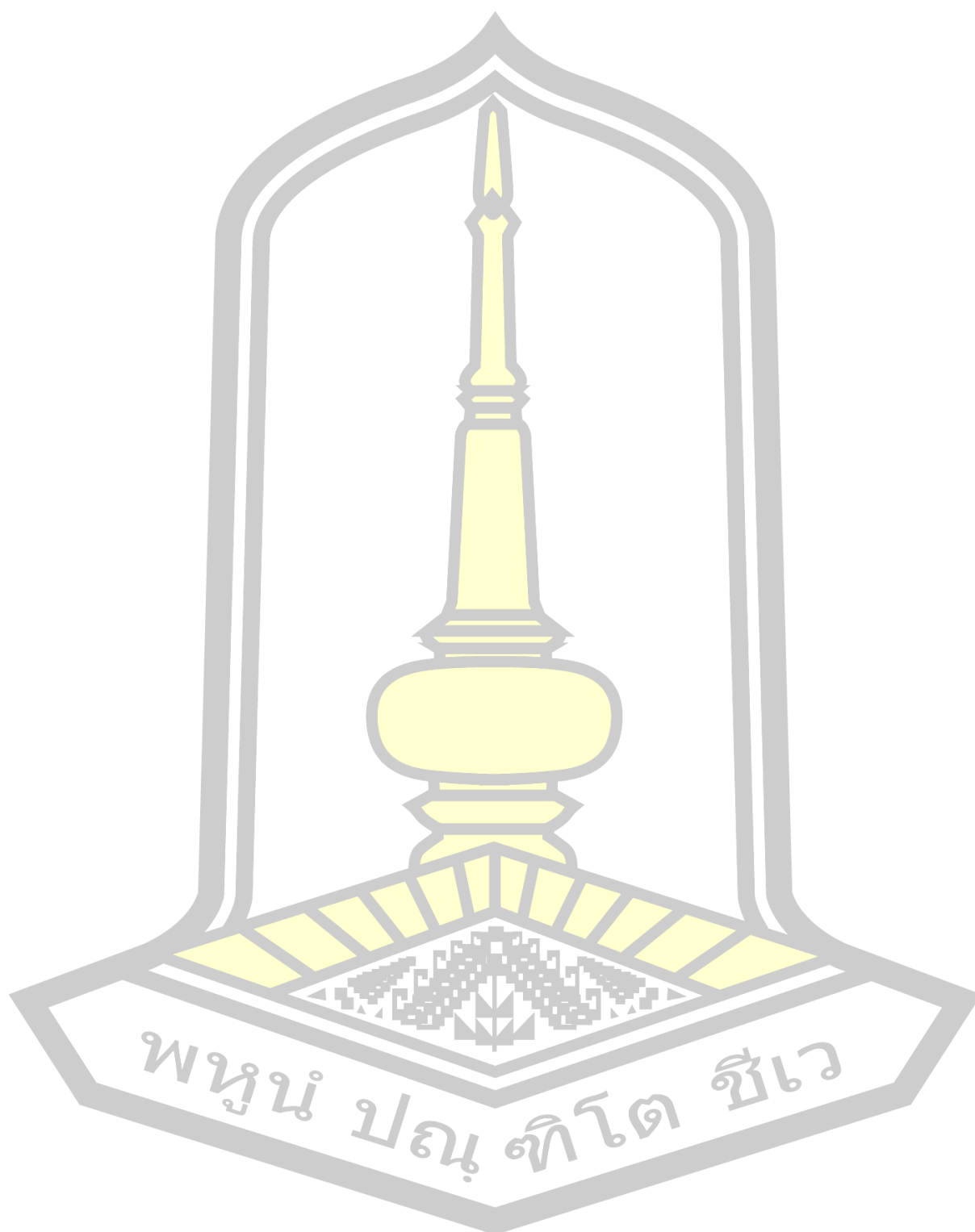
Sawinee Srikwa

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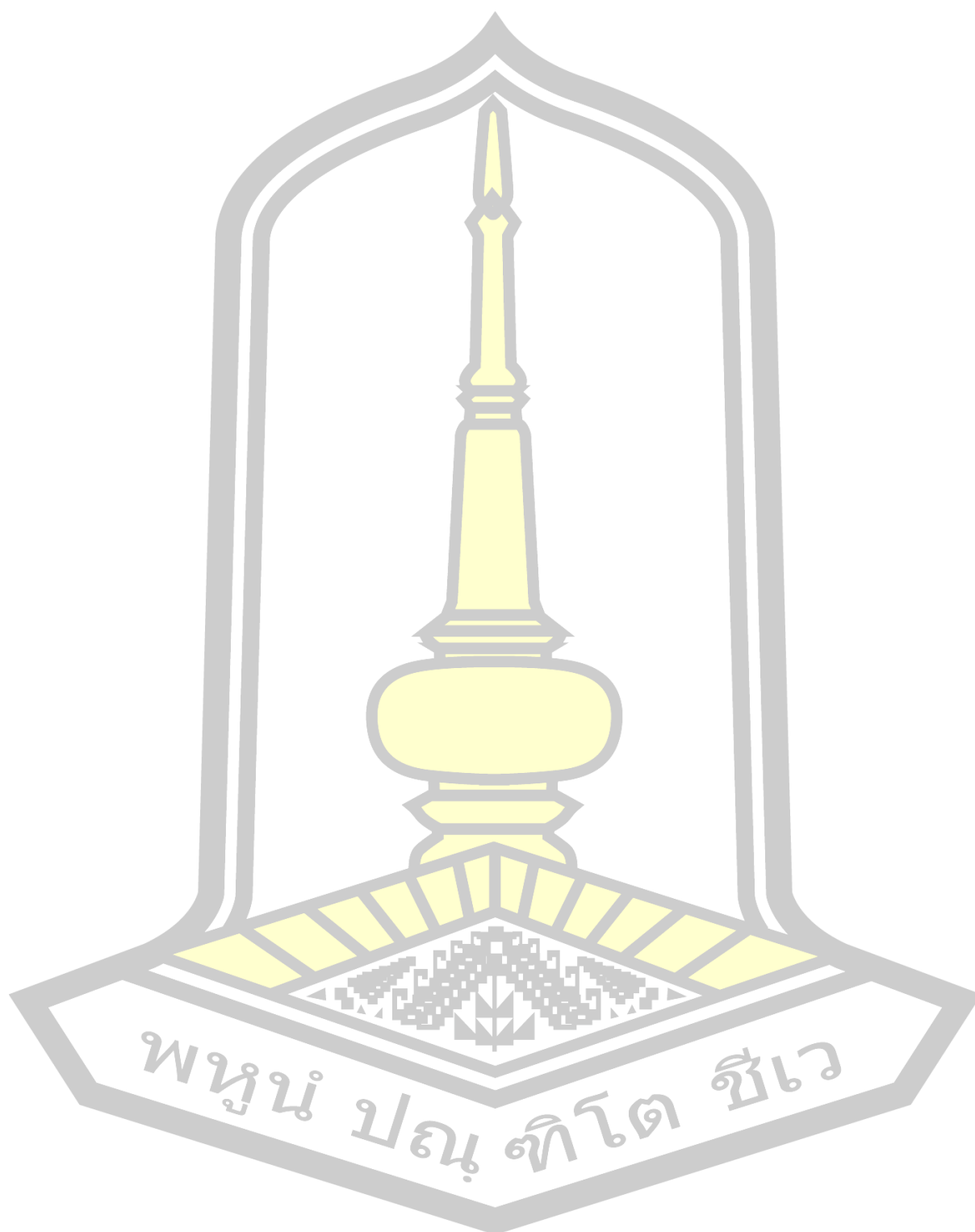
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CHAPTER 1

INTRODUCTION

1.1 Background and Motivation

A cornerstone of universal algebra is a fundamental theorem by G. Birkhoff [4], which states that a class of similar algebras is a *variety* meaning it can be defined by a set of equations if and only if it is closed under the formation of homomorphic images (H), subalgebras (S), and direct products (P). From the proof of Birkhoff's theorem, it follows that the smallest variety containing a class of algebras $\mathcal{K}$, denoted $V(\mathcal{K})$, coincides with the class $HSP(\mathcal{K})$. This profound connection between equational logic and algebraic closure operators forms the foundation of the field.

The primary objects of study in universal algebra are algebraic structures, or *algebras* for short. Traditionally, an algebra $\mathcal{A}$ is defined as a pair $(A, \mathcal{F})$, where A is a non-empty set called the *universe*, and $\mathcal{F}$ is a set of *finitary* operations on A . This classical definition, established in the 1930s, is broad enough to encompass most familiar mathematical structures while remaining amenable to systematic study. Within this framework, two of the most important invariants of any algebra are its lattice of subalgebras and its lattice of congruences [11].

Formally, a classical algebra of type $\tau = (n_i)_{i \in \mathcal{I}}$ is a structure $\mathcal{A} := (A; (f_i^A)_{i \in \mathcal{I}})$, where for each index $i \in \mathcal{I}$, f_i^A is an n_i -ary operation on A . The sequence of arities τ determines the type of the algebra, and $Alg(\tau)$ denotes the class of all algebras of that type [11]. Building upon this structure, concepts such as subalgebras, congruence relations, and homomorphisms are defined. For instance, a congruence is an equivalence relation that is compatible with all operations, and the set of all congruences on an algebra forms a complete lattice [3].

In recent decades, mathematicians have extended these classical structures. Arworn [18] and Denecke [11] introduced concepts like groupoids of hypersubstitutions and G-solid varieties, while Leeratanavalee [12] investigated these ideas in the context of groupoid structures. These advancements highlight a continuous effort to generalize the foundational concepts of universal algebra to new and more abstract settings. The study of homomorphisms and isomorphisms remains central to understanding the relationships between these generalized structures [4, 5, 7, 11, 14].

This research continues in that spirit of generalization. We propose a new

structure, the **generalized sigma algebra**, which extends the classical notion of an algebra by allowing operations to have arbitrary index sets as their arities, rather than being restricted to natural numbers. We define an operation $f_{\Sigma_j}^A$ as a map from A^{Σ_j} to A , where Σ_j is an arbitrary non-empty index set and A^{Σ_j} is the set of all functions from Σ_j to A . An algebra of this new type is a pair $\mathcal{A}_\Sigma := (A; (f_{\Sigma_j}^A)_{j \in \Omega})$, where Ω is an index set for the operations. The type of this algebra is the sequence of cardinalities of these index sets, $\tau := (|\Sigma_j|)_{j \in \Omega}$. This framework allows for the study of operations of potentially uncountable arity, moving beyond the traditional finitary constraint.

The primary goal of this thesis is to systematically develop the foundational theory for these generalized sigma algebras. We will extend the classical definitions of subalgebras, congruence relations, homomorphisms, and isomorphisms to this new, more general setting. By doing so, we aim to investigate which fundamental properties and theorems from classical universal algebra continue to hold in this broader context and to discover new characterizations unique to these structures.

1.2 Objectives of the Research

The objectives of this research are as follows:

1. To define the concept of a generalized sigma algebra, provide illustrative examples, and investigate its fundamental properties.
2. To define and investigate the properties of generalized sigma subalgebras.
3. To define and investigate the properties of congruence relations on generalized sigma algebras.
4. To define and investigate the properties of homomorphisms and isomorphisms between two generalized sigma algebras.

1.3 Procedure of the Research

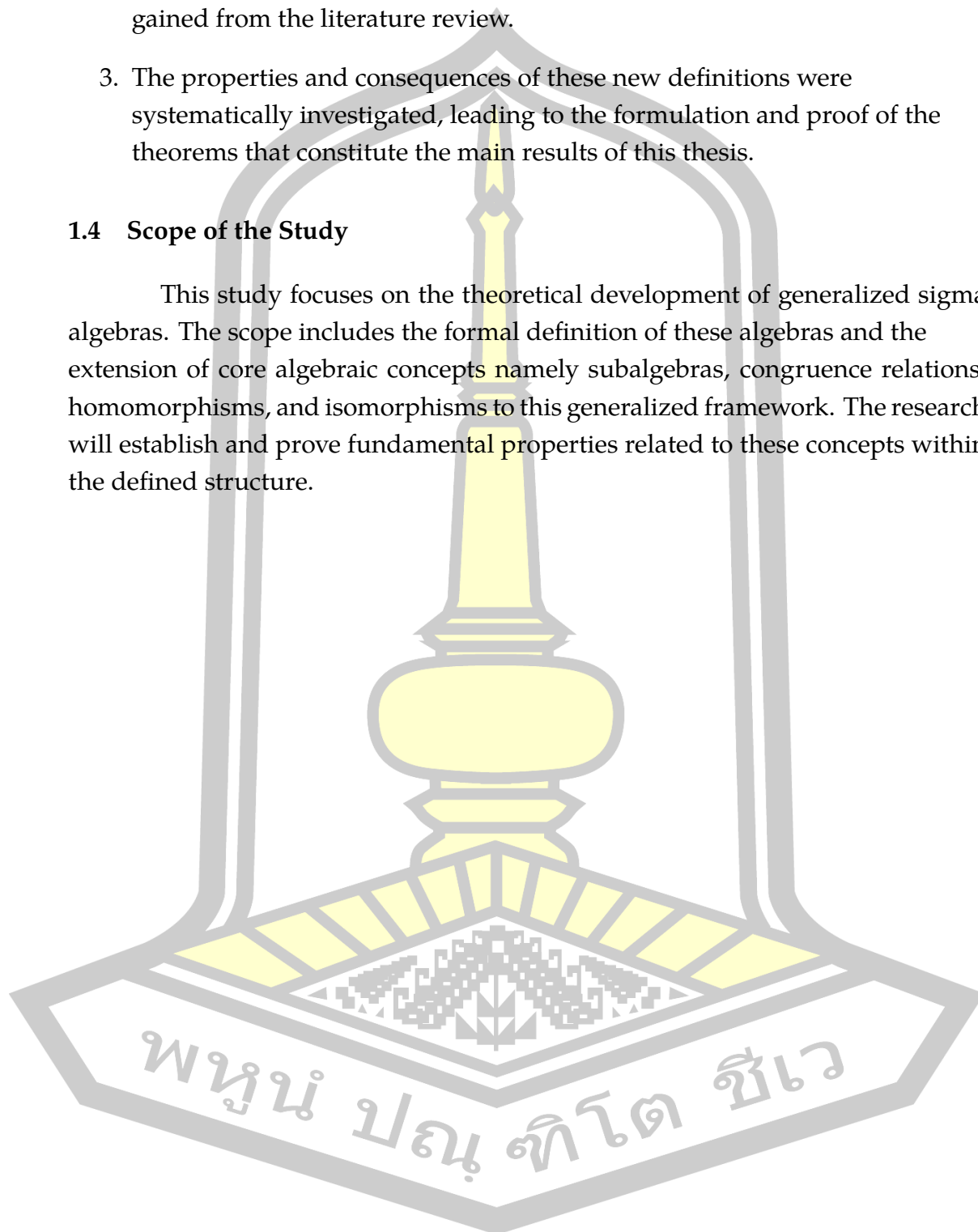
The research presented in this thesis was conducted according to the following procedure:

1. A comprehensive review of the existing literature on classical universal algebra and related generalized structures was conducted to identify foundational concepts and potential areas for extension.

2. New definitions for generalized sigma algebras, subalgebras, congruence relations, and homomorphisms were formulated based on the insights gained from the literature review.
3. The properties and consequences of these new definitions were systematically investigated, leading to the formulation and proof of the theorems that constitute the main results of this thesis.

1.4 Scope of the Study

This study focuses on the theoretical development of generalized sigma algebras. The scope includes the formal definition of these algebras and the extension of core algebraic concepts namely subalgebras, congruence relations, homomorphisms, and isomorphisms to this generalized framework. The research will establish and prove fundamental properties related to these concepts within the defined structure.



CHAPTER 2

PRELIMINARIES

In this chapter, we will give some definitions, notations, dealing with some preliminaries and some useful results that will be duplicated in later chapter.

2.1 Operations and Algebra

The first section we will introduce the concept of a function from a given set (or product of sets) to a fixed set is called operation as the following definition :

Definition 2.1.1. [4] Let A be a set, and let $n \geq 1$ be a natural number. A function $f : A^n \rightarrow A$ is called an n -ary operation defined on A , and is said to have arity n . We let $O^n(A)$ be the set of all n -ary operations defined on A and let

$$O(A) := \bigcup_{n=1}^{\infty} O^n(A).$$

be the set of all finitary operations defined on A .

Definition 2.1.2. [11] Let A be a non-empty set. Let $\mathcal{I}$ be any non-empty index set, and let $(f_i^A)_{i \in \mathcal{I}}$ be a function which assigns to every element of $\mathcal{I}$ an n_i -ary operation f_i^A defined on A . Then the pair $\mathcal{A} := (A; (f_i^A)_{i \in \mathcal{I}})$, is called an (indexed) algebra (indexed by the set $\mathcal{I}$). The set A is called the base or carrier set or universe of A , and f_i^A is called the sequence of fundamental operations of A . For each $i \in \mathcal{I}$ the natural number n_i is called the arity of $(f_i^A)_{i \in \mathcal{I}}$. The sequence $\tau := (n_i)_{i \in \mathcal{I}}$ of all the arities is called the type of the algebra $\mathcal{A}$. We use the name $\text{Alg}(\tau)$ for the class of all algebras of a given type τ .

Example 2.1.3. An algebra $(G; \cdot)$ of type $\tau = (2)$, with one binary operation, is called a groupoid. Here the single binary operation f is denoted by $\cdot$ or even just by juxtaposition, so we write $x \cdot y$ or just xy instead of $f(x, y)$.

2.2 Subalgebra

Definition 2.2.1. Let $\mathcal{B} := (B; (f_i^B)_{i \in \mathcal{I}})$, be an algebra of type τ . Then an algebra $\mathcal{A}$ is called a subalgebra of $\mathcal{B}$, written as $\mathcal{A} \leq \mathcal{B}$, if the following conditions are satisfied:

- (i) $\mathcal{A} := (A, (f_i^A)_{i \in \mathcal{I}})$, is an algebra of type τ
- (ii) $A \subseteq B$.
- (iii) For all $i \in \mathcal{I}$ the graph of f_i^A is a subset of the graph of f_i^B .

Whether one algebra is a subalgebra of another algebra can be checked by the following criterion:

Lemma 2.2.2. (Subalgebra Criterion)

Let $\mathcal{B} := (B; (f_i^B)_{i \in \mathcal{I}})$, be an algebra of type τ and let $A \subseteq B$, be a subset B for which $f_i^A = f_i^B|_A$ for all $i \in \mathcal{I}$. Then $\mathcal{A} := (A, (f_i^A)_{i \in \mathcal{I}})$, is a subalgebra of $\mathcal{B} := (B; (f_i^B)_{i \in \mathcal{I}})$, iff A is closed with respect to all the operations f_i^B for $i \in \mathcal{I}$; that is, if $f_i^B(A^{n_i}) \subseteq A$ for all $i \in \mathcal{I}$.

Proof. See [11, p.15.] □

Corollary 2.2.3. [1] Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be algebras of type τ . Then:

- (i) If $(\mathcal{A} \leq \mathcal{B}) \wedge (\mathcal{B} \leq \mathcal{C})$, then $(\mathcal{A} \leq \mathcal{C})$.
- (ii) If $(A \subseteq B \subseteq C) \wedge (\mathcal{A} \leq \mathcal{C}) \wedge (\mathcal{B} \leq \mathcal{C})$, then $(\mathcal{A} \leq \mathcal{B})$.

Proof. See [11, p.15.] □

Corollary 2.2.4. [1] The non-empty intersection $\mathcal{A}$ of a non-empty family $\{\mathcal{A}_j \mid j \in J\}$ of subalgebras of an algebra $\mathcal{B}$ of type τ is a subalgebra of $\mathcal{B}$.

Proof. See [11, p.15.] □

2.3 Homomorphisms

Consider the absolute value function, $h : \mathbb{R} \rightarrow \mathbb{R}^+$, which assigns to every real number $a \in \mathbb{R}$ its absolute value $|a|$. This function is compatible with the multiplicative structure of $\mathbb{R}$ because it satisfies the property $|a \cdot b| = |a| \cdot |b|$. This identity shows that we obtain the same result whether we first multiply the numbers and then apply the mapping, or first map the numbers and then multiply their images.

This concept of a structure-preserving map is a fundamental theme in algebra. Similar properties can be observed in other contexts; for instance, assigning to each square matrix its determinant satisfies $|\mathbf{A} \cdot \mathbf{B}| = |\mathbf{A}| \cdot |\mathbf{B}|$, and assigning to each permutation its sign satisfies $\text{sgn}(s_1 \circ s_2) = \text{sgn}(s_1) \cdot \text{sgn}(s_2)$. Such mappings, known as homomorphisms, are central to the study of algebraic structures. In each of these examples, we have a mapping between the carrier sets of two algebras of the same type, which is compatible with the operations of the algebras. If the mapping between the carrier sets of the two algebras is also a bijection, then the difference between the two algebras amounts only to a relabelling of the elements.

This can be seen for instance in the following example. We consider the algebras

$$\mathcal{A}_3 = (\{(1), (123), (132)\}; \circ) \text{ and } \mathbb{Z}_3 = (\{[0]_3, [1]_3, [2]_3\}; +),$$

the group of all even permutations (the alternating group) of order three and the cyclic group of order three, respectively. Both algebras have type $\tau = (2)$, and it is easy to verify that both are groups. From their Cayley tables

$\circ$	(1)	(123)	(132)
(1)	(1)	(123)	(132)
(123)	(123)	(132)	(1)
(132)	(132)	(1)	(123)

Table 2.1: The Cayley table of the group $\mathcal{A}_3$ of type $\tau = (2)$

it appears that the bijection

$$h : \{(1), (123), (132)\} \rightarrow \{[0]_3, [1]_3, [2]_3\},$$

+	$[0]_3$	$[1]_3$	$[2]_3$
$[0]_3$	$[0]_3$	$[1]_3$	$[2]_3$
$[1]_3$	$[1]_3$	$[2]_3$	$[0]_3$
$[2]_3$	$[2]_3$	$[0]_3$	$[1]_3$

Table 2.2: The Cayley table of the group $\mathbb{Z}_3$ of type $\tau = (2)$

defined by the mapping $(1) \mapsto [0]_3$, $(123) \mapsto [1]_3$ and $(132) \mapsto [2]_3$, is compatible with the structure. For instance, we have

$$h((123) \circ (132)) = h((1)) = [0]_3 = [1]_3 + [2]_3 = h((123)) + h((132)),$$

and similarly for all other pairs of elements. Functions or bijections between algebras which have this compatibility property are called *homomorphisms* or *isomorphisms*. The following example shows that this compatibility property occurs in very "practical" cases. An industrial automaton is designed to construct more complex parts of a machine from certain simpler parts. There are working phases in which two parts are combined into a new part, other phases in which three parts are combined into a new one, and so on. These correspond to binary operations, ternary operations, etc. If the parts are combined in a different order, represented by a mapping h , then the automaton will continue to work in the correct way when h has our compatibility property. Thus the concepts of homomorphism and isomorphism may be said to model mathematically the "artificial intelligence" we expect from such an automaton. In this section, we define *homomorphisms* and *isomorphisms*, and present several important theorems outlining their properties and connecting them with quotient algebras and congruences.

We begin with the definitions of homomorphism and isomorphism.

Definition 2.3.1. Let $\mathcal{A} = (A; (f_i^A)_{i \in \mathcal{I}})$ and $\mathcal{B} = (B; (f_i^B)_{i \in \mathcal{I}})$ be an algebra of type τ . Then a function $h : A \rightarrow B$ is called a *homomorphism* $h : \mathcal{A} \rightarrow \mathcal{B}$ of $\mathcal{A}$ into $\mathcal{B}$ if for all $i \in \mathcal{I}$ we have

$$h(f_i^A(a_1, a_2, \dots, a_{n_i})) = f_i^B(h(a_1), h(a_2), \dots, h(a_{n_i})),$$

for all $(a_1, a_2, \dots, a_{n_i}) \in A$. In the special case that $n_i = 0$, this equation means that $h(f_i^A(\emptyset)) = f_i^B(\emptyset)$. That is, the element designated by the nullary operation f_i^A in A must be mapped to the corresponding element f_i^B in B .

If the function h is bijective, that is both one-to-one (injective) and "onto" (surjective), then the homomorphism $h : \mathcal{A} \rightarrow \mathcal{B}$ is called an *isomorphism* from $\mathcal{A}$ onto $\mathcal{B}$. An *injective homomorphism* from $\mathcal{A}$ into $\mathcal{B}$ is also called an *embedding* of $\mathcal{A}$ into $\mathcal{B}$.

A homomorphism $h : \mathcal{A} \rightarrow \mathcal{A}$ of an algebra $\mathcal{A}$ into itself is called an *endomorphism* of $\mathcal{A}$, and an *isomorphism* $h : \mathcal{A} \rightarrow \mathcal{A}$ from $\mathcal{A}$ onto $\mathcal{A}$ is called an *automorphism* of $\mathcal{A}$.

Example 2.3.2. (1) It is easy to prove that for every algebra $\mathcal{A}$, the identical mapping $id : \mathcal{A} \rightarrow \mathcal{A}$, defined by $id_{\mathcal{A}}(x) = x$ for all $x \in \mathcal{A}$, is an automorphism of $\mathcal{A}$.

(2) Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be algebras of the same type, and let $h_1 : \mathcal{A} \rightarrow \mathcal{B}$ and $h_2 : \mathcal{B} \rightarrow \mathcal{C}$, be homomorphisms. The composition function $h_2 \circ h_1 : \mathcal{A} \rightarrow \mathcal{C}$, is defined by $(h_2 \circ h_1)(x) = h_2(h_1(x))$, for all $x \in \mathcal{A}$. It is immediately to verify that this composition is also a homomorphism, and that when both h_1 and h_2 are surjective, injective or bijective, then the composition has the same property.

(3) If θ is a congruence relation on $\mathcal{A}$ and if $\mathcal{A}/\theta$ is the corresponding quotient algebra, then $h : \mathcal{A} \rightarrow \mathcal{A}/\theta$, defined by $a \mapsto [a]_{\theta}$ is a surjective homomorphism. The definition of operations on $\mathcal{A}/\theta$ from the congruence lattice theorem gives us

$$\begin{aligned} h(f_i^{\mathcal{A}}(a_1, a_2, \dots, a_{n_i})) &= [f_i^{\mathcal{A}}(a_1, a_2, \dots, a_{n_i})]_{\theta} \\ &= f_1^{A/\theta}([a_1]_{\theta}, [a_2]_{\theta}, \dots, [a_{n_i}]_{\theta}) \\ &= f_1^{A/\theta}(h(a_1), h(a_2), \dots, h(a_{n_i})). \end{aligned}$$

for all $i \in \mathcal{I}$ This homomorphism is called the *natural homomorphism* induced by θ on $\mathcal{A}$, and is usually denoted by $nat \theta$.

It is useful to consider the behaviour of subalgebras under homomorphic mappings.

Theorem 2.3.3. Let $h : \mathcal{A} \rightarrow \mathcal{B}$ be a homomorphism of the algebra $\mathcal{A}$ of type τ into the algebra $\mathcal{B}$ of type τ . Then we have:

- (i) The image $\mathcal{B}_1 = h(\mathcal{A}_1)$ of a subalgebra $\mathcal{A}_1$ of $\mathcal{A}$ under the homomorphism h is a subalgebra of $\mathcal{B}$.

(ii) The preimage $h^{-1}(\mathcal{B}') = \mathcal{A}'$ of a subalgebra $\mathcal{B}'$ of $h(\mathcal{A}) \subseteq \mathcal{B}$ is a subalgebra of $\mathcal{A}$.

(iii) For any subset $X \subseteq A$, we have $\langle h(X) \rangle_{\mathcal{B}} = \langle h(X) \rangle_{\mathcal{A}}$.

Proof. See [11, p.50.] □

Before we continue with general properties of homomorphisms, we will examine more closely the automorphisms on an algebra. The set of all automorphisms of the algebra $\mathcal{A}$ is denoted by $\text{Aut } \mathcal{A}$, and the set of all its endomorphisms is denoted by $\text{End } \mathcal{A}$. Since the composition $\circ$ of two automorphisms (endomorphisms)

of $\mathcal{A}$ is again an automorphism (endomorphism) of $\mathcal{A}$ and since $\circ$ is associative, $(\text{Aut } \mathcal{A}; \circ, id_{\mathcal{A}})$ and $(\text{End } \mathcal{A}; \circ, id_{\mathcal{A}})$ are monoids, with the former a submonoid of the latter. It is also true that if $h : \mathcal{A} \rightarrow \mathcal{A}$ is an automorphism of $\mathcal{A}$, the mapping $h^{-1} : \mathcal{A} \rightarrow \mathcal{A}$ is again an automorphism of $\mathcal{A}$. This can be seen as follows:

$$\begin{aligned} h^{-1}(f_i^A(b_1, b_2, \dots, b_{n_i})) &= h^{-1}(f_i^A(h(a_1), h(a_2), \dots, h(a_{n_i}))) \\ &= h^{-1}(h(f_i^A(a_1, a_2, \dots, a_{n_i}))) \\ &= f_i^A(a_1, a_2, \dots, a_{n_i}) \\ &= f_i^A(h^{-1}(b_1), h^{-1}(b_2), \dots, h^{-1}(b_{n_i})). \end{aligned}$$

This gives the following result.

Lemma 2.3.4. The set of all automorphisms of an algebra $\mathcal{A}$ forms a group, $(\text{Aut } \mathcal{A}; \circ, ^{-1}, id_{\mathcal{A}})$, called the automorphism group of $\mathcal{A}$.

Proof. Let $\mathcal{A} = (A; (f_i^A)_{i \in I})$ be an algebra of type τ and let $h : \mathcal{A} \rightarrow \mathcal{A}$ be an automorphism of $\mathcal{A}$. An element $a \in A$ is called a fixed point of h , if $h(a) = a$. Every $a \in A$ is of course a fixed point of the identical automorphism $id_{\mathcal{A}}$ on $\mathcal{A}$. □

Lemma 2.3.5. The set of all fixed points of an automorphism h of $\mathcal{A}$ is a subalgebra of $\mathcal{A}$.

Proof. See [11, p.51.] □

Definition 2.3.6. Let $\mathcal{A}$ and $\mathcal{A}'$ be algebras of the same type τ and let $\mathcal{B}$ be a subalgebra of both $\mathcal{A}$ and $\mathcal{A}'$. An isomorphism $h : \mathcal{A} \rightarrow \mathcal{A}'$ with $h(b) = b$ for all $b \in \mathcal{B}$ is called a relative isomorphism between $\mathcal{A}$ and $\mathcal{A}'$ with respect to $\mathcal{B}$. When h is an automorphism on $\mathcal{A}$, it is then called a relative automorphism of $\mathcal{A}$ with respect to the subalgebra $\mathcal{B}$.

We can consider such a relative isomorphism, $h : \mathcal{A} \rightarrow \mathcal{A}'$ with respect to a common subalgebra $\mathcal{B}$, in two ways. From "above," we say that the restriction of h to $\mathcal{B}$ is the identity isomorphism on $\mathcal{B}$, while from "below," we call h an extension of the identity isomorphism on $\mathcal{B}$.

More generally, let $\mathcal{B}$ and $\mathcal{B}'$ be subalgebras of the algebras $\mathcal{A}$ and $\mathcal{A}'$, respectively. Let $g : \mathcal{B} \rightarrow \mathcal{B}'$ and $h : \mathcal{A} \rightarrow \mathcal{A}'$ be isomorphisms. Then we say that the isomorphism h is an extension of g , if for all $b \in \mathcal{B}$ the equation $g(b) = h(b)$ is satisfied. The following result is easy to verify.

Lemma 2.3.7. The set $Aut_{rel \mathcal{B}} \mathcal{A}$ of all relative automorphisms of $\mathcal{A}$ with respect to a subalgebra $\mathcal{B} \leq$ forms a subgroup of the automorphism group of $\mathcal{A}$.

Proof. See [11, p.53.] □

We have seen that any automorphism h on $\mathcal{A}$ determines a subalgebra of $\mathcal{A}$, consisting of the fixed points of h . If we take h to be a relative automorphism with respect to some subalgebra $\mathcal{B}$, this fixed point subalgebra contains $\mathcal{B}$. Now let $\mathcal{G}$ be a subgroup of the group of all relative automorphisms of $\mathcal{A}$ with respect to $\mathcal{B}$. We form the set

$$\mathcal{B}' := \{b \in \mathcal{A} \mid s(b) = b \text{ for all } s \in \mathcal{G}\}.$$

of the elements of $\mathcal{A}$ which are fixed points of all the automorphisms in $\mathcal{G}$. Lemma 2.3.5 tells us that $\mathcal{B}'$ is a subalgebra of $\mathcal{A}$, which again has $\mathcal{B}$ as a subalgebra. This gives a way of associating to every subgroup $\mathcal{G} \leq Aut_{rel \mathcal{B}} \mathcal{A}$ an algebra between $\mathcal{B}$ and $\mathcal{A}$, so $\mathcal{B} \leq \mathcal{B}' \leq \mathcal{A}$. We can ask whether the converse holds: does every algebra $\mathcal{B} \leq \mathcal{B}' \leq \mathcal{A}$ determine a subgroup of $Aut_{rel \mathcal{B}} \mathcal{A}$ which consists exactly of the automorphisms of $\mathcal{A}$ fixing the elements from a $\mathcal{B}'$.

Notice: We will have noticed that these interconnections amount to a Galois-connection between the sets $\mathcal{A}$ and $Aut_{rel \mathcal{B}} \mathcal{A}$. We have a basic relation

$$R := \{(a, s) \mid a \in \mathcal{A} \text{ and } s \in Aut_{rel \mathcal{B}} \mathcal{A} \text{ and } s(a) = a\},$$

which induces the maps

$$\begin{aligned}\sigma(X) &:= \{s \in \text{Aut}_{\text{real } \mathcal{B}} \mathcal{A} \mid \text{for all } a \in X (s(a) = a)\} \text{ and} \\ \sigma(Y) &:= \{a \in A \mid \text{for all } s \in Y (s(a) = a)\},\end{aligned}$$

for all $X \subseteq A$ and $Y \subseteq \text{Aut}_{\text{real } \mathcal{B}} \mathcal{A}$. We now return to the general theory of homomorphisms, with some theorems that relate homomorphisms to congruences. Because that every congruence relation θ on an algebra $\mathcal{A}$ determines a homomorphism, namely the natural homomorphism

$$\text{nat } \theta : A \rightarrow A/\theta.$$

onto the quotient algebra. Now we will show that conversely every homomorphism of an algebra $\mathcal{A}$ also determines a congruence relation on $\mathcal{A}$.

Let $h : \mathcal{A} \rightarrow \mathcal{B}$ be a homomorphism. Since the function $h : \mathcal{A} \rightarrow \mathcal{B}$, is in general not injective, we may have different elements with the same image. We investigate the equivalence relation corresponding to the partition of the set A into classes consisting of elements having the same image, that is, the kernel of h .

Definition 2.3.8. Let $\mathcal{A}$ and $\mathcal{A}'$ be algebras of the same type τ and let $h : \mathcal{A} \rightarrow \mathcal{B}$ be a homomorphism. The following binary relation is called the *kernel of the homomorphism h* :

$$\ker h := \{(a, b) \in A^2 \mid h(a) = h(b)\}.$$

We may alternately express this as $\ker h = h^{-1} \circ h$, where h^{-1} is the inverse relation of h .

Lemma 2.3.9. The kernel of any homomorphism $h : \mathcal{A} \rightarrow \mathcal{B}$, is a congruence relation on $\mathcal{A}$.

Proof. see [11, p.54] □

~ Suppose we have a homomorphism $h : \mathcal{A} \rightarrow \mathcal{B}$. We have seen that $\ker h$ is a congruence on $\mathcal{A}$, so we can form the quotient algebra $\mathcal{A}/\ker h$, along with the natural homomorphism $\text{nat } (\ker h) : A \rightarrow \mathcal{A}/\ker h$, which maps the algebra $\mathcal{A}$, onto this quotient algebra. Now we have two homomorphic images of $\mathcal{A}$: the original $h(\mathcal{A})$ and the new quotient $\mathcal{A}/\ker h$. What connection is there between these two homomorphic images? The answer to this question is a special case of the following *General Homomorphism Theorem*.

Theorem 2.3.10. (General Homomorphism Theorem)

Let $h : \mathcal{A} \rightarrow \mathcal{B}$ and $g : \mathcal{A} \rightarrow \mathcal{C}$ be homomorphisms, and let g be surjective. Then there exists a homomorphism $f : \mathcal{C} \rightarrow \mathcal{B}$ which satisfies $f \circ g = h$ iff $\ker g \subseteq \ker h$.

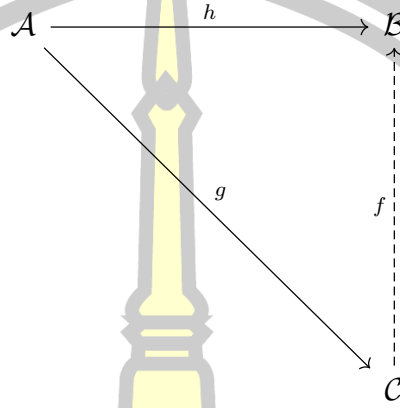


Figure 2.1: Commutativity Diagram $f \circ g = h$

Here $f \circ g = h$ means "commutativity" of the diagram. When this f exists, it has the following properties:

- (i) the homomorphism f is uniquely defined by $f = h \circ g^{-1}$;
- (ii) f is injective if $\ker g = \ker h$;
- (iii) f is surjective if h is surjective.

Proof. See [11, p.56.] □

As mentioned above, as a specific case of the General Homomorphism Theorem we can consider the surjective natural homomorphism associated to any congruence. We have the following result, which tells us that any homomorphic image of an algebra is isomorphic to a quotient algebra of that algebra by the kernel of the homomorphism.

Theorem 2.3.11. (homomorphic image theorem) Let $h : \mathcal{A} \rightarrow \mathcal{B}$ be a surjective homomorphism. Then there exists a unique isomorphism f from $\mathcal{A}/\ker h$ onto $\mathcal{B}$ with $f \circ \text{nat}(\ker h) = h$.

Proof. See [11, p.57.] □

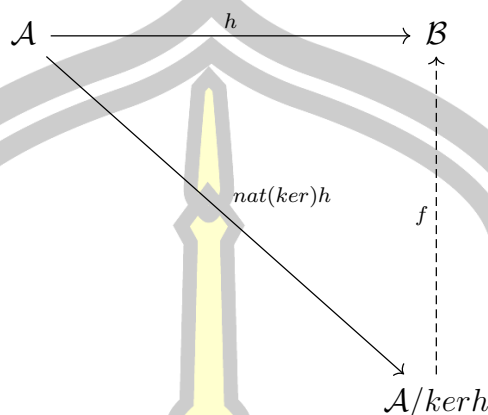


Figure 2.2: Diagram of Homomorphic Image Theorem

The basic idea of the Homomorphism Theorem is that whenever we have a surjective homomorphism h from an algebra $\mathcal{A}$ onto an algebra $\mathcal{B}$, the image $\mathcal{B}$ is isomorphic to the quotient algebra $\mathcal{A}/\ker h$. This means that any homomorphic image of an algebra, which is typically considered "outside" of the original algebra, can in fact be characterized up to isomorphism "inside" the algebra as a quotient determined by the kernel of the homomorphism. Therefore, to know all possible homomorphic images of a given algebra, it is sufficient to find all of its congruence relations and their corresponding quotient algebras.

2.4 Congruence relations

Every function $\varphi : A \rightarrow B$ from a set A into a set B defines a partition of the set A into classes of elements having the same image. The equivalence relation $\ker \varphi$ corresponding to this partition of A is called the kernel of the function φ . That is, for elements $a, b \in A$ we define

$$(a, b) \in \ker \varphi : \Leftrightarrow \varphi(a) = \varphi(b).$$

This definition of the kernel makes it clear that the kernel is an equivalence relation on A . It is also easy to show that $\ker \varphi$ is equal to the composition $\varphi^{-1} \circ \varphi$.

For instance, let $\mathbb{Z}$ be the set of all integers, let m be a natural number, and let $\mathbb{Z}/(m)$ be the set of all equivalence classes modulo m . Then the kernel of the function $\varphi : \mathbb{Z} \rightarrow \mathbb{Z}/(m)$ which is defined by $a \mapsto [a]_m$, that is, which maps

every integer to its class modulo m , is obviously the well-known equivalence or congruence modulo m :

$$a \equiv b(m) :\Leftrightarrow \exists g \in \mathbb{Z}(a - b = gm).$$

(This is also often written as $a \equiv b \pmod{m}$). With respect to addition and multiplication in the ring $(\mathbb{Z}; +, \cdot, 0)$ of all integers, this congruence modulo m has the following additional property:

$$a_1 \equiv b_1(m) \wedge a_2 \equiv b_2(m) \Rightarrow a_1 + a_2 \equiv b_1 + b_2(m) \wedge a_1 \cdot a_2 \equiv b_1 \cdot b_2(m).$$

This tells us that in this example the kernel of the function φ is compatible with the operations of the ring. This observation motivates the following more general definition.

Definition 2.4.1. Let A be a set, let $\theta \subseteq A \times A$ be an equivalence relation on A , and let f be an n -ary operation from $O^n(A)$. Then f is said to be compatible with θ , or to preserve θ , if for all $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in A$, then

$$(a_1, b_1) \in \theta, \dots, (a_n, b_n) \in \theta \text{ implies } (f(a_1, a_2, \dots, a_n), f(b_1, b_2, \dots, b_n)) \in \theta.$$

Remark 2.4.2. This definition can also be applied to arbitrary relations, not just to binary equivalence relations, and we can speak of a function preserving, or being compatible with, any relation. For instance, the monotone increasing real functions $f : \mathbb{R} \rightarrow \mathbb{R}$ are exactly the unary operations on $\mathbb{R}$ which are compatible with the relation $\leq$ on $\mathbb{R}$, since $x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$ for all x_1, x_2 for which f is defined. The preservation of relations by functions will be studied in more detail in the next Chapter.

Definition 2.4.3. [11] Let $\mathcal{A} = (A; (f_i^A)_{i \in I})$ be an algebra of type τ . An equivalence relation θ on A is called a *congruence relation* on $\mathcal{A}$ if all the fundamental operations f_j^A are compatible with θ . We denote by $Con \mathcal{A}$ the set of all congruence relations of the algebra $\mathcal{A}$. For every algebra $\mathcal{A} = (A; (f_i^A)_{i \in I})$ the trivial equivalence relations

$$\Delta := \{(a, a) \mid a \in A\} \text{ and } \nabla_A = A \times A.$$

are congruence relations. An algebra which has no congruence relations except Δ_A and ∇_A is called simple.

Theorem 2.4.4. An equivalence relation θ on an algebra $\mathcal{A}$ is a congruence relation on $\mathcal{A}$ if θ is compatible with all translations of $\mathcal{A}$.

Proof. See [11, p.24.] □

Theorem 2.4.5. The intersection $\theta_1 \cap \theta_2$ of two congruence relations on an algebra $\mathcal{A} = (A; (f_i^A)_{i \in I})$, is again a congruence relation on $\mathcal{A}$.

Proof. See [11, p.24.] □

Remark 2.4.6. Theorem 2.4.5 is also satisfied for arbitrary families of congruence relations on $\mathcal{A}$. But in general, the union of two congruence relations of an algebra $\mathcal{A}$ is not a congruence relation, since this does not hold even for equivalence relations, as the following example shows. Take:

$$\begin{aligned} A &= \{1, 2, 3\}, \\ \theta_1 &= \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}, \\ \theta_2 &= \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}. \end{aligned}$$

The relations θ_1 and θ_2 are equivalence relations, but

$$\theta_1 \cup \theta_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}.$$

is not an equivalence relation in $\mathcal{A}$, since it is not transitive:

$$(1, 2) \in \theta_1 \cup \theta_2, (2, 3) \in \theta_1 \cup \theta_2, \text{ but } (1, 3) \notin \theta_1 \cup \theta_2.$$

But although the union of two congruence relations θ_1 and θ_2 need not be a congruence relation, as in the subalgebra case we can use intersections of congruence to define a smallest congruence generated by the union. This motivates the following definition.

Definition 2.4.7. Let $\mathcal{A}$ be an algebra, and let θ be a binary relation on $\mathcal{A}$. We define the congruence relation $\langle \text{Con } \mathcal{A} \rangle$ on $\mathcal{A}$ generated by θ to be the intersection of all congruence relations θ' on $\mathcal{A}$ which contain θ :

$$\langle \theta_{\text{Con } \mathcal{A}} \rangle := \bigcap \{ \theta' \mid \theta' \in \text{Con } \mathcal{A} \text{ and } \theta \subseteq \theta' \}.$$

It is easy to see that $\langle \theta_{\text{Con } \mathcal{A}} \rangle$ has the three important properties of a closure operator:

$$\begin{aligned}
\theta &\subseteq \langle \theta \rangle_{\text{Con } \mathcal{A}} && \text{(extensivity),} \\
\theta_1 \subseteq \theta_2 &\Rightarrow \langle \theta_1 \rangle_{\text{Con } \mathcal{A}} \subseteq \langle \theta_2 \rangle_{\text{Con } \mathcal{A}} && \text{(monotonicity)} \\
\langle \langle \theta \rangle_{\text{Con } \mathcal{A}} \rangle_{\text{Con } \mathcal{A}} &= \langle \theta \rangle_{\text{Con } \mathcal{A}} && \text{(idempotency)}
\end{aligned}$$

Remark. As we did in this section for subalgebras, we can ask how to derive, from a binary relation θ defined on A , the congruence relation of the algebra $\mathcal{A}$ generated by θ . First we have to enlarge the relation θ to its reflexive and symmetric closure $\Delta_A \cup \theta \cup \theta^{-1}$. Then we form the compatible closure $\langle \Delta_A \cup \theta \cup \theta^{-1} \rangle$. Finally the set produced in this way has to be completed by all pairs needed for transitivity. This is done by using the *transitive hull operator*, which maps any relation α to

$$\alpha_T := \bigcap \{ \beta \subseteq A^2 \mid \beta \supseteq \alpha \text{ and } \beta \text{ is transitive} \}.$$

Altogether, then, we need to use

$$\langle \theta \rangle_{\text{Con } \mathcal{A}} = \langle \langle \Delta_A \cup \theta' \cup (\theta')^{-1} \rangle \rangle_T.$$

This construction allows us to produce, for two congruences on A , the least congruence on $\mathcal{A}$ which contains them both. This gives us the two binary operations on $\text{Con}(\mathcal{A})$ we wanted,

$$\begin{aligned}
\wedge : \text{Con } \mathcal{A} \times \text{Con } \mathcal{A} &\rightarrow \text{Con } \mathcal{A} && \text{defined by } (\theta_1, \theta_2) \mapsto \theta_1 \cap \theta_2, \\
\vee : \text{Con } \mathcal{A} \times \text{Con } \mathcal{A} &\rightarrow \text{Con } \mathcal{A} && \text{defined by } (\theta_1, \theta_2) \mapsto \langle \theta_1 \cup \theta_2 \rangle_{\text{Con } \mathcal{A}},
\end{aligned}$$

Since congruence relations are equivalence relations, each congruence on an algebra $\mathcal{A}$ induces a partition on the set A . What makes congruence relations significant is that the partition set induced by a congruence can also be made into the universe of an algebra. For each $i \in \mathcal{I}$, we define an n_i -ary operation $f_i^{A/\theta}$ on the quotient set A/θ , by

$$f_i^{A/\theta} : (A/\theta)^{n_i} \rightarrow A/\theta,$$

defined by

$$([a_1]_\theta, \dots, [a_{n_i}]_\theta) \mapsto f_i^{A/\theta}([a_1]_\theta, \dots, [a_{n_i}]_\theta) := \left[f_i^{A/\theta}(a_1, \dots, a_{n_i}) \right]_\theta.$$

As always when we define operations on equivalence classes, we must verify that our operation are well-defined, that is, that they are independent of the representatives chosen. But as the following proof shows, this is exactly what the compatibility property of a congruence relation means.

2.5 Isomorphism

In this section we apply our General Homomorphism Theorem to two other situations, to produce two theorems which are usually called the First and Second Isomorphism Theorems. For the first one, we consider subalgebras of an algebra.

Theorem 2.5.1. (First Isomorphism Theorem) Let $\mathcal{A}$ and $\mathcal{B}$ be algebras of the same type τ , and let $h : \mathcal{A} \rightarrow \mathcal{B}$ be a homomorphism. Let $\mathcal{A}_1$ be a subalgebra of $\mathcal{A}$ and $h(\mathcal{A}_1) \leq \mathcal{B}$ its image. We also assume that $\mathcal{A}_1^*$ is the preimage of $h(\mathcal{A}_1)$ and that $h_1 = h|_{\mathcal{A}_1}$ is the restriction of h to $\mathcal{A}_1$, and we take $h_1^* = h|_{\mathcal{A}_1^*}$ to be the restriction of h to $\mathcal{A}_1^*$. Then

$$\varphi : \mathcal{A}_1/\ker h_1 \rightarrow \mathcal{A}_1^*/\ker h_1^*, \text{ defined by } [a]_{\ker h_1} \mapsto [a]_{\ker h_1^*},$$

is an isomorphism from $\mathcal{A}_1/\ker h_1$ onto $\mathcal{A}_1^*/\ker h_1^*$.

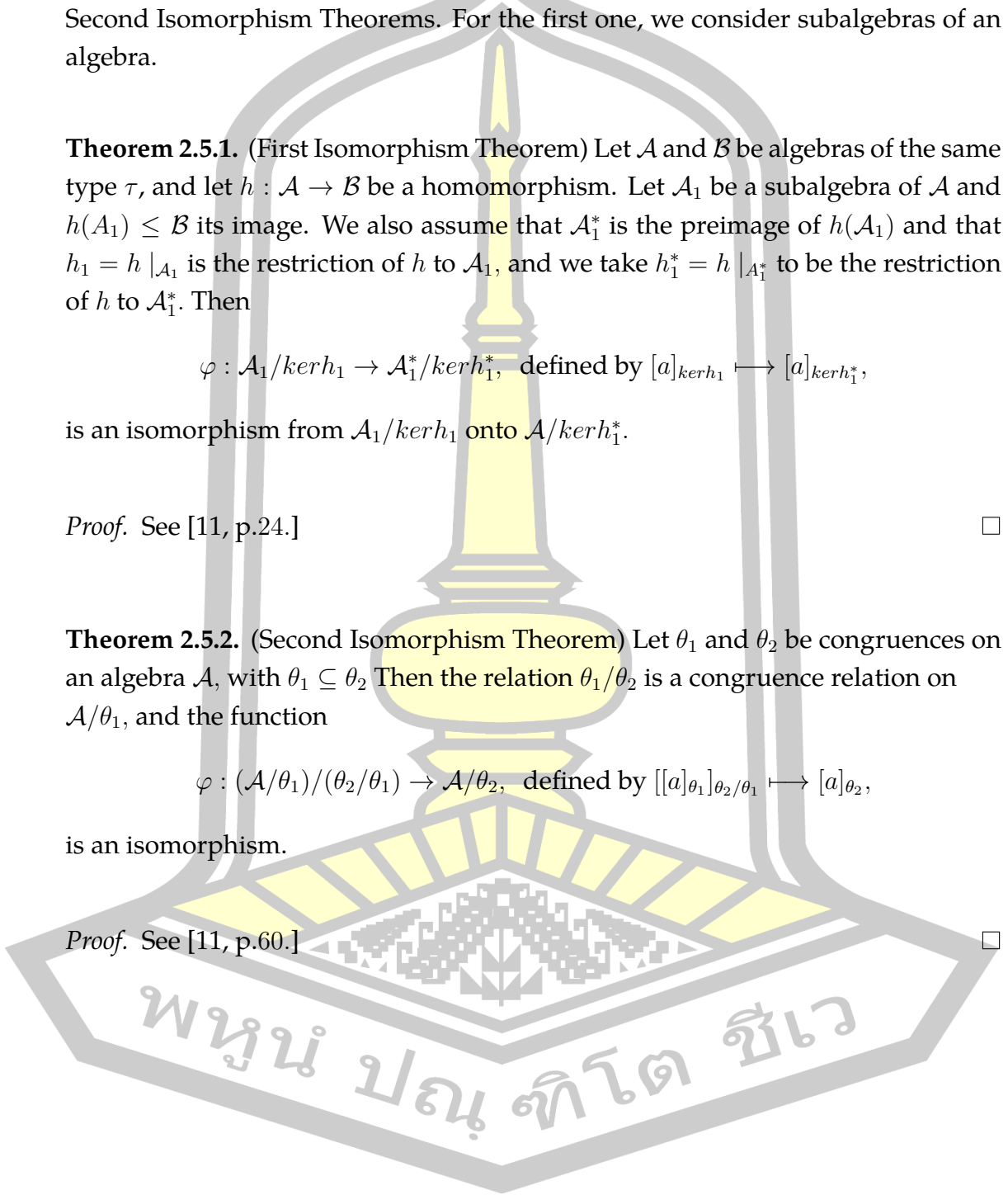
Proof. See [11, p.24.] □

Theorem 2.5.2. (Second Isomorphism Theorem) Let θ_1 and θ_2 be congruences on an algebra $\mathcal{A}$, with $\theta_1 \subseteq \theta_2$. Then the relation θ_1/θ_2 is a congruence relation on $\mathcal{A}/\theta_1$, and the function

$$\varphi : (\mathcal{A}/\theta_1)/(\theta_2/\theta_1) \rightarrow \mathcal{A}/\theta_2, \text{ defined by } [[a]_{\theta_1}]_{\theta_2/\theta_1} \mapsto [a]_{\theta_2},$$

is an isomorphism.

Proof. See [11, p.60.] □



CHAPTER 3

GENERALIZED SIGMA ALGEBRA

In this chapter, we will introduce the concepts of a generalized sigma algebra, and give an example to show that a generalized sigma algebra is distinct from a classical algebra and every classical algebra can be obtained as a generalized sigma algebra. We will investigate the properties of generalized sigma algebra.

3.1 Generalized Sigma Algebra

Before providing the definition of a generalized sigma algebra, we will first review and summarize the construction process of a classical algebra.

Example 3.1.1. (A classical algebra)

Let the components of our algebra be:

$$\Omega = \{1, 2\}$$

$$\text{Base set: } A = \{a, b\}$$

$$\text{Arity sets } \Sigma_1 = \{x, y\}$$

$$\text{and } \Sigma_2 = \{m, n, p\}$$

$$A^{\Sigma_1} : \{\varphi | \varphi : \Sigma_1 \rightarrow A\}$$

$$A^{\Sigma_2} : \{\psi | \psi : \Sigma_2 \rightarrow A\}$$

Therefore, the fundamental operations can be defined as the binary operation

$$f_{\Sigma_1}^A : A^{\Sigma_1} \rightarrow A$$

base on the projection onto the first component:

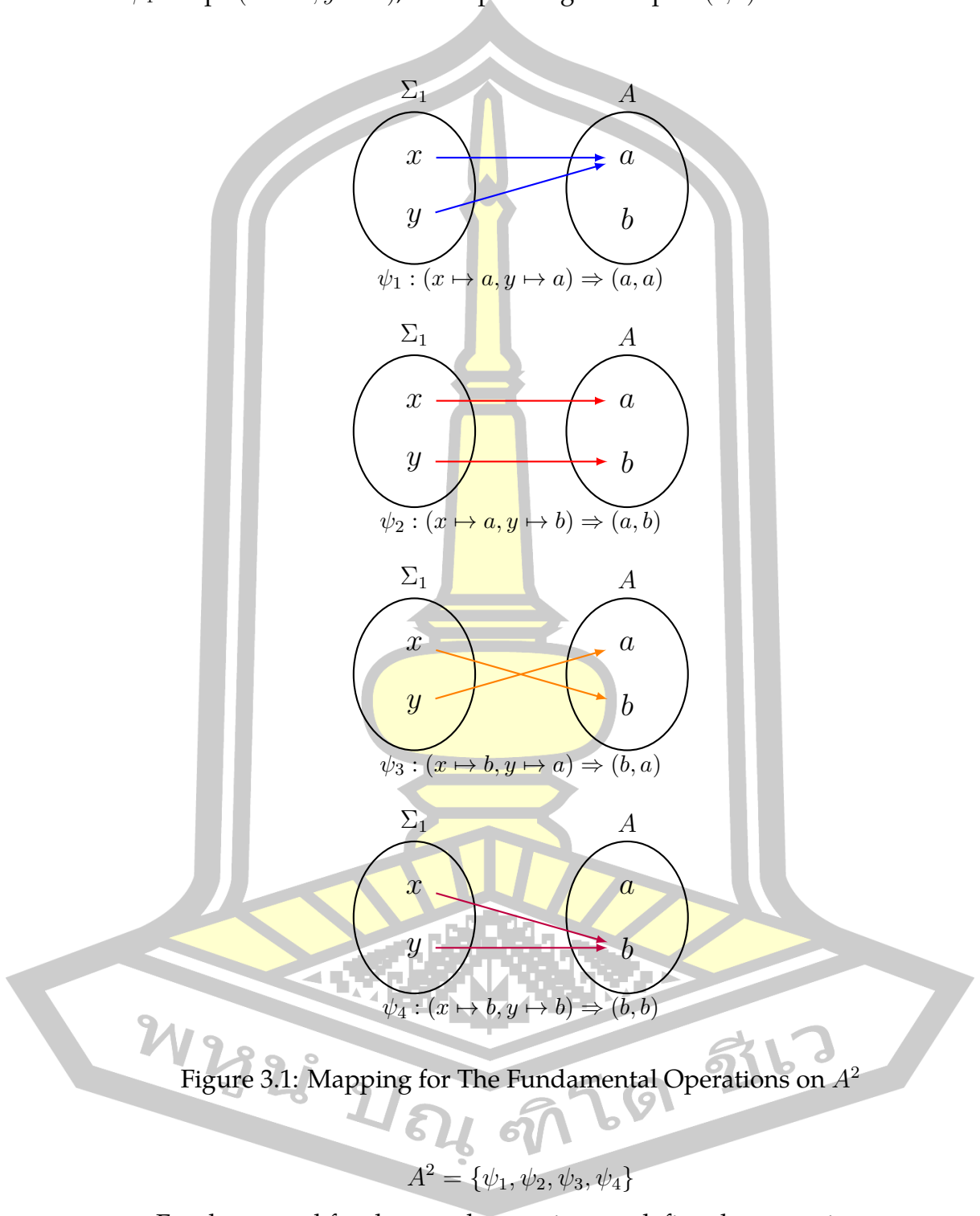
$$f_{\Sigma_1}^A(x, y) = x \text{ for all } x, y \in A$$

For example, $f_{\Sigma_1}^A(a, b) = a$ and $f_{\Sigma_1}^A(b, a) = b$.

This is the set of all functions $\psi : \{x, y\} \rightarrow \{a, b\}$. There are $|A|^{| \Sigma_1 |} = 2^2 = 4$ such functions:

- ψ_1 : maps $(x \mapsto a, y \mapsto a)$, corresponding to the pair (a, a) .
- ψ_2 : maps $(x \mapsto a, y \mapsto b)$, corresponding to the pair (a, b) .

- ψ_3 : maps $(x \mapsto b, y \mapsto a)$, corresponding to the pair (b, a) .
- ψ_4 : maps $(x \mapsto b, y \mapsto b)$, corresponding to the pair (b, b) .



For the second fundamental operation we define the operations $f_{\Sigma_2}^A : A^3 \rightarrow A$, based on the second component: $f_{\Sigma_2}^A(x, y, z) = y$ for all $x, y, z \in A$. For example,

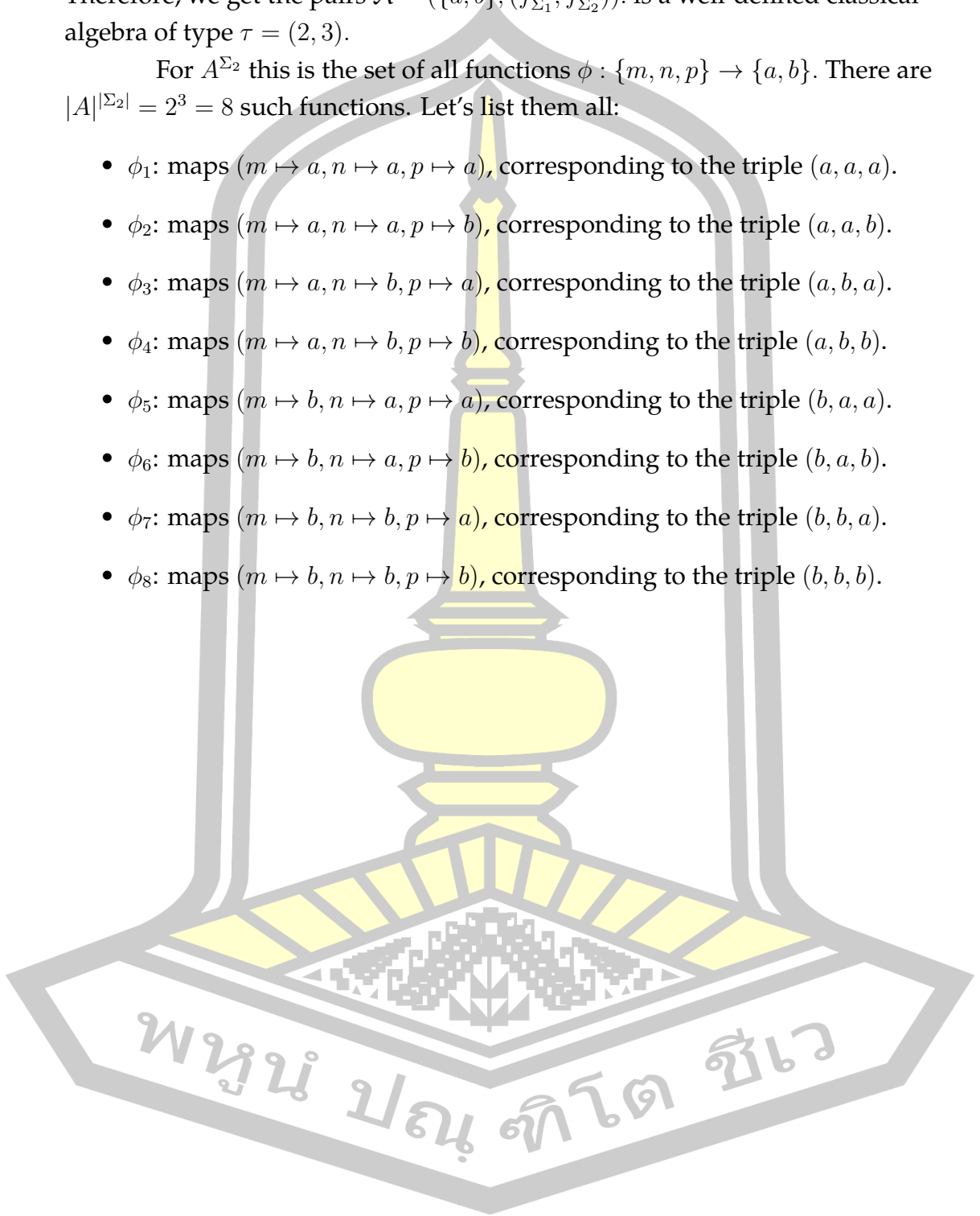
$$f_{\Sigma_2}^A(b, a, b) = a.$$

$$\text{and } f_{\Sigma_2}^A(a, b, a) = b.$$

Therefore, we get the pairs $\mathcal{A} = (\{a, b\}; (f_{\Sigma_1}^A, f_{\Sigma_2}^A))$. is a well-defined classical algebra of type $\tau = (2, 3)$.

For A^{Σ_2} this is the set of all functions $\phi : \{m, n, p\} \rightarrow \{a, b\}$. There are $|A|^{| \Sigma_2 |} = 2^3 = 8$ such functions. Let's list them all:

- ϕ_1 : maps $(m \mapsto a, n \mapsto a, p \mapsto a)$, corresponding to the triple (a, a, a) .
- ϕ_2 : maps $(m \mapsto a, n \mapsto a, p \mapsto b)$, corresponding to the triple (a, a, b) .
- ϕ_3 : maps $(m \mapsto a, n \mapsto b, p \mapsto a)$, corresponding to the triple (a, b, a) .
- ϕ_4 : maps $(m \mapsto a, n \mapsto b, p \mapsto b)$, corresponding to the triple (a, b, b) .
- ϕ_5 : maps $(m \mapsto b, n \mapsto a, p \mapsto a)$, corresponding to the triple (b, a, a) .
- ϕ_6 : maps $(m \mapsto b, n \mapsto a, p \mapsto b)$, corresponding to the triple (b, a, b) .
- ϕ_7 : maps $(m \mapsto b, n \mapsto b, p \mapsto a)$, corresponding to the triple (b, b, a) .
- ϕ_8 : maps $(m \mapsto b, n \mapsto b, p \mapsto b)$, corresponding to the triple (b, b, b) .



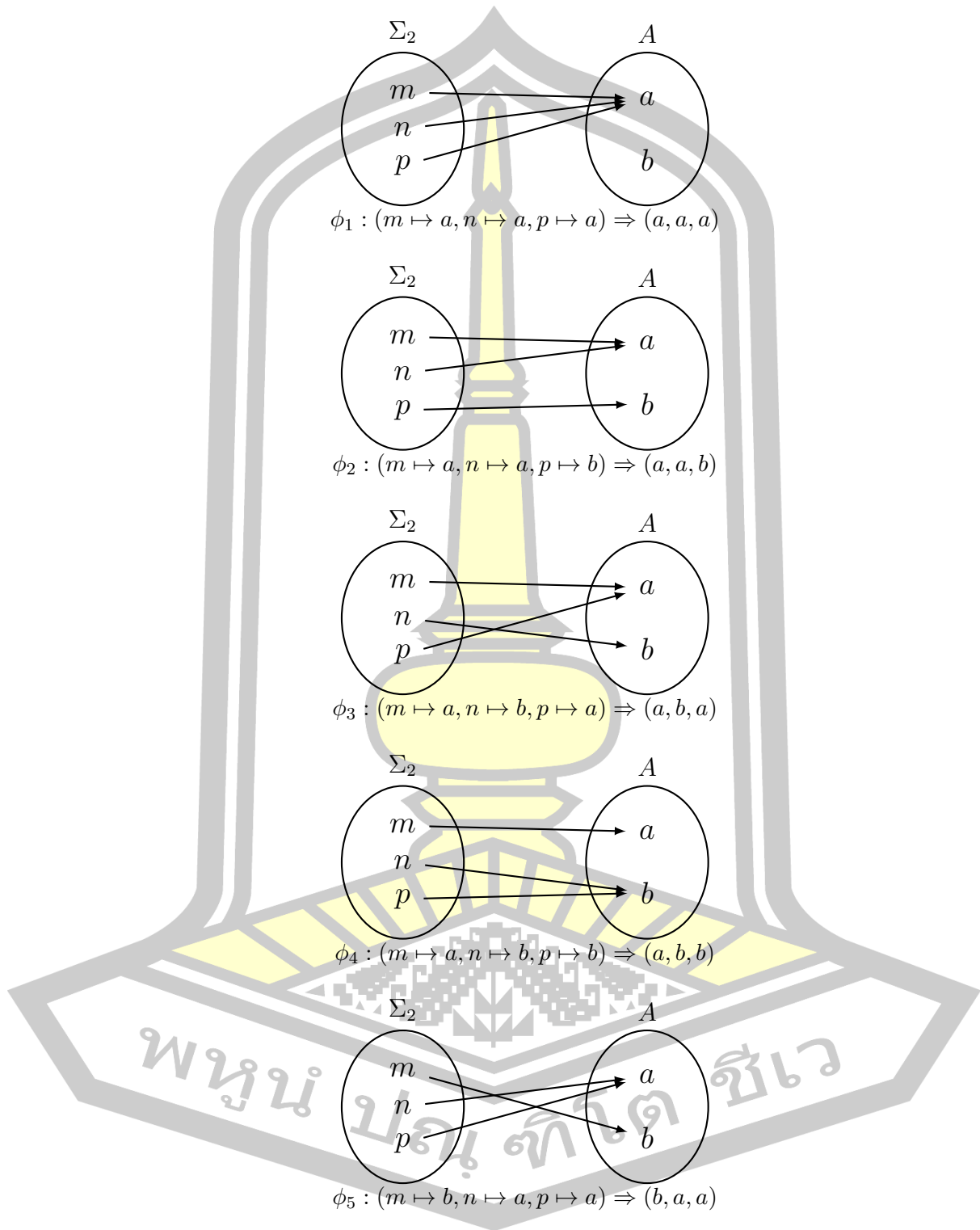


Figure 3.2: Examples of mappings for the sigma operation on A^{Σ_2} .

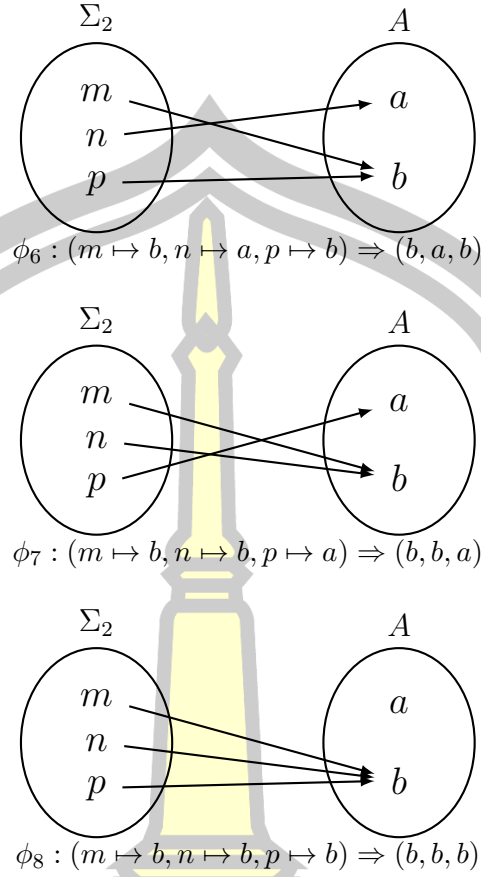


Figure 3.3: Examples of mappings for the sigma operation on A^{Σ_2} . (Continue)

For the operation $f_{\Sigma_2}^A$ with arity 3, the domain is

$$\begin{aligned} A^3 &= A \times A \times A \\ &= \{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7, \phi_8\}. \end{aligned}$$

Its elements are 3-tuples ordered.

Therefore, $(A, (f_{\Sigma_1}^A, f_{\Sigma_2}^A))$ is algebra of type $\tau = (|\Sigma_1|, |\Sigma_2|)$.

Example 3.1.2. Let $\mathcal{A} = (A, (f_i^A)_{i \in \mathcal{I}})$ be an algebra of type $\tau = (n_i)_{i \in \mathcal{I}}$. An algebra $\mathcal{A}$ can be considered as a generalized algebra of type Σ -algebra as follows :

For each $i \in \mathcal{I}$, Let $\Sigma_i = \{1, 2, 3, \dots, n_i\}$, $(|\Sigma_i| = n_i)$.

Then

$$\begin{aligned} A^{\Sigma_i} &= \{f_\rho | f_\rho : \Sigma_i \rightarrow A, \rho \in \{1, 2, 3, \dots, 2^{|\Sigma_i|}\}\}. \\ f_\rho &= (f_\rho(1), f_\rho(2), f_\rho(3), \dots, f_\rho(n_i)). \end{aligned}$$

Since $f : \Sigma_i \rightarrow A$ is a mapping and $\Sigma_i = \{1, 2, 3, \dots, n_i\}$ we given f as an n_i -tuples $(f(1), f(2), \dots, f(n_i))$. Then

$$A^{\Sigma_i} \cong A^{n_i} \text{ for all } i \in \mathcal{I}.$$

Therefore, $\forall i \in \mathcal{I}$ we define $f_{\Sigma_i}^A : A^{\Sigma_i} \rightarrow A$ by

$$f_{\Sigma_i}^A(f_\rho) = f_i^A(f_\rho(1), f_\rho(2), f_\rho(3), \dots, f_\rho(n_i)) \quad (f_i^A \cong f_{\Sigma_i}^A) \quad \text{for all } f_\rho \in A^{\Sigma_i},$$

$\rho \in \{1, 2, 3, \dots, 2^{|\Sigma_i|}\}$. Then $\left(A, (f_{\Sigma_i}^A)_{i \in \mathcal{I}}\right)$ is a generalized algebra of type $\tau = (|\Sigma_i|)_{i \in \mathcal{I}}$.

Definition 3.1.3. Let A be a nonempty set, and let Ω be a any nonempty index set and let $\Sigma = \{\Sigma_j : j \in \Omega\}$ nonempty index set. We defined $f_{\Sigma_j}^A : A^{\Sigma_j} \rightarrow A$, where

$$\begin{aligned} A^{\Sigma_j} &= \prod_{\beta \in \Sigma_j} A_\beta ; A_\beta = A \text{ for all } \beta \in \Sigma_j \\ &= \{\sigma_\beta | \sigma_\beta : \Sigma_j \rightarrow A, \sigma_\beta \in \Sigma_j\}. \end{aligned}$$

We called $f_{\Sigma_j}^A : A^{\Sigma_j} \rightarrow A$ for all $j \in \Omega$ the $|\Sigma_j|$ -ary operation defined on A , and is said to have *arity* $|\Sigma_j|$. We let $\left(f_{\Sigma_j}^A\right)_{j \in \Omega}$ be a function which assigns to every element of Ω and $|\Sigma_j|$ -ary operation $f_{\Sigma_j}^A$ defined on A^{Σ_j} . Then the pair $\mathcal{A}_\Sigma := \left(A; \left(f_{\Sigma_j}^A\right)_{j \in \Omega}\right)$ is called (indexed) *Generalized algebra* (indexed by the set Σ_j) for all $\Sigma_j \in \Sigma$; for all $j \in \Omega$. The set A is called the *base* or *carrier set* or *universe* of A and $\left(f_{\Sigma_j}^A\right)_{j \in \Omega}$ is called the *sequence of fundamental operations* of A , for each $j \in \Omega$ the index Σ_j is called *arity* of $\left(f_{\Sigma_j}^A\right)_{j \in \Omega}$. The sequence $\tau := (|\Sigma_j|_{j \in \Omega})$ of all arities is called *type* of the generalized algebra. We use the name $\text{Alg}(\Sigma)$ for the calss of all algebras of a given type $\tau := (|\Sigma_j|_{j \in \Omega})$.

Remark 3.1.4. From the definition 3.1.3, since $\bigcup_{\sigma \in \Sigma} A_\sigma = \bigcup A = A$, then we have

$$\begin{aligned} A^{\Sigma_j} &= \prod_{\beta \in \Sigma_j} A_\beta \\ &= \left\{ \sigma : \Sigma_j \rightarrow \bigcup_{\beta \in \Sigma_j} A_\beta \mid \sigma(\beta) \in A_\beta \text{ for all } \beta \in \Sigma_j \right\} \\ &= \{ \sigma : \Sigma_j \rightarrow A \mid \sigma \text{ is a function} \}. \end{aligned}$$

Now we will give some example of generalized sigma algebra with uncountable arity and that is not classical algebras.

Example 3.1.5. (An Algebra with Uncountable Arity)

This example demonstrates a generalized sigma algebra that cannot be

represented as a classical algebra due to its uncountable arity.

Let the base set be $A = \{a, b, c\}$. and the index set for the operations be $\Omega = \{1, 2\}$. We defined the arity sets as real intervals:

$$\Sigma_1 = [0, 1]$$

$$\Sigma_2 = [2, 3]$$

Observe that, the arity of our operations is determined by sets of real numbers, not natural numbers. If we given that for each $j \in \Omega$, the carrier of the operation

$$A^{\Sigma_j} = \{\varphi | \varphi : \Sigma_j \rightarrow \{a, b, c\}\}.$$

So we have

$$A^{\Sigma_1} = \{\varphi | \varphi : \Sigma_1 \rightarrow \{a, b, c\}\}.$$

$$A^{\Sigma_2} = \{\varphi | \varphi : \Sigma_2 \rightarrow \{a, b, c\}\}.$$

Hence A^{Σ_1} and A^{Σ_2} are the sets of all functions from the arity set Σ_j to the base set A , where $j = 1, 2$.

For the arity set $\Sigma_1 = [0, 1]$, we define $f_{\Sigma_1}^A$ based on the value of the input function φ at the specific point $\beta = 1$:

$$f_{\Sigma_1}^A(\varphi) = \begin{cases} a & \text{if } \varphi(1) = a \\ b & \text{otherwise.} \end{cases}$$

For the arity set $\Sigma_2 = [2, 3]$, we define $f_{\Sigma_2}^A$ as follows:

$$f_{\Sigma_2}^A(\varphi) = \begin{cases} b & \text{if } \varphi(2) = b \\ c & \text{otherwise.} \end{cases}$$

Therefore,

$$\mathcal{A}_{\Sigma} = (\{a, b, c\}; (f_{[0,1]}^A, f_{[2,3]}^A))$$

is a well-defined generalized sigma algebra of type $\tau = (|\mathbb{R}|, |\mathbb{R}|) = |\mathbb{R}|$

Since the arity of the operation $f_{[0,1]}^A$ is the set $[0, 1]$. The cardinality of this set is $|[0, 1]| = |\mathbb{R}|$, which is an uncountable infinity. Since classical algebras require all arities to be natural numbers ($n \in \mathbb{N}$), there is no way to represent this structure in the classical framework. This example clearly demonstrates that the class of Generalized Sigma Algebras is a proper extension of the class of classical algebras.

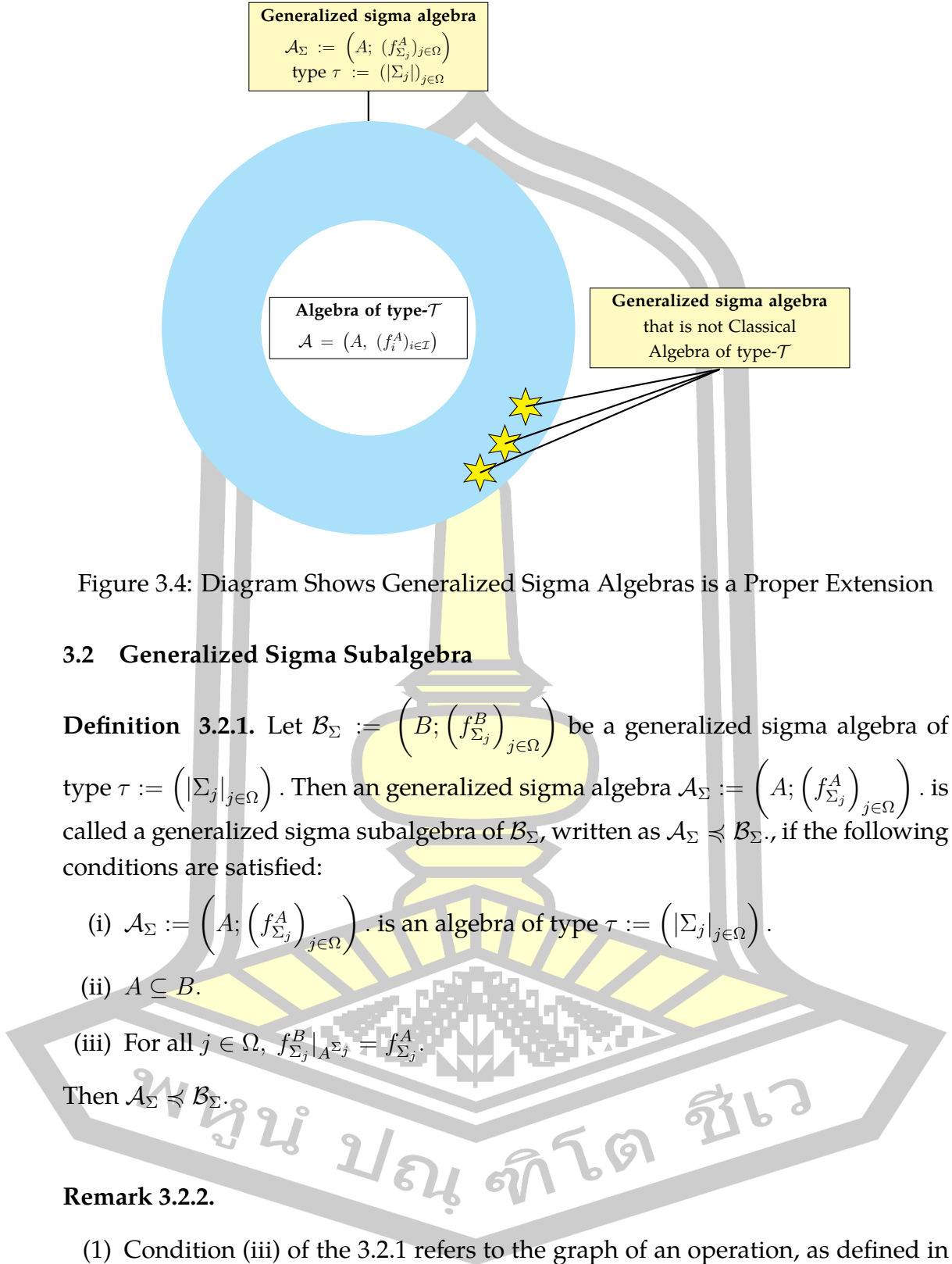


Figure 3.4: Diagram Shows Generalized Sigma Algebras is a Proper Extension

3.2 Generalized Sigma Subalgebra

Definition 3.2.1. Let $\mathcal{B}_\Sigma := (B; (f_{\Sigma_j}^B)_{j \in \Omega})$ be a generalized sigma algebra of type $\tau := (|\Sigma_j|)_{j \in \Omega}$. Then an generalized sigma algebra $\mathcal{A}_\Sigma := (A; (f_{\Sigma_j}^A)_{j \in \Omega})$ is called a generalized sigma subalgebra of $\mathcal{B}_\Sigma$, written as $\mathcal{A}_\Sigma \preceq \mathcal{B}_\Sigma$, if the following conditions are satisfied:

- (i) $\mathcal{A}_\Sigma := (A; (f_{\Sigma_j}^A)_{j \in \Omega})$ is an algebra of type $\tau := (|\Sigma_j|)_{j \in \Omega}$.
- (ii) $A \subseteq B$.
- (iii) For all $j \in \Omega$, $f_{\Sigma_j}^B|_{A^{\Sigma_j}} = f_{\Sigma_j}^A$.

Then $\mathcal{A}_\Sigma \preceq \mathcal{B}_\Sigma$.

Remark 3.2.2.

- (1) Condition (iii) of the 3.2.1 refers to the graph of an operation, as defined in Remark of graph This condition means that the graph of $f_{\Sigma_j}^A$ for all $j \in \Omega$ is the restriction of the graph of $f_{\Sigma_j}^B$ for all $j \in \Omega$ to $A^{\Sigma_j} \subseteq B^{\Sigma_j}$. We write $f_{\Sigma_j}^B|_{A^{\Sigma_j}} = f_{\Sigma_j}^A$, for all $j \in \Omega$, using $f_{\Sigma_j}^B|_{A^{\Sigma_j}}$, to denote the restriction of $f_{\Sigma_j}^B$ to A^{Σ_j} .

(2) If $f_{\Sigma_j}^A$ is a nullary operation, then

$$\begin{aligned}
 f_{\Sigma_j}^A &= (A^0 \times B_{\Sigma_j}) \cap f_{\Sigma_j}^B \\
 &= (\{\emptyset\} \times B_{\Sigma_j}) \cap f_{\Sigma_j}^B \\
 &= \left\{ \left(\emptyset, f_{\Sigma_j}^B(\emptyset) \right) \right\} \\
 &= f_{\Sigma_j}^B.
 \end{aligned}$$

That is, a nullary operation $f_{\Sigma_j}^A$ must designate the same element in each subalgebra A_{Σ} of B_{Σ} .

Whether one generalized sigma algebra is a generalized sigma subalgebra of another generalized sigma algebra can be checked by the following criterion:

Lemma 3.2.3. (Generalized Sigma Subalgebra Criterion)

Let $B_{\Sigma} := \left(B; \left(f_{\Sigma_j}^B \right)_{j \in \Omega} \right)$ be a generalized sigma algebra of type $\tau := \left(|\Sigma_j|_{j \in \Omega} \right)$ and let $A \subseteq B$, then $A_{\Sigma} := \left(A; \left(f_{\Sigma_j}^A \right)_{j \in \Omega} \right)$ is a generalized sigma subalgebra of B_{Σ} iff A is closed with respect to all the operations $f_{\Sigma_j}^B$ for all $j \in \Omega$; that is, if $f_{\Sigma_j}^B|_{A^{\Sigma_j}}(\sigma) \in A$ for all $\sigma \in A^{\Sigma_j}$ and for all $j \in \Omega$

Proof. ($\Rightarrow$) Assume that $A_{\Sigma} := \left(A; \left(f_{\Sigma_j}^A \right)_{j \in \Omega} \right)$ is a generalized sigma subalgebra of B_{Σ} . We need to show that A is closed with respect to all the operations $f_{\Sigma_j}^B$ for all $j \in \Omega$.

Closure Property: Consider any $j \in \Omega$ and any $\sigma \in A^{\Sigma_j}$. Since A_{Σ} is a generalized sigma subalgebra, we know that $f_{\Sigma_j}^A(\sigma) \in A$. By definition, $f_{\Sigma_j}^A(\sigma) = f_{\Sigma_j}^B|_{A^{\Sigma_j}}(\sigma)$. Therefore, we have shown that $f_{\Sigma_j}^B|_{A^{\Sigma_j}}(\sigma) \in A$. Thus, A is closed with respect to all the operations $f_{\Sigma_j}^B$ for all $j \in \Omega$.

($\Leftarrow$) Assume that A is closed with respect to all the operations $f_{\Sigma_j}^B$ for all $j \in \Omega$. This means, for all $j \in \Omega$ and for all $\sigma \in A^{\Sigma_j}$, $f_{\Sigma_j}^B|_{A^{\Sigma_j}}(\sigma) \in A$.

We need to show that $A_{\Sigma} := \left(A; \left(f_{\Sigma_j}^A \right)_{j \in \Omega} \right)$ is a generalized sigma subalgebra of B_{Σ} .

For each $j \in \Omega$, $f_{\Sigma_j}^A(\sigma) = f_{\Sigma_j}^B|_{A^{\Sigma_j}}(\sigma)$ for all $\sigma \in A^{\Sigma_j}$. Because we assumed A is closed under $f_{\Sigma_j}^B$, this ensures that $f_{\Sigma_j}^A(\sigma)$ is always in A . Therefore, A_{Σ} is a generalized sigma subalgebra of B_{Σ} . $\square$

Example 3.2.4. (A Generalized Sigma Subalgebra with Uncountable Arity) We demonstrate the concept of a generalized sigma subalgebra using the algebra with uncountable arity from Example 3.1.5. Let this "larger" algebra be denoted by B_{Σ} .

The components of $\mathcal{B}_\Sigma$ are:

- The **base set** is $B = \{a, b, c\}$.
- The **operations** are $f_{[0,1]}^B$ and $f_{[2,3]}^B$ defined as:

$$f_{[0,1]}^B(\sigma) = \begin{cases} a & \text{if } \sigma(1) = a \\ b & \text{otherwise.} \end{cases}$$

$$f_{[2,3]}^B(\sigma) = \begin{cases} b & \text{if } \sigma(2) = b \\ c & \text{otherwise.} \end{cases}$$

Thus, the algebra is $\mathcal{B}_\Sigma = (\{a, b, c\}; (f_{[0,1]}^B, f_{[2,3]}^B))$.

Now, we propose a candidate for a subalgebra, $\mathcal{A}_\Sigma$, constructed on the carrier set $A = \{b, c\}$. To formally verify that $\mathcal{A}_\Sigma$ is a generalized sigma subalgebra of $\mathcal{B}_\Sigma$, we must check the three conditions outlined in Definition 3.2.1.

- Type: We intend to define the operations for $\mathcal{A}_\Sigma$ as the restrictions of the operations from $\mathcal{B}_\Sigma$. By this construction, $\mathcal{A}_\Sigma$ will have the same index sets and thus the same type $\tau = (|\mathbb{R}|, |\mathbb{R}|)$ as $\mathcal{B}_\Sigma$. This condition is satisfied by our construction.
- Subset: The proposed carrier set is $A = \{b, c\}$. It is clear that $A \subseteq B$, since $\{b, c\} \subseteq \{a, b, c\}$. This condition is satisfied.
- Restriction of Operations. The third condition requires that for each $j \in \Omega$, the operation $f_{\Sigma_j}^A$ is the restriction of $f_{\Sigma_j}^B$ to A^{Σ_j} . For this restriction to be a well-defined operation on A , its codomain must be A . That is, for any input $\sigma \in A^{\Sigma_j}$, the output $f_{\Sigma_j}^B(\sigma)$ must be an element of A . This is precisely the closure property, which now we will show that.

- (1) Closure under $f_{[0,1]}^B$: Let $\sigma \in A^{\Sigma_1}$, meaning $\sigma : [0, 1] \rightarrow \{b, c\}$.

Consequently, $\sigma(1)$ can only be b or c ; it can never be a . According to the definition of the operation, the result $f_{[0,1]}^B(\sigma)$ is a only if $\sigma(1) = a$. Since this case is impossible for any $\sigma \in A^{\Sigma_1}$, the result always falls into the "otherwise" case, yielding the element b . Since $b \in A$, the set A is closed under $f_{[0,1]}^B$.

- (2) Closure under $f_{[2,3]}^B$: Let $\sigma \in A^{\Sigma_2}$, meaning $\sigma : [2, 3] \rightarrow \{b, c\}$. We consider the possible values of $\sigma(2)$.

- If $\sigma(2) = b$, the operation yields b , which is in A .

- If $\sigma(2) = c$, this falls into the "otherwise" case, and the operation yields c , which is in A .

In all possible cases, the output is an element of A . Thus, A is also closed under $f_{[2,3]}^B$.

Since the closure property holds for all fundamental operations, the restricted operations are well-defined on A , and Condition (iii) is satisfied.

Since all three conditions from Definition 3.2.1 have been verified, we conclude that $\mathcal{A}_\Sigma = (\{b, c\}; (f_{[0,1]}^B|_A, f_{[2,3]}^B|_A))$ is a well-defined proper generalized sigma subalgebra of $\mathcal{B}_\Sigma$, and we may write $\mathcal{A}_\Sigma \preccurlyeq \mathcal{B}_\Sigma$.

Corollary 3.2.5. Let $\mathcal{A}_\Sigma, \mathcal{B}_\Sigma$ and $\mathcal{C}_\Sigma$ be generalized sigma algebra of type $\tau := (|\Sigma_j|_{j \in \Omega} \cdot)$. Then:

- (i) If $(\mathcal{A}_\Sigma \preccurlyeq \mathcal{B}_\Sigma)$ and $(\mathcal{B}_\Sigma \preccurlyeq \mathcal{C}_\Sigma)$, then $\mathcal{A}_\Sigma \preccurlyeq \mathcal{C}_\Sigma$.
- (ii) If $(A \subseteq B \subseteq C)$ and $(\mathcal{A}_\Sigma \preccurlyeq \mathcal{C}_\Sigma)$ and $(\mathcal{B}_\Sigma \preccurlyeq \mathcal{C}_\Sigma)$, then $\mathcal{A}_\Sigma \preccurlyeq \mathcal{B}_\Sigma$.

Proof. (i) Assume $(\mathcal{A}_\Sigma \preccurlyeq \mathcal{B}_\Sigma)$ and $(\mathcal{B}_\Sigma \preccurlyeq \mathcal{C}_\Sigma)$. This means:

- (i) $A \subseteq B$ and $B \subseteq C$.
- (ii) $f_{\Sigma_j}^A(\sigma) = f_{\Sigma_j}^B|_{A^{\Sigma_j}}(\sigma)$ for all $\sigma \in A^{\Sigma_j}$ and all $j \in \Omega$. and $f_{\Sigma_j}^B(\sigma) = f_{\Sigma_j}^C|_{B^{\Sigma_j}}(\sigma)$ for all $\sigma \in B^{\Sigma_j}$ and all $j \in \Omega$.

We will show that $\mathcal{A}_\Sigma \preccurlyeq \mathcal{C}_\Sigma$. To do this, we need to show:

- (1.) $A \subseteq C$: This follows directly from the given inclusions: $A \subseteq B \subseteq C$.
- (2.) $f_{\Sigma_j}^A(\sigma) = f_{\Sigma_j}^C|_A(\sigma)$ for all $\sigma \in A^{\Sigma_j}$ and all $j \in \Omega$.
Since $\sigma \in A^{\Sigma_j}$, and $A \subseteq B$, we have $\sigma \in B^{\Sigma_j}$.

Therefore, using the given equality, we have:

$$\begin{aligned} f_{\Sigma_j}^A(\sigma) &= f_{\Sigma_j}^B|_A(\sigma) \quad (\text{Since } \mathcal{A}_\Sigma \preccurlyeq \mathcal{B}_\Sigma) \\ f_{\Sigma_j}^B(\sigma) &= f_{\Sigma_j}^C|_B(\sigma) \quad (\text{Since } \mathcal{B}_\Sigma \preccurlyeq \mathcal{C}_\Sigma) \\ \Rightarrow f_{\Sigma_j}^B|_{A^{\Sigma_j}}(\sigma) &= f_{\Sigma_j}^C|_{A^{\Sigma_j}}(\sigma). \end{aligned}$$

Hence, we get $f_{\Sigma_j}^A(\sigma) = f_{\Sigma_j}^C|_A(\sigma)$. Thus, $\mathcal{A}_\Sigma \preccurlyeq \mathcal{C}_\Sigma$, proving (i).

- (ii) Assume $(A \subseteq B \subseteq C)$ and $(\mathcal{A}_\Sigma \preccurlyeq \mathcal{C}_\Sigma)$ and $(\mathcal{B}_\Sigma \preccurlyeq \mathcal{C}_\Sigma)$. This implies:

$$A \subseteq B \subseteq C, \quad f_{\Sigma_j}^A(\sigma) = f_{\Sigma_j}^C|_A(\sigma), \quad f_{\Sigma_j}^B(\sigma) = f_{\Sigma_j}^C|_B(\sigma).$$

for all $\sigma \in A^{\Sigma_j}$ and all $j \in \Omega$.
need to show:

We need to show that $\mathcal{A}_\Sigma \preccurlyeq \mathcal{B}_\Sigma$, meaning we

(1) $A \subseteq B$. This is given in the assumptions.

(2) $f_{\Sigma_j}^A(\sigma) = f_{\Sigma_j}^B|_A(\sigma)$ for all $\sigma \in A^{\Sigma_j}$ and all $j \in \Omega$. Since $\sigma \in A^{\Sigma_j}$ and $A \subseteq C$, we have $\sigma \in C^{\Sigma_j}$.

Therefore, using the given equalities:

$$\begin{aligned} f_{\Sigma_j}^A(\sigma) &= f_{\Sigma_j}^C|_A(\sigma) \quad (\text{Since } \mathcal{A}_\Sigma \preceq \mathcal{C}_\Sigma) \\ f_{\Sigma_j}^B(\sigma) &= f_{\Sigma_j}^C|_B(\sigma) \quad (\text{Since } \mathcal{B}_\Sigma \preceq \mathcal{C}_\Sigma) \\ \Rightarrow f_{\Sigma_j}^B|_{A^{\Sigma_j}}(\sigma) &= f_{\Sigma_j}^C|_{A^{\Sigma_j}}(\sigma). \end{aligned}$$

Hence, we get $f_{\Sigma_j}^A(\sigma) = f_{\Sigma_j}^B|_A(\sigma)$. Thus, $\mathcal{A}_\Sigma \preceq \mathcal{B}_\Sigma$, proving (ii). $\square$

The next corollary we will show that the non-empty intersection of a non-empty family of generalized sigma subalgebra is a generalized sigma subalgebra of its whole generalized sigma algebra, i.e. suppose that $\mathcal{B}_\Sigma := \left(B; \left(f_{\Sigma_j}^B \right)_{j \in \Omega} \right)$ be a generalized sigma algebra of type $\tau := \left(|\Sigma_j|_{j \in \Omega} \right)$ and $\{\mathcal{A}_{k,\Sigma} \mid k \in \Gamma\}$ is a family of subalgebras of $\mathcal{B}_\Sigma$. We define a set $\mathbf{A} := \bigcap_{k \in \Gamma} A_k$ and operations $f_{\Sigma_j}^{\mathbf{A}} = f_{\Sigma_j}^B|_{\mathbf{A}^{\Sigma_j}}$.

We will verify that when the intersection set $\mathbf{A} := \bigcap_{k \in \Gamma} A_k$ is non-empty, the algebra

$\mathcal{A}_{k,\Sigma} := \left(\mathbf{A}; \left(f_{\Sigma_j}^{\mathbf{A}} \right)_{j \in \Omega} \right)$ indeed an algebra of type $\tau := \left(|\Sigma_j|_{j \in \Omega} \right)$, to be called the *intersection* of the algebras $\mathcal{A}_\Sigma$ and denoted by $\mathcal{A}_{k,\Sigma} := \bigcap_{k \in \Gamma} \mathcal{A}_{k,\Sigma}$.

Corollary 3.2.6. Let $\mathcal{B}_\Sigma := \left(B; \left(f_{\Sigma_j}^B \right)_{j \in \Omega} \right)$ be a generalized sigma algebra of type τ . The non-empty intersection of any non-empty family of its generalized sigma subalgebras, $\{\mathcal{A}_{k,\Sigma} \mid k \in \Gamma\}$, is also a generalized sigma subalgebra of $\mathcal{B}_\Sigma$.

Proof. Let $\mathcal{B}_\Sigma = (B; (f_{\Sigma_j}^B)_{j \in \Omega})$ be a generalized sigma algebra, and let $\{\mathcal{A}_{k,\Sigma} \mid k \in \Gamma\}$ be a non-empty family of its subalgebras.

Let $\mathcal{A}_{k,\Sigma}$ denote the intersection algebra, defined on the carrier set $\mathbf{A} := \bigcap_{k \in \Gamma} A_k$. We aim to prove that $\mathcal{A}_{k,\Sigma}$ is a generalized sigma subalgebra of $\mathcal{B}_\Sigma$. We will use the Generalized Subalgebra Criterion (Lemma 3.2.3) to do so. This requires us to verify two conditions.

1. Verifying the Subset Condition ($\mathbf{A} \subseteq B$):

By assumption, for each $k \in \Gamma$, $\mathcal{A}_{k,\Sigma}$ is a subalgebra of $\mathcal{B}_\Sigma$. By definition of a subalgebra, this implies that its carrier set A_k is a subset of B for all $k \in \Gamma$.

Since every set A_k in the family is a subset of B , their intersection must also be a subset of B . Therefore,

$$\mathbf{A} := \bigcap_{k \in \Gamma} A_k \subseteq B$$

This condition is satisfied.

2. Verifying Closure Under Operations:

We must show that the set $\mathbf{A}$ is closed under every fundamental operation $f_{\Sigma_j}^B$ of the algebra $\mathcal{B}_\Sigma$.

Let $j \in \Omega$ be an arbitrary operation index, and let σ be an arbitrary element from the domain $\mathbf{A}^{\Sigma_j}$. We need to show that the result $f_{\Sigma_j}^B(\sigma)$ is an element of $\mathbf{A}$.

First, by the definition of the intersection set $\mathbf{A}$, an element σ being in $\mathbf{A}^{\Sigma_j}$ means that for any $\beta \in \Sigma_j$, the value $\sigma(\beta)$ is in $\mathbf{A}$. Because $\mathbf{A} \subseteq A_k$ for every $k \in \Gamma$, this implies that $\sigma(\beta)$ is also in A_k for every k . Therefore, the function σ is an element of the domain $(A_k)^{\Sigma_j}$ for all $k \in \Gamma$.

Now, we use the fact that each $\mathcal{A}_{k,\Sigma}$ is a subalgebra. According to the Subalgebra Criterion, this means each carrier set A_k is closed under the operation $f_{\Sigma_j}^B$. Since we have established that $\sigma \in (A_k)^{\Sigma_j}$, it follows that the result of the operation must be in A_k :

$$f_{\Sigma_j}^B(\sigma) \in A_k$$

This statement is true for every $k \in \Gamma$.

If the element $f_{\Sigma_j}^B(\sigma)$ belongs to every set A_k in the family (i.e., it is in A_k for all $k \in \Gamma$), then by the definition of set intersection, it must belong to their intersection, $\mathbf{A}$.

$$f_{\Sigma_j}^B(\sigma) \in \bigcap_{k \in \Gamma} A_k = \mathbf{A}$$

Since we have shown that for an arbitrary operation $f_{\Sigma_j}^B$ and an arbitrary input $\sigma \in \mathbf{A}^{\Sigma_j}$, the result $f_{\Sigma_j}^B(\sigma)$ is in $\mathbf{A}$, we have proven that the set $\mathbf{A}$ is closed under all fundamental operations.

Conclusion:

Since the carrier set $\mathbf{A}$ is a subset of B and is closed under all fundamental operations of $\mathcal{B}_\Sigma$, we conclude by the Subalgebra Criterion that the intersection algebra $\mathcal{A}_{k,\Sigma}$ is indeed a generalized sigma subalgebra of $\mathcal{B}_\Sigma$. $\square$

CHAPTER 4

HOMOMORPHISMS, CONGRUENCES, AND ISOMORPHISMS

4.1 Generalized Homomorphisms

A fundamental pursuit in any algebraic theory is not merely to study individual structures in isolation, but to understand the rich web of relationships that connect them. The primary tools for this exploration are structure-preserving maps, known as generalized homomorphisms [3, 6]. These maps allow us to compare different algebras, identify shared properties, and understand how one structure can be mapped into another.

However, generalized homomorphisms are more than just external comparisons; they reveal profound internal structures within their domain algebra. Every generalized homomorphism naturally gives rise to an equivalence relation on its domain and its kernel which partitions the set into classes of elements that are indistinguishable from the perspective of a target algebra. Crucially, this kernel is not just any partition; it possesses a special compatibility with the algebraic operations, making it a congruence relation [3, 17].

This observation is the cornerstone of a celebrated generalized homomorphism theorems, a central pillar of universal algebra [3, 6, 17]. These theorems establish a deep and beautiful duality: the external view of an algebra (its generalized homomorphic images) is completely determined, up to isomorphism, by its internal structure (its congruence relations). Specifically, any generalized homomorphic image of an algebra $\mathcal{A}$ is generalized isomorphic to a quotient algebra $\mathcal{A}/\theta$, where θ is the congruence induced by the generalized homomorphism.

This chapter extends this fundamental trilogy of concepts generalized homomorphisms, generalized congruences, and generalized isomorphisms to the setting of generalized sigma algebras. Our primary objective is to demonstrate that this powerful interplay, which forms the bedrock of classical universal algebra, holds with equal significance in this more general framework. We will begin by formally defining generalized homomorphisms for generalized sigma algebras and establishing how they preserve the generalized operations. We will then show how every such generalized homomorphism induces a generalized congruence relation, paving the way to construct the corresponding quotient algebra.

The culmination of this chapter will be the formulation and proof of the generalized homomorphism Theorems for generalized sigma algebras, thereby solidifying the foundational link between the external comparisons and internal structures of these complex algebraic objects.

Definition 4.1.1. (Generalized Homomorphisms.)

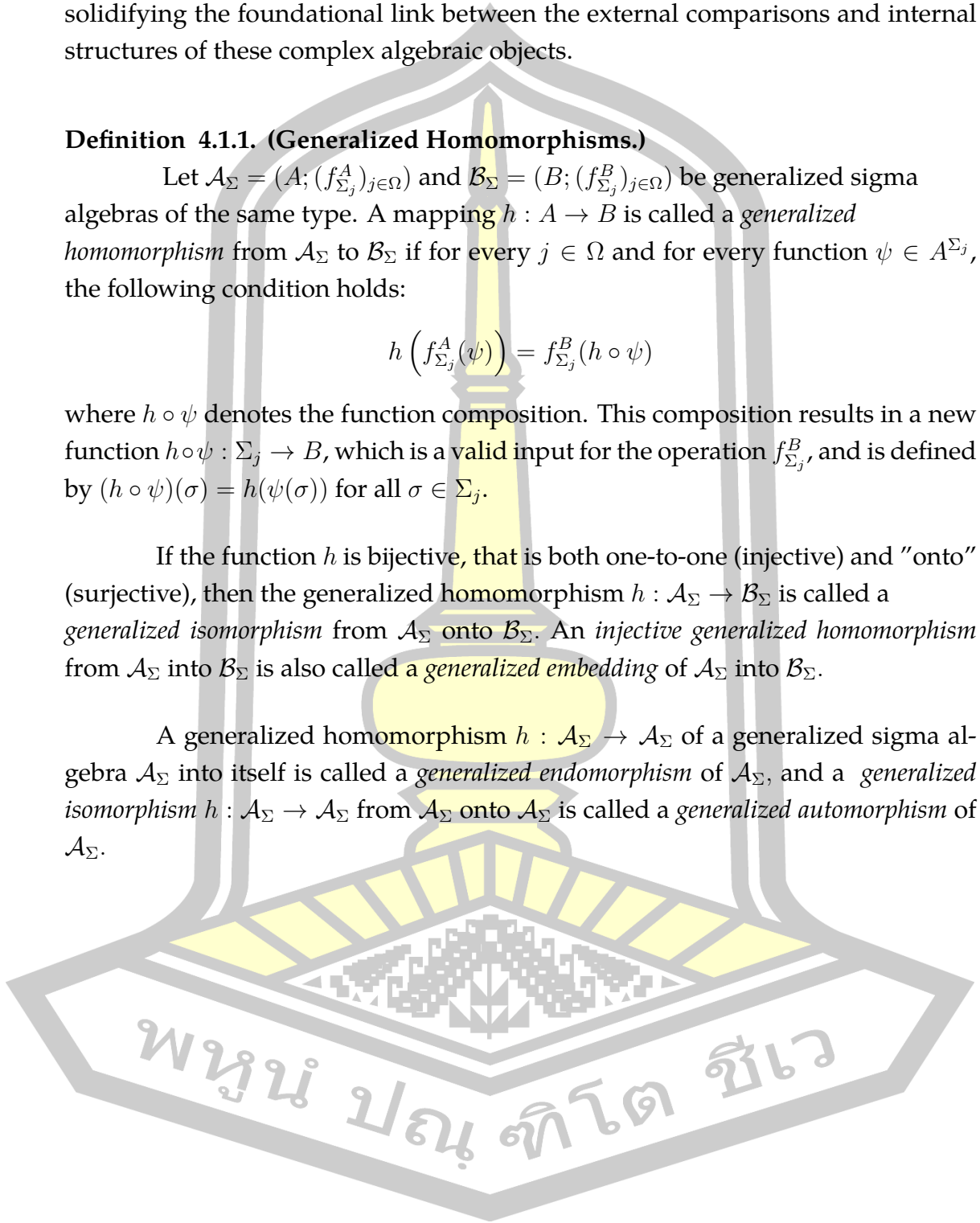
Let $\mathcal{A}_\Sigma = (A; (f_{\Sigma_j}^A)_{j \in \Omega})$ and $\mathcal{B}_\Sigma = (B; (f_{\Sigma_j}^B)_{j \in \Omega})$ be generalized sigma algebras of the same type. A mapping $h : A \rightarrow B$ is called a *generalized homomorphism* from $\mathcal{A}_\Sigma$ to $\mathcal{B}_\Sigma$ if for every $j \in \Omega$ and for every function $\psi \in A^{\Sigma_j}$, the following condition holds:

$$h \left(f_{\Sigma_j}^A(\psi) \right) = f_{\Sigma_j}^B(h \circ \psi)$$

where $h \circ \psi$ denotes the function composition. This composition results in a new function $h \circ \psi : \Sigma_j \rightarrow B$, which is a valid input for the operation $f_{\Sigma_j}^B$, and is defined by $(h \circ \psi)(\sigma) = h(\psi(\sigma))$ for all $\sigma \in \Sigma_j$.

If the function h is bijective, that is both one-to-one (injective) and "onto" (surjective), then the generalized homomorphism $h : \mathcal{A}_\Sigma \rightarrow \mathcal{B}_\Sigma$ is called a *generalized isomorphism* from $\mathcal{A}_\Sigma$ onto $\mathcal{B}_\Sigma$. An *injective generalized homomorphism* from $\mathcal{A}_\Sigma$ into $\mathcal{B}_\Sigma$ is also called a *generalized embedding* of $\mathcal{A}_\Sigma$ into $\mathcal{B}_\Sigma$.

A generalized homomorphism $h : \mathcal{A}_\Sigma \rightarrow \mathcal{A}_\Sigma$ of a generalized sigma algebra $\mathcal{A}_\Sigma$ into itself is called a *generalized endomorphism* of $\mathcal{A}_\Sigma$, and a *generalized isomorphism* $h : \mathcal{A}_\Sigma \rightarrow \mathcal{A}_\Sigma$ from $\mathcal{A}_\Sigma$ onto $\mathcal{A}_\Sigma$ is called a *generalized automorphism* of $\mathcal{A}_\Sigma$.



Example 4.1.2. (Examples of Generalized Homomorphisms and Generalized Isomorphisms.)

To provide concrete examples for the definitions of generalized homomorphism, generalized isomorphism, generalized embedding, and generalized automorphism, we will revisit and expand upon the algebras defined in example 3.1.5.

Let us recall the two generalized sigma algebras, $\mathcal{A}_\Sigma$ and $\mathcal{B}_\Sigma$. Let the universe for both be $A = \{a, b, c\}$. Set for Operations is $\Omega = \{1, 2\}$. Let $\Sigma_1 = [0, 1]$ and $\Sigma_2 = [2, 3]$. be arities. Then we have the fundamental operations for algebra $\mathcal{A}_\Sigma = (A; (f_{\Sigma_1}^A, f_{\Sigma_2}^A))$ are:

$f_{\Sigma_1}^A : A^{[0,1]} \rightarrow A$ where $f_{\Sigma_1}^A(\psi) = \psi(1)$ (Projection onto coordinate 1),
and $f_{\Sigma_2}^A : A^{[2,3]} \rightarrow A$ is defined by

$$f_{\Sigma_2}^A(\psi) = \begin{cases} b & \text{if } \psi(2) = a \\ a & \text{if } \psi(2) = b \\ c & \text{Otherwise.} \end{cases}$$

Note: This operation is closed on $\{a, b, c\}$. Hence the fundamental operations for algebra $\mathcal{B}_\Sigma = (A; (g_{\Sigma_1}^B, g_{\Sigma_2}^B))$ are:

- $g_{\Sigma_1}^B : A^{[0,1]} \rightarrow A$ is defined by $g_{\Sigma_1}^B(\psi) = \psi(0)$ (Projection onto coordinate 0).
- $g_{\Sigma_2}^B : A^{[2,3]} \rightarrow A$ is defined by $g_{\Sigma_2}^B(\psi) = \psi(3)$ (Projection onto coordinate 3).

Case 1: The mapping is not a generalized homomorphism.

Let us consider the identity map $h : A \rightarrow A$ and test if it is a homomorphism from $\mathcal{A}_\Sigma$ to $\mathcal{B}_\Sigma$. The condition is $h(f_{\Sigma_j}^A(\psi)) = g_{\Sigma_j}^B(h \circ \psi)$.

- For $j = 1$: The condition becomes $h(f_{\Sigma_1}^A(\psi)) = g_{\Sigma_1}^B(h \circ \psi)$. The LHS is $h(\psi(1)) = \psi(1)$. The RHS is $g_{\Sigma_1}^B(\psi) = \psi(0)$. The requirement $\psi(1) = \psi(0)$ for all functions $\psi : [0, 1] \rightarrow A$ is not met. For instance, a function with $\psi(0) = a$ and $\psi(1) = b$ violates the condition.

Therefore, the identity map is not a generalized homomorphism from $\mathcal{A}_\Sigma$ to $\mathcal{B}_\Sigma$.

Case 2: A generalized automorphism of an algebra.

Now, let's consider a generalized homomorphism from $\mathcal{A}_\Sigma$ to itself.

Let $id_A : A \rightarrow A$ be the identity map. We check if id_A is a generalized homomorphism from $\mathcal{A}_\Sigma$ to $\mathcal{A}_\Sigma$. The condition is $id_A(f_{\Sigma_j}^A(\psi)) = f_{\Sigma_j}^A(id_A \circ \psi)$.

- The LHS is $id_A(f_{\Sigma_j}^A(\psi)) = f_{\Sigma_j}^A(\psi)$.
- The RHS is $f_{\Sigma_j}^A(id_A \circ \psi) = f_{\Sigma_j}^A(\psi)$, since $id_A \circ \psi = \psi$.

Since LHS = RHS for all operations, the identity map is a generalized homomorphism.

- A generalized homomorphism from an algebra to itself, like id_A , is called a generalized endomorphism.
- Since id_A is bijective, it is also a generalized isomorphism.
- A generalized isomorphism from an algebra onto itself is a generalized automorphism.

Therefore, the identity map is a generalized automorphism of $\mathcal{A}_\Sigma$.

Case 3: A generalized embedding

To illustrate a generalized embedding, let's define a subalgebra of $\mathcal{A}_\Sigma$.

- Let the subset be $C = \{a, b\}$. This subset is closed under the operations of $\mathcal{A}_\Sigma$:
 - For $f_{\Sigma_1}^A$, if $\psi : [0, 1] \rightarrow C$, then the output $\psi(1)$ is in C .
 - For $f_{\Sigma_2}^A$, if $\psi : [2, 3] \rightarrow C$, the input $\psi(2.5)$ can only be a or b , so the output is b or a , which is in C .
- Thus, we can define a subalgebra $\mathcal{C}_\Sigma = (C; (f_{\Sigma_1}^A|_C, f_{\Sigma_2}^A|_C))$, where the operations are the restrictions of those in $\mathcal{A}_\Sigma$.

Now, consider the inclusion map $k : C \rightarrow A$ defined by $k(x) = x$. Let's check if k is a homomorphism from $\mathcal{C}_\Sigma$ to $\mathcal{A}_\Sigma$.

- The condition is $k(f_{\Sigma_j}^C(\psi)) = f_{\Sigma_j}^A(k \circ \psi)$ for any $\psi : \Sigma_j \rightarrow C$.
- The LHS is $k(f_{\Sigma_j}^C(\psi)) = f_{\Sigma_j}^C(\psi)$.
- The RHS is $f_{\Sigma_j}^A(k \circ \psi)$. Since k is the inclusion map, $k \circ \psi$ is just ψ considered as a map into A . Because $f_{\Sigma_j}^C$ is the restriction of $f_{\Sigma_j}^A$, the outputs are identical. The condition holds.

Therefore, the map k is a generalized homomorphism. Since k is injective (one-to-one), it is a generalized embedding of $\mathcal{C}_\Sigma$ into $\mathcal{A}_\Sigma$. It is not a generalized isomorphism because it is not surjective (the element $c \in A$ is not in the image of k).

Theorem 4.1.3. Let $\mathcal{A}_\Sigma, \mathcal{B}_\Sigma$, and $\mathcal{C}_\Sigma$ be generalized sigma algebras of the same type.

1. The identity map on A , denoted $id_A : A \rightarrow A$, is a generalized homomorphism from $\mathcal{A}_\Sigma$ to $\mathcal{A}_\Sigma$ (i.e., a generalized endomorphism).
2. If $h : \mathcal{A}_\Sigma \rightarrow \mathcal{B}_\Sigma$, and $k : \mathcal{B}_\Sigma \rightarrow \mathcal{C}_\Sigma$, are generalized homomorphisms, then their composition $k \circ h : \mathcal{A}_\Sigma \rightarrow \mathcal{C}_\Sigma$, is also a generalized homomorphism.

Proof. Let $j \in \Omega$ be an arbitrary index for an operation. We will show that satisfying :

1. Identity Map: Let $id_A : A \rightarrow A$ be the identity map. To show it is a generalized homomorphism, we must verify that for any function $\psi \in A^{\Sigma_j}$, the following condition holds:

$$id_A(f_{\Sigma_j}^A(\psi)) = f_{\Sigma_j}^A(id_A \circ \psi).$$

We analyze both sides of the equation:

- LHS : By the definition of the identity map, $id_A(x) = x$ for any element $x \in A$. Since $f_{\Sigma_j}^A(\psi)$ is an element of A , we have:

$$id_A(f_{\Sigma_j}^A(\psi)) = f_{\Sigma_j}^A(\psi).$$

- RHS : Consider the input function for the operation, which is the composition $id_A \circ \psi$. For any $\sigma \in \Sigma_j$, we have

$$(id_A \circ \psi)(\sigma) = id_A(\psi(\sigma)) = \psi(\sigma).$$

This means that the function $id_A \circ \psi$ is identical to the function ψ .

Therefore:

$$f_{\Sigma_j}^A(id_A \circ \psi) = f_{\Sigma_j}^A(\psi).$$

Since LHS = RHS, the identity map is a generalized homomorphism. This is the property we demonstrated in Case 2 of Example 4.1.2.

Hence if we given that $h : \mathcal{A}_\Sigma \rightarrow \mathcal{B}_\Sigma$ and $k : \mathcal{B}_\Sigma \rightarrow \mathcal{C}_\Sigma$, be generalized homomorphisms. We must show that for any function $\psi \in A^{\Sigma_j}$, the map $k \circ h$ satisfies the generalized homomorphism condition:

$$(k \circ h)(f_{\Sigma_j}^A(\psi)) = f_{\Sigma_j}^C((k \circ h) \circ \psi).$$

We start from the left-hand side and transform it step-by-step using the properties of composition and generalized homomorphisms. We have

$$(k \circ h)(f_{\Sigma_j}^A(\psi)) = k \left(h(f_{\Sigma_j}^A(\psi)) \right) \quad (\text{by definition of composition 'o'})$$

$$\begin{aligned}
&= k \left(f_{\Sigma_j}^B(h \circ \psi) \right) \quad (\text{since } h \text{ is a generalized homomorphism}) \\
&= f_{\Sigma_j}^C(k \circ (h \circ \psi)) \quad (\text{since } k \text{ is a generalized homomorphism}) \\
&= f_{\Sigma_j}^C((k \circ h) \circ \psi). \quad (\text{by associativity of composition 'o'})
\end{aligned}$$

The final expression is the right-hand side of the condition we needed to prove. Thus, $k \circ h$ is a generalized homomorphism. □

These properties are fundamental to the study of generalized homomorphisms and generalized isomorphisms in algebraic structures, including generalized sigma algebras. They are crucial for understanding how these mappings behave and how they contribute to the analysis and classification of algebraic systems.

Corollary 4.1.4. (The Monoid of Generalized Endomorphisms)

Let $\mathcal{A}_\Sigma$ be a generalized sigma algebra. The set of all generalized endomorphisms of $\mathcal{A}_\Sigma$, denoted $\text{End}(\mathcal{A}_\Sigma)$, forms a **monoid** under the operation of function composition ($\circ$).

Proof. To prove that $(\text{End}(\mathcal{A}_\Sigma), \circ)$ forms a monoid, we must verify three properties: closure, associativity, and the existence of an identity element.

To Show closure property: Let $h, k \in \text{End}(\mathcal{A}_\Sigma)$. This means that $h : \mathcal{A}_\Sigma \rightarrow \mathcal{A}_\Sigma$ and $k : \mathcal{A}_\Sigma \rightarrow \mathcal{A}_\Sigma$ are generalized homomorphisms. By Theorem 4.1.3(2), the composition $h \circ k$ is also a generalized homomorphism from $\mathcal{A}_\Sigma$ to $\mathcal{A}_\Sigma$. Therefore, $h \circ k \in \text{End}(\mathcal{A}_\Sigma)$, and the set is closed under composition.

To show associativity: The operation of function composition is inherently associative. For any three generalized endomorphisms $f, g, h \in \text{End}(\mathcal{A}_\Sigma)$, it is a fundamental property of functions that $(f \circ g) \circ h = f \circ (g \circ h)$. Thus, the operation is associative.

To show it has identity element: By Theorem 4.1.3(1), the identity map $id_A : A \rightarrow A$ is a generalized homomorphism, and thus $id_A \in \text{End}(\mathcal{A}_\Sigma)$. For any $h \in \text{End}(\mathcal{A}_\Sigma)$, we have $h \circ id_A = h$ and $id_A \circ h = h$. Therefore, id_A serves as the identity element for the operation of composition.

Since the set of generalized endomorphisms is closed under an associative operation and contains an identity element, $(\text{End}(\mathcal{A}_\Sigma), \circ)$ is a monoid. $\square$

Remark. (The Category of Generalized Sigma Algebras)

The properties established in Theorem 4.1.3 are precisely the axioms required to demonstrate that generalized sigma algebras form a category.

Specifically, for a fixed type $\tau = (|\Sigma_j|)_{j \in \Omega}$, we can define the category GenAlg_τ as follows:

1. Objects: The objects of the category are all generalized sigma algebras of type τ .
2. Morphisms: For any two objects $\mathcal{A}_\Sigma$ and $\mathcal{B}_\Sigma$, the morphisms from $\mathcal{A}_\Sigma$ to $\mathcal{B}_\Sigma$ are the generalized homomorphisms $h : \mathcal{A}_\Sigma \rightarrow \mathcal{B}_\Sigma$.

Theorem 4.1.3 confirms that this construction satisfies the two fundamental axioms of a category:

1. Existence of Identity Morphisms: For any object $\mathcal{A}_\Sigma$, there exists an identity morphism, which is the identity map $id_A : \mathcal{A}_\Sigma \rightarrow \mathcal{A}_\Sigma$ (from part 1 of the theorem).
2. Composition of Morphisms: The composition of any two morphisms

$$h : \mathcal{A}_\Sigma \rightarrow \mathcal{B}_\Sigma$$

and $k : \mathcal{B}_\Sigma \rightarrow \mathcal{C}_\Sigma$, is also a morphism in the category, $k \circ h : \mathcal{A}_\Sigma \rightarrow \mathcal{C}_\Sigma$. (from part 2 of the theorem). The associativity of this composition is inherited from the standard associativity of function composition.

Therefore, the class of all generalized sigma algebras of a given type, together with their generalized homomorphisms, constitutes a well-defined mathematical category.

Theorem 4.1.5. (Generalized Homomorphisms and Subalgebras.)

Let $h : \mathcal{A}_\Sigma \rightarrow \mathcal{B}_\Sigma$, be a generalized homomorphism between two generalized sigma algebras of the same type.

- (i) If $\mathcal{A}' = (A', (f_{\Sigma_j}^A|_{A'})_{j \in \Omega})$, is a subalgebra of $\mathcal{A}_\Sigma$, then its image, the algebra $\mathcal{B}' = (h(A'), (f_{\Sigma_j}^B|_{h(A')})_{j \in \Omega})$, is a subalgebra of $\mathcal{B}_\Sigma$.

- (ii) If $B' = (B', (f_{\Sigma_j}^B|_{B'})_{j \in \Omega})$, is a subalgebra of $\mathcal{B}_\Sigma$, then its preimage, the set $A' = h^{-1}(B')$, is the carrier set of a subalgebra of $\mathcal{A}_\Sigma$.

Proof. Let $j \in \Omega$ be an arbitrary index for an operation. To prove that a set is the carrier of a subalgebra, we must show that it is non-empty (which is trivial if the source algebra is non-empty) and that it is closed under all fundamental operations.

- (i) The image of a subalgebra is a subalgebra. Let $\mathcal{A}'$ be a subalgebra with carrier set $A' \subseteq A$. We need to show that the image set $h(A')$ is closed under every operation $f_{\Sigma_j}^B$ of $\mathcal{B}_\Sigma$.

Let $\varphi : \Sigma_j \rightarrow h(A')$ be an arbitrary function whose range is contained in the image set. Our goal is to prove that the result $f_{\Sigma_j}^B(\varphi)$ is also an element of $h(A')$.

By the definition of the image set $h(A')$, for every $\sigma \in \Sigma_j$, the element $\varphi(\sigma) \in h(A')$ implies that there exists some element $a_\sigma \in A'$ such that $h(a_\sigma) = \varphi(\sigma)$. We can use these elements to define a new function $\psi : \Sigma_j \rightarrow A'$ such that $\psi(\sigma) = a_\sigma$, for each $\sigma \in \Sigma_j$.

By this construction, we have $h(\psi(\sigma)) = h(a_\sigma) = \varphi(\sigma)$, for all $\sigma \in \Sigma_j$. This means the function φ is identical to the function composition $h \circ \psi$.

Now we can evaluate the result of the operation:

$$\begin{aligned} f_{\Sigma_j}^B(\varphi) &= f_{\Sigma_j}^B(h \circ \psi) && \text{(substituting } \varphi = h \circ \psi) \\ &= h(f_{\Sigma_j}^A(\psi)). && \text{(since } h \text{ is a generalized homomorphism)} \end{aligned}$$

Let's examine the argument of h in the final expression: $f_{\Sigma_j}^A(\psi)$. Since the function ψ maps into the set A' , and $\mathcal{A}'$ is a subalgebra of $\mathcal{A}_\Sigma$, its carrier set A' must be closed under the operation $f_{\Sigma_j}^A$. Therefore, the result $f_{\Sigma_j}^A(\psi)$ is an element of A' .

Finally, since $f_{\Sigma_j}^A(\psi) \in A'$, applying the map h to this element gives a result, $h(f_{\Sigma_j}^A(\psi))$, which by definition must be in the image set $h(A')$.

Thus, we have shown that $f_{\Sigma_j}^B(\varphi) \in h(A')$. This holds for any operation $j \in \Omega$, so $h(A')$ is the carrier set of a subalgebra of $\mathcal{B}_\Sigma$.

- (ii) The preimage of a subalgebra is a subalgebra. Let B' be a subalgebra with carrier set $B' \subseteq B$. Let $A' = h^{-1}(B')$ be the preimage of this set. We need to show that A' is closed under every operation $f_{\Sigma_j}^A$ of $\mathcal{A}_\Sigma$.

Let $\psi : \Sigma_j \rightarrow A'$ be an arbitrary function whose range is contained in the preimage set A' . Our goal is to prove that the result $f_{\Sigma_j}^A(\psi)$ is also an element of $A' = h^{-1}(B')$.

By the definition of a preimage, showing that $f_{\Sigma_j}^A(\psi) \in h^{-1}(B')$ is equivalent to showing that $h(f_{\Sigma_j}^A(\psi)) \in B'$. This is our new target.

Using the generalized homomorphism property of h , we can transform the expression:

$$h(f_{\Sigma_j}^A(\psi)) = f_{\Sigma_j}^B(h \circ \psi).$$

Now, let's analyze the input function on the right-hand side, $h \circ \psi$. For any $\sigma \in \Sigma_j$, we have $\psi(\sigma) \in A' = h^{-1}(B')$. By definition of preimage, this means $h(\psi(\sigma)) \in B'$. Since this is true for all $\sigma \in \Sigma_j$, the range of the composite function $h \circ \psi$ is entirely contained within the set B' .

So, the expression $f_{\Sigma_j}^B(h \circ \psi)$ represents the application of the operation $f_{\Sigma_j}^B$ to a function that maps into B' . Since B' is a subalgebra, its carrier set B' is closed under this operation. Therefore, the result must be an element of B' :

$$f_{\Sigma_j}^B(h \circ \psi) \in B'.$$

Combining our steps, we have shown that $h(f_{\Sigma_j}^A(\psi)) \in B'$. This directly implies that $f_{\Sigma_j}^A(\psi) \in h^{-1}(B') = A'$.

This holds for any operation $j \in \Omega$, so $h^{-1}(B')$ is the carrier set of a subalgebra of $\mathcal{A}_\Sigma$. □

A central theme in algebra is understanding how a homomorphism $h : \mathcal{A}_\Sigma \rightarrow \mathcal{B}_\Sigma$ naturally partitions its domain, $\mathcal{A}_\Sigma$. The mapping h groups elements of the set A together based on whether they are sent to the same element in B . This induced partition is not arbitrary; it is a highly structured equivalence relation that is compatible with the algebra's operations, a concept we will later define as a congruence relation.

The binary relation that defines this partition is called the kernel of the homomorphism. To properly build our framework, we first establish a key property of generalized isomorphisms, which will be essential for developing the Generalized Homomorphism Theorems.

Lemma 4.1.6. (Inverse of a Generalized Isomorphism)

If $h : \mathcal{A}_\Sigma \rightarrow \mathcal{B}_\Sigma$ is a generalized isomorphism between two generalized sigma algebras, then its inverse map $h^{-1} : B \rightarrow A$ is also an isomorphism from $\mathcal{B}_\Sigma$ to $\mathcal{A}_\Sigma$.

Proof. Since h is a generalized isomorphism, it is a bijective generalized homomorphism. It is a standard result from set theory that the inverse of a bijective function is also a bijection. Therefore, we only need to prove that the map h^{-1} is a generalized homomorphism.

Let $j \in \Omega$ be an arbitrary index for an operation. We must show that for any function $\varphi \in B^{\Sigma_j}$, the following condition holds:

$$h^{-1}(f_{\Sigma_j}^B(\varphi)) = f_{\Sigma_j}^A(h^{-1} \circ \varphi).$$

Since h is a bijection (and therefore surjective), for any function $\varphi : \Sigma_j \rightarrow B$, there exists a unique function $\psi : \Sigma_j \rightarrow A$ such that $h \circ \psi = \varphi$. Applying the inverse map h^{-1} to this function equation gives us $\psi = h^{-1} \circ \varphi$.

Now, we can prove the generalized homomorphism condition for h^{-1} by starting with the left-hand side:

$$\begin{aligned} h^{-1}(f_{\Sigma_j}^B(\varphi)) &= h^{-1}(f_{\Sigma_j}^B(h \circ \psi)) && \text{(substituting } \varphi = h \circ \psi) \\ &= h^{-1}(h(f_{\Sigma_j}^A(\psi))) && \text{(since } h \text{ is a generalized homomorphism)} \\ &= (h^{-1} \circ h)(f_{\Sigma_j}^A(\psi)) && \text{(by definition of composition)} \\ &= id_A(f_{\Sigma_j}^A(\psi)) && \text{(since } h^{-1} \circ h = id_A) \\ &= f_{\Sigma_j}^A(\psi) && \text{(by definition of identity map)} \\ &= f_{\Sigma_j}^A(h^{-1} \circ \varphi). && \text{(substituting back } \psi = h^{-1} \circ \varphi) \end{aligned}$$

The final expression is the right-hand side of the condition we needed to prove. Thus, h^{-1} is a generalized homomorphism. Since it is also a bijection, h^{-1} is a generalised isomorphism. $\square$

With the properties of generalized homomorphisms established, we can now formally define the kernel.

Definition 4.1.7. (Kernel of a Generalized Homomorphism.)

Let $\mathcal{A}_\Sigma = (A; (f_{\Sigma_j}^A)_{j \in \Omega})$ and $\mathcal{B}_\Sigma = (B; (f_{\Sigma_j}^B)_{j \in \Omega})$ be generalized sigma algebras of the same type, and let $h : A \rightarrow B$ be a generalized homomorphism. The **kernel of a generalized homomorphism** h , denoted by $\ker h$, is the binary relation on the base set A defined as follows:

$$\ker h := \{(a_1, a_2) \in A \times A \mid h(a_1) = h(a_2)\}.$$

Remark. The kernel is an equivalence relation on the carrier set A . It partitions A into equivalence classes, where each class consists of all the elements that are mapped by h to a single element in the image set $h(A)$. As we will see later, this relation is not just an equivalence relation but a **congruence relation**, which is a cornerstone of the Homomorphism Theorems.

4.1.1 The Generalized Homomorphism Theorem

The concept of a generalized homomorphism, its kernel, and the resulting quotient algebra all culminate in one of the most fundamental results in universal algebra. Given any generalized homomorphism $h : \mathcal{A}_\Sigma \rightarrow \mathcal{B}_\Sigma$, we have constructed a canonical generalized homomorphic image of $\mathcal{A}_\Sigma$, namely quotient algebra $\mathcal{A}_\Sigma / \ker h$. A natural and crucial question arises: What is the relationship between this internally constructed quotient algebra and the external generalized homomorphic image, $h(\mathcal{A}_\Sigma)$?

The answer lies in the first Isomorphism theorem (often called the fundamental generalized homomorphism theorem), which states that these two algebras are, in essence, the same they are isomorphic. This theorem provides a canonical factorization for any homomorphism.

Theorem 4.1.8. (First Generalized Isomorphism Theorem for Generalized Sigma Algebras.)

Let $h : \mathcal{A}_\Sigma \rightarrow \mathcal{B}_\Sigma$ be a generalized homomorphism. Then the quotient algebra $\mathcal{A}_\Sigma / \ker h$ is isomorphic to the image algebra $h(\mathcal{A}_\Sigma)$. Specifically, there exists a unique isomorphism

$$f : \mathcal{A}_\Sigma / \ker h \rightarrow h(\mathcal{A}_\Sigma).$$

such that the following diagram commutes, i.e., $h|_{\text{im}(h)} = f \circ \text{nat}(\ker h)$, where $\text{nat}(\ker h)$ is the natural generalized homomorphism and $h|_{\text{im}(h)}$ is the generalized homomorphism h with its codomain restricted to its image.

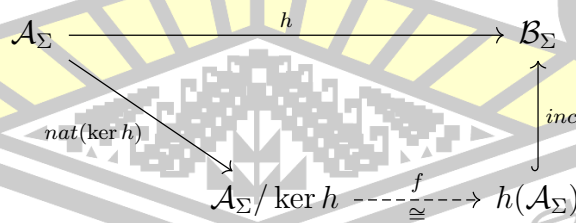


Figure 4.1: The First Generalized Isomorphism Diagram

Proof. Let $\theta = \ker h$. To prove the theorem, we must construct the map f and show that it is satisfying (1) well-defined, (2) a generalized homomorphism, (3) injective, and (4) surjective.

To check well-defined: We defined the map f .

Let an element of the quotient algebra $\mathcal{A}_\Sigma / \theta$ be denoted by an equivalence

class $[a]_\theta$ for some $a \in A$. We define the map $f : \mathcal{A}_\Sigma/\theta \rightarrow h(A)$, as follows:

$$f([a]_\theta) := h(a).$$

We must show that the definition of f does not depend on the choice of representative for the equivalence class. Assume $[a]_\theta = [b]_\theta$ for $a, b \in A$.

$$\begin{aligned} [a]_\theta = [b]_\theta &\iff (a, b) \in \theta && \text{(by definition of equivalence classes)} \\ &\iff (a, b) \in \ker h && \text{(since } \theta = \ker h) \\ &\iff h(a) = h(b) && \text{(by definition of kernel)} \\ &\iff f([a]_\theta) = f([b]_\theta). && \text{(by our definition of } f) \end{aligned}$$

The forward implication shows that if we pick different representatives for the same class, the output of f is the same. Thus, f is well-defined.

To check f is a generalized homomorphism : Let $j \in \Omega$ and let Ψ be a function from Σ_j into the quotient algebra $\mathcal{A}_\Sigma/\theta$. The operation in the quotient algebra is defined as $f_{\Sigma_j}^{A/\theta}(\Psi) = [f_{\Sigma_j}^A(\psi)]_\theta$, where ψ is any function such that $\text{nat}(\theta) \circ \psi = \Psi$. We need to show $f(f_{\Sigma_j}^{A/\theta}(\Psi)) = f_{\Sigma_j}^{h(A)}(f \circ \Psi)$.

$$\begin{aligned} f(f_{\Sigma_j}^{A/\theta}(\Psi)) &= f([f_{\Sigma_j}^A(\psi)]_\theta) && \text{(def. of operation in quotient algebra)} \\ &= h(f_{\Sigma_j}^A(\psi)) && \text{(def. of } f) \\ &= f_{\Sigma_j}^B(h \circ \psi). && \text{(since } h \text{ is a generalized homomorphism)} \end{aligned}$$

Note that $(f \circ \Psi)(\sigma) = f(\Psi(\sigma)) = f([\psi(\sigma)]_\theta) = h(\psi(\sigma))$. Thus, the function $f \circ \Psi$ is identical to $h \circ \psi$. The operation $f_{\Sigma_j}^{h(A)}$ is just the restriction of $f_{\Sigma_j}^B$ to the subalgebra $h(\mathcal{A}_\Sigma)$. So, $f_{\Sigma_j}^B(h \circ \psi) = f_{\Sigma_j}^{h(A)}(f \circ \Psi)$. The condition holds.

To check f is injective (one-to-one) : Assume $f([a]_\theta) = f([b]_\theta)$. From the chain of equivalences in step 2, the reverse implication shows:

$$f([a]_\theta) = f([b]_\theta) \implies h(a) = h(b) \implies (a, b) \in \ker h \implies [a]_\theta = [b]_\theta.$$

Thus, f is injective.

To check f is surjective (onto) : Let y be an arbitrary element in the codomain $h(A)$. By definition of the image set, there must exist some $a \in A$ such that $h(a) = y$. The element $[a]_\theta$ is in the domain $\mathcal{A}_\Sigma/\theta$, and by our definition of f , we have $f([a]_\theta) = h(a) = y$. Thus, for any element y in the codomain, we have found a preimage. So, f is surjective.

Therefore, f is a well-defined, bijective generalized homomorphism, it is a generalized isomorphism. The uniqueness follows from the construction. This completes the proof. $\square$

Corollary 4.1.9. (Generalized Homomorphic Image Theorem.)

If a generalized homomorphism $h : \mathcal{A}_\Sigma \rightarrow \mathcal{B}_\Sigma$ is surjective, then $\mathcal{A}_\Sigma / \ker h \cong \mathcal{B}_\Sigma$.

Proof. This is a direct consequence of Theorem 4.1.8. If h is surjective, then its image is the entire codomain, i.e., $h(\mathcal{A}_\Sigma) = \mathcal{B}_\Sigma$. The theorem then directly states that $\mathcal{A}_\Sigma / \ker h \cong \mathcal{B}_\Sigma$. $\square$

4.2 Congruence Relations

Definition 4.2.1. (Compatibility in Generalized Sigma Algebras)

Let A be a set, let θ be an equivalence relation on A , and let $f_{\Sigma_j}^A : A^{\Sigma_j} \rightarrow A$ be a generalized operation.

We say that $f_{\Sigma_j}^A$ is **compatible** with θ if for all functions $\alpha, \beta \in A^{\Sigma_j}$, the following implication holds:

$$\text{If } (\alpha(\sigma), \beta(\sigma)) \in \theta \text{ for all } \sigma \in \Sigma_j, \text{ then } (f_{\Sigma_j}^A(\alpha), f_{\Sigma_j}^A(\beta)) \in \theta.$$

Remark 4.2.2. (From Congruences to General Relation Preservation)

The concept of compatibility, which is central to the definition of a congruence, is a specific instance of a more general and powerful idea in universal algebra: **Relation preservation**. While congruences deal exclusively with equivalence relations, the notion of an operation being compatible with a relation can be extended to any arbitrary relation.

This broader perspective allows us to characterize algebraic structures not only by their congruences but also by the entire set of relations they preserve. To illustrate this, let us examine the compatibility of an operation with a non-equivalence relation, specifically the subset relation ($\subseteq$).

Example 4.2.3. (Compatibility on a Classical Algebra)

Let's use a classical algebra from example 3.1.1 to illustrate the definition of compatibility.

First we will defined the classical algebra as follows : Let base set $A = \{a, b, c\}$ and a single binary operation $g : A^2 \rightarrow A$ defined as the first projection

$$g(u, v) := u \quad \text{for all } u, v \in A.$$

$\mathcal{A} = (A; g)$. Next we will define an equivalence relation on A . Let θ be an equivalence relation on A that partitions the set into two classes: $\{a\}$ and $\{b, c\}$.

$$\theta := \Delta_A \cup \{(b, c), (c, b)\}.$$

where $\Delta_A = \{(a, a), (b, b), (c, c)\}$ is the identity relation. We will verification that θ is an equivalence relations, we must first verify that the defined relation θ is indeed an equivalence relation. First, let's explicitly list the elements of θ . Given $A = \{a, b, c\}$:

$$\theta = \Delta_A \cup \{(b, c), (c, b)\} = \{(a, a), (b, b), (c, c), (b, c), (c, b)\}.$$

We now check the three required properties.

Reflexivity: We must show that for every element $x \in A$, the pair (x, x) is in θ .

- For $x = a$: $(a, a) \in \theta$ (since it is in Δ_A).
- For $x = b$: $(b, b) \in \theta$ (since it is in Δ_A).
- For $x = c$: $(c, c) \in \theta$ (since it is in Δ_A).

The relation is reflexive.

Symmetry: We must show that if $(x, y) \in \theta$, then (y, x) must also be in θ . We only need to check the non-diagonal pairs. Let $(b, c) \in \theta$. We have $(c, b) \in \theta$.

The other non-diagonal pair is (c, b) , whose symmetric counterpart (b, c) is in the set. Hence the relation is symmetric.

Transitivity: We must show that if $(x, y) \in \theta$ and $(y, z) \in \theta$, then (x, z) must also be in θ . We check all non-trivial. Let: $(b, c) \in \theta$ and $(c, b) \in \theta$. This requires that $(b, b) \in \theta$. and let $(c, b) \in \theta$ and $(b, c) \in \theta$. This requires that $(c, c) \in \theta$.

Other cases, like (b, b) and (b, c) , require $(b, c) \in \theta$, which is trivially true. So we have checked all necessary cases. The relation is transitive.

Since the relation θ is reflexive, symmetric, and transitive, it is a valid equivalence relation on the set A . The equivalence classes (the partition) induced by θ are indeed $\{a\}$ and $\{b, c\}$. There fore θ is an equivalnce relation.

Last check for compatibility: To determine if θ is a congruence relation on $\mathcal{A}$, we must check if the operation g is compatible with θ . According to the definition of compatibility for a binary operation, we must verify the following implication:

If $(u_1, u_2) \in \theta$ and $(v_1, v_2) \in \theta$, then does it follow that $(g(u_1, v_1), g(u_2, v_2)) \in \theta$?

- Let's assume the premise is true: $(u_1, u_2) \in \theta$ and $(v_1, v_2) \in \theta$.
- Now we evaluate the conclusion. By the definition of our operation g :
 - $g(u_1, v_1) = u_1$
 - $g(u_2, v_2) = u_2$

This is clear that $(u_1, u_2) \in \theta$.

The implication holds true.

Since the operation g is compatible with the equivalence relation θ , we conclude that θ is a congruence relation on the algebra $\mathcal{A}$.

Example 4.2.4. (A Congruence on an Algebra with Uncountable Arity)

To illustrate the concept of a congruence relation, we will use the generalized sigma algebra defined in Example 3.1.5.

Let us recall the structure of our algebra $\mathcal{A}_\Sigma$: The base set is $A = \{a, b, c\}$. The index set for the operations is $\Omega = \{1, 2\}$. The arity sets are the real intervals $\Sigma_1 = [0, 1]$ and $\Sigma_2 = [2, 3]$. The fundamental operations are defined as:

- $f_{[0,1]}^A(\psi) = \begin{cases} a & \text{if } \psi(1) = a \\ b & \text{otherwise.} \end{cases}$
- $f_{[2,3]}^A(\psi) = \begin{cases} b & \text{if } \psi(2) = b \\ c & \text{otherwise.} \end{cases}$

The resulting algebra is $\mathcal{A}_\Sigma = (\{a, b, c\}; (f_{[0,1]}^A, f_{[2,3]}^A))$.

We will define an equivalence relation : Let us define an equivalence relation θ on the base set A . For this example, we will consider the universal relation, denoted by ∇_A . This relations relates every element to every other element:

$$\theta = \nabla_A = A \times A = \{a, b, c\} \times \{a, b, c\}.$$

It is clear that ∇_A is an equivalence relation.

We will show that is compatibility: To determine if $\theta = \nabla_A$ is a congruence relation, we must verify that it is compatible with all fundamental operations of $\mathcal{A}_\Sigma$.

Fore $f_{[0,1]}^A$: Let $\alpha, \beta : [0, 1] \rightarrow A$ be two functions. Assume that for all $\sigma \in [0, 1]$, $(\alpha(\sigma), \beta(\sigma)) \in \theta$. Since $\theta = \nabla_A$, this premise is always true for any pair of functions.

We need to check if $(f_{[0,1]}^A(\alpha), f_{[0,1]}^A(\beta)) \in \theta$.

The outputs $f_{[0,1]}^A(\alpha)$ and $f_{[0,1]}^A(\beta)$ can only be a or b . Since all pairs from $\{a, b\}$ are in ∇_A , the pair of outputs will always be in θ . Thus, the condition holds.

For $f_{[2,3]}^A$: Let $\alpha, \beta : [2, 3] \rightarrow A$ be two functions. Assume that for all $\sigma \in [2, 3]$, $(\alpha(\sigma), \beta(\sigma)) \in \theta$. This is always true since $\theta = \nabla_A$.

We need to check if $(f_{[2,3]}^A(\alpha), f_{[2,3]}^A(\beta)) \in \theta$.

The outputs $f_{[2,3]}^A(\alpha)$ and $f_{[2,3]}^A(\beta)$ can only be b or c . Since all pairs from $\{b, c\}$ are in ∇_A , the pair of outputs will always be in θ . Thus, the condition holds.

Since the compatibility condition holds for all fundamental operations, we conclude that the universal relation $\theta = \nabla_A$ is a congruence relation on this generalized sigma algebra.

Definition 4.2.5. (Generalized Congruence Relations)

Let $\mathcal{A}_\Sigma := \left(A; \left(f_{\Sigma_j}^A \right)_{j \in \Omega} \right)$ be a generalized sigma algebra of type $\tau := (|\Sigma_j|_{j \in \Omega})$. An equivalence relation θ on A is called a *generalized congruence relations* on $\mathcal{A}_\Sigma$ if all the fundamental operations $f_{\Sigma_j}^A$ are compatible with θ . We denote by $Con\mathcal{A}_\Sigma$ the set of all generalized congruence relations of the generalized sigma algebra $\mathcal{A}_\Sigma$.

For every generalized sigma algebra $\mathcal{A}_\Sigma = \left(A; \left(f_{\Sigma_j}^A \right)_{j \in \Omega} \right)$, the trivial equivalence relations:

$$\Delta_A := \{(a, a) \mid a \in A\} \text{ and } \nabla_A = A \times A.$$

are congruence relations.

An algebra which has no congruence relations except Δ_A and ∇_A is called **simple**.

Theorem 4.2.6. (Intersection of Congruences)

Let θ_1 and θ_2 be two generalized congruence relations on a generalized sigma algebra $\mathcal{A}_\Sigma = (A; (f_{\Sigma_j}^A)_{j \in \Omega})$. Then their set-theoretic intersection, $\theta_1 \cap \theta_2$, is also a generalized congruence relation on $\mathcal{A}_\Sigma$.

Proof. Let $\theta = \theta_1 \cap \theta_2$. To prove that θ is a congruence relation, we must verify two conditions: that θ is an equivalence relation on A , and that it is compatible with all fundamental operations of $\mathcal{A}_\Sigma$.

To show that θ is an Equivalence Relation:

It is a standard result from set theory that the intersection of two (or any

number of) equivalence relations on a set is also an equivalence relation on that set. Since both θ_1 and θ_2 are equivalence relations on A , their intersection θ is also an equivalence relation on A . (This covers reflexivity, symmetry, and transitivity automatically).

To show that θ is Compatible with Operations:

Let $j \in \Omega$ be an arbitrary index, and let $f_{\Sigma_j}^A$ be the corresponding fundamental operation. We need to show that $f_{\Sigma_j}^A$ is compatible with θ .

Let $\psi : \Sigma_j \rightarrow A$ and $\varphi : \Sigma_j \rightarrow A$ be two functions. We start with the premise for compatibility, which is:

Assume that $(\psi(\sigma), \varphi(\sigma)) \in \theta$ for all $\sigma \in \Sigma_j$.

Our goal is to prove that this implies $(f_{\Sigma_j}^A(\psi), f_{\Sigma_j}^A(\varphi)) \in \theta$.

From our assumption and the definition of set intersection, the premise $(\psi(\sigma), \varphi(\sigma)) \in \theta_1 \cap \theta_2$ for all $\sigma \in \Sigma_j$ is equivalent to two separate statements:

- (a) $(\psi(\sigma), \varphi(\sigma)) \in \theta_1$ for all $\sigma \in \Sigma_j$.
- (b) $(\psi(\sigma), \varphi(\sigma)) \in \theta_2$ for all $\sigma \in \Sigma_j$.

Now, we use the fact that θ_1 and θ_2 are generalized congruence relations. So we have :

- (i) Since θ_1 is a generalized congruence, it is compatible with $f_{\Sigma_j}^A$. From statement (a), it follows directly that:

$$(f_{\Sigma_j}^A(\psi), f_{\Sigma_j}^A(\varphi)) \in \theta_1$$

- (ii) Similarly, since θ_2 is a generalized congruence, it is compatible with $f_{\Sigma_j}^A$. From statement (b), it follows directly that:

$$(f_{\Sigma_j}^A(\psi), f_{\Sigma_j}^A(\varphi)) \in \theta_2$$

Since the pair $(f_{\Sigma_j}^A(\psi), f_{\Sigma_j}^A(\varphi))$ belongs to both the relation θ_1 and the relation θ_2 , it must, by definition, belong to their intersection, $\theta_1 \cap \theta_2$.

$$(f_{\Sigma_j}^A(\psi), f_{\Sigma_j}^A(\varphi)) \in \theta_1 \cap \theta_2 = \theta.$$

This concludes the check for compatibility. Since this holds for any arbitrary operation $f_{\Sigma_j}^A$, the relation θ is compatible with all operations of $\mathcal{A}_\Sigma$.

Since $\theta_1 \cap \theta_2$ is both an equivalence relation on A and is compatible with all operations of $\mathcal{A}_\Sigma$, it is a congruence relation on $\mathcal{A}_\Sigma$. $\square$

Remark. This theorem can be extended by induction to show that the intersection of any finite number of congruence relations is a congruence. Furthermore, it can be shown that the intersection of *any* (possibly infinite) collection of congruence relations is also a congruence. This property is crucial as it guarantees that the set of all congruences on an algebra, $\text{Con}(\mathcal{A}_\Sigma)$, forms a complete lattice under the subset ordering.

Example 4.2.7. (Intersection of Congruences on a Non-trivial Algebra)

To provide a concrete illustration for Theorem 4.2.6, we will use the generalized sigma algebra previously discussed and define two distinct congruence relations on it.

Let our algebra be $\mathcal{A}_\Sigma = (A; (f_1^A, f_2^A))$, where, base set: $A = \{a, b, c\}$.
Operations: $f_1^A : A^{[0,1]} \rightarrow A$ is defined by

$$f_1^A(\psi) = \begin{cases} a & \text{if } \psi(1) = a \\ b & \text{otherwise.} \end{cases}$$

$f_2^A : A^{[2,3]} \rightarrow A$ is defined by

$$f_2^A(\psi) = \begin{cases} b & \text{if } \psi(2) = b \\ c & \text{otherwise.} \end{cases}$$

Defined two generalized congruence relations : θ_1 (The Universal Relation):

Let $\theta_1 = \nabla_A = A \times A$. As established in Example 4.2.4, the universal relation is always a generalized congruence relations on any algebra.

Generalized congruence relations θ_2 :

Let's define a non-trivial equivalence relation θ_2 that partitions A into the classes $\{a\}$ and $\{b, c\}$.

$$\theta_2 := \{(a, a), (b, b), (c, c), (b, c), (c, b)\}$$

We must first verify that θ_2 is satisfying conditions of congruence relation.

(i) Equivalence Relation: θ_2 is clearly reflexive, symmetric, and transitive.

(ii) Compatibility with f_1^A : Assume $(\psi(\sigma), \varphi(\sigma)) \in \theta_2$ for all $\sigma \in [0, 1]$. We need to show $(f_1^A(\psi), f_1^A(\varphi)) \in \theta_2$. The output of f_1^A is always an element of $\{a, b\}$. Let's check the possible output pairs: $(a, a), (a, b), (b, a), (b, b)$. None of these pairs relate an element from the class $\{a\}$ to an element from the class $\{b, c\}$.

In fact, any pair of outputs from f_1^A will *never* be (b, c) or (c, b) . Let's be more rigorous. If $f_1^A(\psi) \neq f_1^A(\varphi)$, it must be that one is a and the other is b . For this to happen, one input function (say ψ) must have $\psi(1) = a$, and the other (φ) must have $\varphi(1) \neq a$. But if $\varphi(1) = b$ or $\varphi(1) = c$, then $(\psi(1), \varphi(1)) = (a, b)$ or (a, c) , neither of which is in θ_2 . This contradicts our premise. Therefore, if the inputs are related by θ_2 at coordinate 1, the outputs must be identical. Thus, the operation is compatible.

- (iii) Compatibility with f_2^A : Assume $(\psi(\sigma), \varphi(\sigma)) \in \theta_2$ for all $\sigma \in [2, 3]$. We need to show $(f_2^A(\psi), f_2^A(\varphi)) \in \theta_2$. The output of f_2^A is always an element of $\{b, c\}$. Any pair of elements from this set, whether identical or not, is contained in θ_2 (e.g., $(b, b), (c, c), (b, c), (c, b) \in \theta_2$). So, the compatibility condition always holds.

Since θ_2 is an equivalence relation compatible with all operations, it is a generalized congruence relation.

To show the intersection:

$$\theta = \theta_1 \cap \theta_2$$

Now we find the intersection of these two generalized congruences.

$$\theta = \theta_1 \cap \theta_2 = \nabla_A \cap \theta_2$$

The intersection of the universal relation with any other relation is simply that other relation.

$$\theta = \theta_2 = \{(a, a), (b, b), (c, c), (b, c), (c, b)\}.$$

To check the Intersection is a generalized congruence: According to theorem 4.2.6, the resulting relation θ must also be a congruence. In our case, the intersection is θ_2 itself. Since we have already proven in Step 2 that θ_2 is a generalized congruence relation, the theorem holds for this example. The intersection of the two generalized congruences is indeed a congruence.

Theorem 4.2.8. (Arbitrary Intersection of Generalized Congruences)

Let $\mathcal{A}_\Sigma = (A; (f_{\Sigma_j}^A)_{j \in \Omega})$ be a generalized sigma algebra. Let $\{\theta_i\}_{i \in I}$ be any non-empty family of generalized congruence relations on $\mathcal{A}_\Sigma$, indexed by the set I . Then their intersection,

$$\theta := \bigcap_{i \in I} \theta_i.$$

is also a generalized congruence relation on $\mathcal{A}_\Sigma$.

Proof. To prove that θ is a generalized congruence relation, we must verify two conditions: that θ is an equivalence relation on A , and that it is compatible with all fundamental operations of $\mathcal{A}_\Sigma$.

To show that θ is an equivalence relation:

It is a standard result from set theory that the intersection of any non-empty family of equivalence relations on a set is also an equivalence relation on that set. Since each θ_i for $i \in I$ is an equivalence relation on A , their intersection θ is also an equivalence relation on A .

To show that θ is compatible with operations:

Let $j \in \Omega$ be an arbitrary index, and let $f_{\Sigma_j}^A$ be the corresponding fundamental operation. We need to show that $f_{\Sigma_j}^A$ is compatible with the intersection θ .

Let $\psi : \Sigma_j \rightarrow A$ and $\varphi : \Sigma_j \rightarrow A$ be two functions. We start with the premise for compatibility:

Assume that $(\psi(\sigma), \varphi(\sigma)) \in \theta$ for all $\sigma \in \Sigma_j$.

Our goal is to prove that this implies $(f_{\Sigma_j}^A(\psi), f_{\Sigma_j}^A(\varphi)) \in \theta$.

By the definition of an arbitrary intersection, our premise $(\psi(\sigma), \varphi(\sigma)) \in \bigcap_{i \in I} \theta_i$ means that the pair belongs to *every* relation in the family. That is, for any chosen index $i_0 \in I$, it must be true that:

$$(\psi(\sigma), \varphi(\sigma)) \in \theta_{i_0} \quad \text{for all } \sigma \in \Sigma_j.$$

Since each θ_{i_0} is a generalized congruence relation on $\mathcal{A}_\Sigma$, it must be compatible with the operation $f_{\Sigma_j}^A$. Therefore, the premise above directly implies that the results of an operation are also related by θ_{i_0} :

$$(f_{\Sigma_j}^A(\psi), f_{\Sigma_j}^A(\varphi)) \in \theta_{i_0}.$$

Now, let's review what we have shown. We picked an arbitrary index i_0 from the index set I , and we proved that the pair $(f_{\Sigma_j}^A(\psi), f_{\Sigma_j}^A(\varphi))$ belongs to the relation θ_{i_0} . Since our choice of i_0 was arbitrary, this conclusion must hold for all $i \in I$.

This means that the pair $(f_{\Sigma_j}^A(\psi), f_{\Sigma_j}^A(\varphi))$ belongs to every single relation θ_i in the family. By the definition of intersection, this implies that the pair must belong to their intersection:

$$(f_{\Sigma_j}^A(\psi), f_{\Sigma_j}^A(\varphi)) \in \bigcap_{i \in I} \theta_i = \theta.$$

Since $\bigcap_{i \in I} \theta_i$ is both an equivalence relation on A and is compatible with all operations of $\mathcal{A}_\Sigma$, it is a generalized congruence relation on $\mathcal{A}_\Sigma$. $\square$

Definition 4.2.9. (Quotient Algebra)

Let $\mathcal{A}_\Sigma = (A; (f_{\Sigma_j}^A)_{j \in \Omega})$, be a generalized sigma algebra, and let θ be a generalized congruence relation on $\mathcal{A}_\Sigma$. The **quotient algebra** of $\mathcal{A}_\Sigma$ by θ , denoted $\mathcal{A}_\Sigma/\theta$, is a new algebra of the same type, defined as follows:

The Carrier Set (Universe):

The carrier set of the quotient algebra is the set of all θ -equivalence classes of A , denoted by A/θ .

$$A/\theta := \{[a]_\theta \mid a \in A\}.$$

where $[a]_\theta = \{b \in A \mid (a, b) \in \theta\}$. is the equivalence class containing the element a .

The Fundamental Operations :

For each operation index $j \in \Omega$, we define a new operation $f_{\Sigma_j}^{A/\theta}$ on the set A/θ . The domain of this new operation is the set of functions from Σ_j to the new carrier set, i.e., $(A/\theta)^{\Sigma_j}$.

Let $\Psi : \Sigma_j \rightarrow A/\theta$ be an arbitrary function in this domain. Each value $\Psi(\sigma)$ is an equivalence class. The operation $f_{\Sigma_j}^{A/\theta}$ is defined by the following rule:

$$f_{\Sigma_j}^{A/\theta}(\Psi) := [f_{\Sigma_j}^A(\psi)]_\theta.$$

where $\psi : \Sigma_j \rightarrow A$ is any function chosen such that it represents Ψ on a component-wise basis. That is, for every $\sigma \in \Sigma_j$, the element $\psi(\sigma)$ is a representative of the class $\Psi(\sigma)$, i.e., $[\psi(\sigma)]_\theta = \Psi(\sigma)$.

Remark. (Well-Definedness of the Operations)

A crucial point for this definition is to ensure that the operation $f_{\Sigma_j}^{A/\theta}$ is **well-defined**. This means we must show that the resulting equivalence class $[f_{\Sigma_j}^A(\psi)]_\theta$ is the same, no matter which representative function ψ we choose for Ψ .

Let ψ and φ be two different functions that both represent Ψ . This means that for all $\sigma \in \Sigma_j$, we have $[\psi(\sigma)]_\theta = \Psi(\sigma)$. and $[\varphi(\sigma)]_\theta = \Psi(\sigma)$. This implies that $[\psi(\sigma)]_\theta = [\varphi(\sigma)]_\theta$, which is equivalent to $(\psi(\sigma), \varphi(\sigma)) \in \theta$ for all $\sigma \in \Sigma_j$.

This is precisely the premise for the compatibility property of a generalized congruence relation. Since θ is a generalized congruence, it is

compatible with $f_{\Sigma_j}^A$, so we can conclude:

$$(f_{\Sigma_j}^A(\psi), f_{\Sigma_j}^A(\varphi)) \in \theta.$$

By the definition of an equivalence class, this means:

$$[f_{\Sigma_j}^A(\psi)]_\theta = [f_{\Sigma_j}^A(\varphi)]_\theta.$$

This confirms that the output of the operation is independent of the choice of the representative function, and therefore the operation $f_{\Sigma_j}^{A/\theta}$ is well-defined. This is the very reason why the relation must be a generalized congruence and not just any equivalence relation.

Theorem 4.2.10. (The Natural Homomorphism)

Let $\mathcal{A}_\Sigma = (A; (f_{\Sigma_j}^A)_{j \in \Omega})$ be a generalized sigma algebra and let θ be a generalized congruence relation on it. The mapping $nat_\theta : A \rightarrow A/\theta$, defined by

$$nat_\theta(a) := [a]_\theta.$$

for all $a \in A$, is a surjective generalized homomorphism from $\mathcal{A}_\Sigma$ onto the quotient algebra $\mathcal{A}_\Sigma/\theta$. This map is called the **natural generalized homomorphism** (or canonical generalized homomorphism) associated with θ .

Proof. By its definition, the map nat_θ is surjective, as every equivalence class $[a]_\theta$ in the codomain A/θ has at least one preimage, namely the element a itself.

To prove that nat_θ is a generalized homomorphism, we must show that for any operation index $j \in \Omega$ and for any function $\psi : \Sigma_j \rightarrow A$, the following condition holds:

$$nat_\theta(f_{\Sigma_j}^A(\psi)) = f_{\Sigma_j}^{A/\theta}(nat_\theta \circ \psi).$$

We analyze both sides of the equation.

LHS : By the definition of the natural map, we apply nat_θ to the element $f_{\Sigma_j}^A(\psi)$:

$$nat_\theta(f_{\Sigma_j}^A(\psi)) = [f_{\Sigma_j}^A(\psi)]_\theta.$$

RHS : The input for the operation $f_{\Sigma_j}^{A/\theta}$ is the composite function

$$\Psi := nat_\theta \circ \psi.$$

Let's analyze this function Ψ . For any $\sigma \in \Sigma_j$, its output is

$$\Psi(\sigma) = nat_\theta(\psi(\sigma)) = [\psi(\sigma)]_\theta.$$

Now we apply the definition of the operation in the quotient algebra (from Definition 4.2.9) to the input function Ψ . The definition states that we can compute the result by choosing any representative function for Ψ . The function ψ itself is a perfect representative for Ψ , since $[\psi(\sigma)]_\theta = \Psi(\sigma)$. for all $\sigma \in \Sigma_j$.

Therefore, according to the definition of the quotient operation:

$$f_{\Sigma_j}^{A/\theta}(\Psi) = f_{\Sigma_j}^{A/\theta}(\text{nat}_\theta \circ \psi) := [f_{\Sigma_j}^A(\psi)]_\theta.$$

Since both the LHS and the RHS simplify to the same expression, $[f_{\Sigma_j}^A(\psi)]_\theta$, the homomorphism condition is satisfied. Thus, nat_θ is a surjective homomorphism. $\square$

Remark. A key consequence of this theorem is that the kernel of the natural homomorphism, $\ker(\text{nat}_\theta)$, is the congruence relation θ itself.

Proof of the Remark. Let $(a, b) \in A \times A$.

$$\begin{aligned} (a, b) \in \ker(\text{nat}_\theta) &\iff \text{nat}_\theta(a) = \text{nat}_\theta(b) && \text{(by definition of kernel)} \\ &\iff [a]_\theta = [b]_\theta && \text{(by definition of } \text{nat}_\theta) \\ &\iff (a, b) \in \theta. && \text{(by property of equivalence classes)} \end{aligned}$$

This confirms that $\ker(\text{nat}_\theta) = \theta$. This fact is essential for the proof of the Homomorphism Theorems. $\square$

Theorem 4.2.11. (Generalized Homomorphism Theorem)

Let $h : \mathcal{A}_\Sigma \rightarrow \mathcal{B}_\Sigma$. and $g : \mathcal{A}_\Sigma \rightarrow \mathcal{C}_\Sigma$. be generalized homomorphisms between generalized sigma algebras of the same type, and assume that g is surjective.

A generalized homomorphism $f : \mathcal{C}_\Sigma \rightarrow \mathcal{B}_\Sigma$. satisfying the condition $f \circ g = h$ exists if and only if $\ker g \subseteq \ker h$.

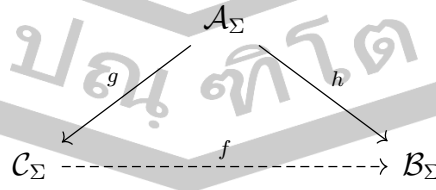


Figure 4.2: The Generalized Homomorphism Diagram

Furthermore, if such a generalized homomorphism f exists, it satisfies the following:

- (i) f is unique.
- (ii) f is injective if and only if $\ker g = \ker h$.
- (iii) f is surjective if and only if h is surjective.

Proof. ($\Rightarrow$) Existence of f implies $\ker g \subseteq \ker h$.

Assume a homomorphism $f : \mathcal{C}_\Sigma \rightarrow \mathcal{B}_\Sigma$ exists such that $f \circ g = h$. Let (a_1, a_2) be an arbitrary pair in $\ker g$.

$$\begin{aligned}
 (a_1, a_2) \in \ker g &\implies g(a_1) = g(a_2) && \text{(by definition of kernel)} \\
 &\implies f(g(a_1)) = f(g(a_2)) && \text{(applying the function } f) \\
 &\implies (f \circ g)(a_1) = (f \circ g)(a_2) && \text{(by definition of composition)} \\
 &\implies h(a_1) = h(a_2) && \text{(since } f \circ g = h) \\
 &\implies (a_1, a_2) \in \ker h. && \text{(by definition of kernel)}
 \end{aligned}$$

Since any pair in $\ker g$ is also in $\ker h$, we conclude that $\ker g \subseteq \ker h$.

($\Leftarrow$) $\ker g \subseteq \ker h$ implies Existence of f .

Assume $\ker g \subseteq \ker h$. We must construct the map f and show it is a well-defined homomorphism.

1. *Definition of the map f .* Let c be an arbitrary element of the set C . Since g is surjective, there exists at least one element $a \in A$ such that $g(a) = c$. We define the map $f : C \rightarrow B$ as follows:

$$f(c) := h(a) \quad \text{where } a \text{ is any element such that } g(a) = c.$$

2. *f is well-defined.* We must show that the definition of $f(c)$ is independent of the choice of the preimage a . Suppose $a_1, a_2 \in A$ are two elements such that $g(a_1) = c$ and $g(a_2) = c$.

$$\begin{aligned}
 g(a_1) = g(a_2) &\implies (a_1, a_2) \in \ker g && \text{(by definition of kernel)} \\
 &\implies (a_1, a_2) \in \ker h && \text{(by our assumption } \ker g \subseteq \ker h) \\
 &\implies h(a_1) = h(a_2). && \text{(by definition of kernel)}
 \end{aligned}$$

This shows that the value of $h(a)$ is the same for all possible choices of a . Thus, f is well-defined.

3. *f is a homomorphism.* Let $j \in \Omega$ and let $\Phi : \Sigma_j \rightarrow C$ be a function. We need to show $f(f_{\Sigma_j}^C(\Phi)) = f_{\Sigma_j}^B(f \circ \Phi)$. Since g is surjective, for the function Φ , there exists a representative function $\psi : \Sigma_j \rightarrow A$ such that $g \circ \psi = \Phi$. Now, let's evaluate the left-hand side (LHS): $f(f_{\Sigma_j}^C(\Phi)) = f(f_{\Sigma_j}^C(g \circ \psi)) = f(g(f_{\Sigma_j}^A(\psi)))$. By the definition of f , this becomes $h(f_{\Sigma_j}^A(\psi))$. For the right-hand side (RHS): We first

analyze the function $f \circ \Phi$. For any $\sigma \in \Sigma_j$, $(f \circ \Phi)(\sigma) = f(\Phi(\sigma)) = f(g(\psi(\sigma)))$. By definition of f , this is $h(\psi(\sigma))$. So, the function $f \circ \Phi$ is identical to $h \circ \psi$. The RHS is therefore $f_{\Sigma_j}^B(f \circ \Phi) = f_{\Sigma_j}^B(h \circ \psi)$. Since h is a generalized homomorphism, we know that $h(f_{\Sigma_j}^A(\psi)) = f_{\Sigma_j}^B(h \circ \psi)$. Since LHS = RHS, we conclude that f is a generalized homomorphism.

Last we proof of properties (i)-(iii).

(i) *Uniqueness*: Assume f_1, f_2 are two generalized homomorphisms such that $f_1 \circ g = h$ and $f_2 \circ g = h$. For any $c \in C$, since g is surjective, there exists an $a \in A$ with $g(a) = c$. Then $f_1(c) = f_1(g(a)) = h(a)$ and $f_2(c) = f_2(g(a)) = h(a)$. Thus $f_1(c) = f_2(c)$ for all $c \in C$, which means $f_1 = f_2$.

(ii) *Injectivity*: For the 'if and only if' statement, we prove both directions. $(\Rightarrow)$ Assume f is injective. Let $(a_1, a_2) \in \ker h$, so $h(a_1) = h(a_2)$. This means $f(g(a_1)) = f(g(a_2))$. Since f is injective, $g(a_1) = g(a_2)$, which means $(a_1, a_2) \in \ker g$. So $\ker h \subseteq \ker g$. Since we already know $\ker g \subseteq \ker h$, we have $\ker g = \ker h$.

$(\Leftarrow)$ Assume $\ker g = \ker h$. Let $c_1, c_2 \in C$ such that $f(c_1) = f(c_2)$. Let $a_1, a_2 \in A$ be preimages such that $g(a_1) = c_1$ and $g(a_2) = c_2$. Then $h(a_1) = f(c_1)$ and $h(a_2) = f(c_2)$. So $h(a_1) = h(a_2)$, which means $(a_1, a_2) \in \ker h = \ker g$. This implies $g(a_1) = g(a_2)$, so $c_1 = c_2$. Thus f is injective.

(iii) *Surjectivity*: $(\Rightarrow)$ Assume f is surjective. For any $b \in B$, there exists $c \in C$ such that $f(c) = b$. Since g is surjective, there exists $a \in A$ such that $g(a) = c$. Then $h(a) = f(g(a)) = f(c) = b$. This shows h is surjective. $(\Leftarrow)$ Assume h is surjective. For any $b \in B$, there exists $a \in A$ such that $h(a) = b$. Let $c = g(a)$. Then $c \in C$ and $f(c) = f(g(a)) = h(a) = b$. This shows f is surjective. $\square$

4.3 The Generalized Isomorphism Theorems

Isomorphisms are fundamental to algebra as they establish when two structures are essentially identical. This section explores the three classical Isomorphism Theorems, which are cornerstones of the subject, adapted to the framework of generalized sigma algebras. These theorems reveal deep structural relationships between subalgebras, quotient algebras, and generalized homomorphic images.

4.3.1 The First Generalized Isomorphism Theorem

We have already established the most critical generalized isomorphism theorem in the previous section on the Generalized Homomorphism Theorem.

We restate it here for completeness.

Theorem 4.3.1. (First Generalized Isomorphism Theorem)

If $h : \mathcal{A}_\Sigma \rightarrow \mathcal{B}_\Sigma$ is a generalized homomorphism, then the quotient algebra of the domain by a kernel is generalized isomorphic to the image of a generalized homomorphism.

$$\mathcal{A}_\Sigma / \ker h \cong h(\mathcal{A}_\Sigma).$$

4.3.2 The Second Generalized Isomorphism Theorem

The Second Generalized Isomorphism Theorem describes the structure of a quotient algebra formed by a subalgebra and a congruence. It reveals how a larger subalgebra "modded out" by a generalized congruence relates to a smaller subalgebra.

Let $\mathcal{A}_\Sigma$ be a generalized sigma algebra, S be a subalgebra of $\mathcal{A}_\Sigma$ with carrier set S , and θ be a generalized congruence on $\mathcal{A}_\Sigma$. We define the set $S \vee \theta$ as the set of elements in A that are related by θ to some element in S :

$$S \vee \theta := \{a \in A \mid \exists s \in S, (a, s) \in \theta\}.$$

It can be shown that $S \vee \theta$ is the carrier set of a subalgebra of $\mathcal{A}_\Sigma$, which we denote $S \vee \theta$.

Theorem 4.3.2. (Second Isomorphism Theorem)

Let $\mathcal{A}_\Sigma$ be a generalized sigma algebra, let S be a subalgebra of $\mathcal{A}_\Sigma$ with carrier set S , and let θ be a congruence on $\mathcal{A}_\Sigma$. Then:

- (i) The set $S \vee \theta := \{a \in A \mid \exists s \in S, (a, s) \in \theta\}$ is the carrier set of a subalgebra of $\mathcal{A}_\Sigma$, denoted $S \vee \theta$.
- (ii) The relation $\theta_S := \theta \cap (S \times S)$ is a congruence on the subalgebra S .
- (iii) The corresponding quotient algebras are isomorphic:

$$(S \vee \theta) / \theta \cong S / \theta_S.$$

Proof. The proofs for parts (i) and (ii) involve straightforward checks of the definitions of subalgebra and congruence, and are omitted here for brevity. We will focus on proving the isomorphism in part (iii).

The strategy is to cleverly define a surjective generalized homomorphism from S onto $(S \vee \theta) / \theta$ and then apply the First Generalized Isomorphism Theorem.

To define the map: Let's define a map $h : S \rightarrow (S \vee \theta)/\theta$ by the rule:

$$h(s) := [s]_\theta \quad \text{for all } s \in S.$$

This map takes an element from the subalgebra S and maps it to its equivalence class in the larger quotient algebra $(S \vee \theta)/\theta$.

To show h is a generalized homomorphism: We must show that h is a generalized homomorphism from the algebra S to $(S \vee \theta)/\theta$. Let $j \in \Omega$ and let $\psi : \Sigma_j \rightarrow S$ be a function.

$$\begin{aligned} h(f_{\Sigma_j}^S(\psi)) &= [f_{\Sigma_j}^S(\psi)]_\theta && \text{(by definition of } h) \\ &= [f_{\Sigma_j}^A(\psi)]_\theta. && \text{(since } f^S \text{ is the restriction of } f^A) \end{aligned}$$

Now, consider the right-hand side, $f_{\Sigma_j}^{(S \vee \theta)/\theta}(h \circ \psi)$. The function :

$$h \circ \psi : \Sigma_j \rightarrow (S \vee \theta)/\theta.$$

maps $\sigma \mapsto [\psi(\sigma)]_\theta$. By the definition of a quotient operation,

$$f_{\Sigma_j}^{(S \vee \theta)/\theta}(h \circ \psi) = [f_{\Sigma_j}^A(\psi)]_\theta.$$

Since LHS = RHS, h is a generalized homomorphism.

To show h is surjective: Let $[x]_\theta$ be an arbitrary element in the codomain $(S \vee \theta)/\theta$. By definition of the set $S \vee \theta$, this means there exists some element $s \in S$ such that $(x, s) \in \theta$. This implies that $[x]_\theta = [s]_\theta$. The element s is in the domain S of our map h . Then $h(s) = [s]_\theta = [x]_\theta$. We have found a preimage $s \in S$ for any element $[x]_\theta$ in the codomain. Thus, h is surjective.

To determine the kernel of h : We now find the kernel of this generalized homomorphism h . Let $s_1, s_2 \in S$.

$$\begin{aligned} (s_1, s_2) \in \ker h &\iff h(s_1) = h(s_2) && \text{(by definition of kernel)} \\ &\iff [s_1]_\theta = [s_2]_\theta && \text{(by definition of } h) \\ &\iff (s_1, s_2) \in \theta. && \text{(by property of equivalence classes)} \end{aligned}$$

Since s_1 and s_2 must both be in S , the condition is that $(s_1, s_2) \in \theta$ and

$$(s_1, s_2) \in S \times S.$$

This means $\ker h = \theta \cap (S \times S)$, which is precisely the definition of θ_S .

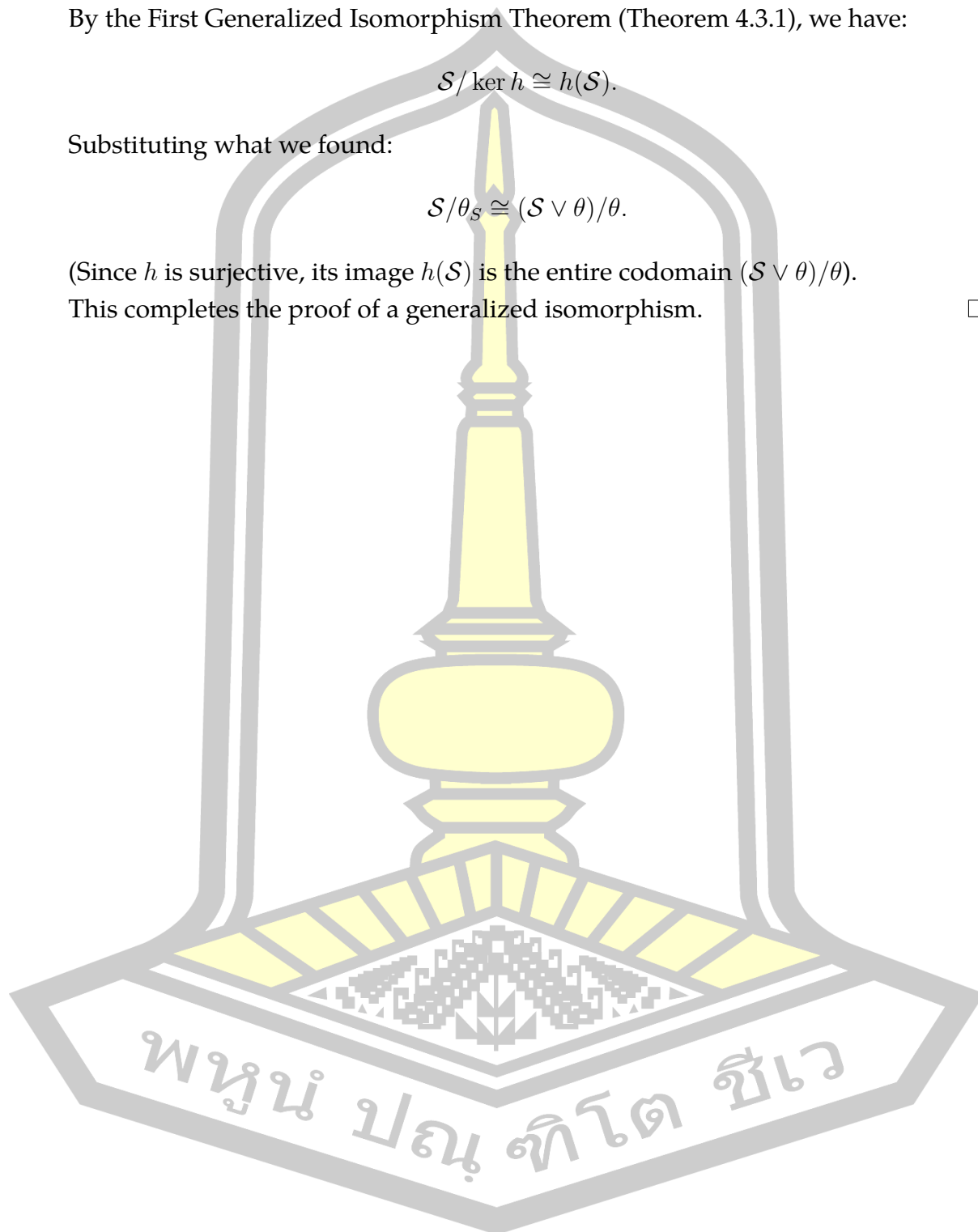
Apply the First Generalized Isomorphism Theorem: We have now established that $h : \mathcal{S} \rightarrow (\mathcal{S} \vee \theta)/\theta$ is a surjective generalized homomorphism with $\ker h = \theta_S$. By the First Generalized Isomorphism Theorem (Theorem 4.3.1), we have:

$$\mathcal{S}/\ker h \cong h(\mathcal{S}).$$

Substituting what we found:

$$\mathcal{S}/\theta_S \cong (\mathcal{S} \vee \theta)/\theta.$$

(Since h is surjective, its image $h(\mathcal{S})$ is the entire codomain $(\mathcal{S} \vee \theta)/\theta$). This completes the proof of a generalized isomorphism. □



CHAPTER 5

CONCLUSION

This research has successfully achieved its primary goal: to introduce and develop the foundational theory of generalized sigma algebras, a novel extension of classical algebraic structures. By generalizing the notion of arity from a natural number to an arbitrary index set, this work provides a framework for studying operations of potentially uncountable arities, thereby broadening the scope of universal algebra.

The core contributions of this thesis are the rigorous definitions and systematic investigation of fundamental algebraic concepts within this new framework. We have defined generalized sigma subalgebras, generalized congruence relations, generalized homomorphisms, and generalized isomorphisms and have proven that many of the cornerstones of classical algebra such as the Isomorphism Theorems are preserved in this more abstract setting. The following sections provide a detailed summary of these new definitions and the key theorems that have been established.

5.1 New Definition

1. Let A be a nonempty set, and let Ω be a any nonempty index set and let $\Sigma = \{ \Sigma_j : j \in \Omega \}$ nonempty index set. We defined $f_{\Sigma_j}^A : A^{\Sigma_j} \rightarrow A$, where

$$\begin{aligned} A^{\Sigma_j} &= \prod_{\beta \in \Sigma_j} A_\beta ; A_\beta = A \text{ for all } \beta \in \Sigma_j \\ &= \{ \sigma : \Sigma_j \rightarrow A \mid \sigma(\beta) \in A \text{ for all } \beta \in \Sigma_j \} \end{aligned}$$

We called $f_{\Sigma_j}^A : A^{\Sigma_j} \rightarrow A$ for all $j \in \Omega$ the $|\Sigma_j|$ -ary operation defined on A , and is said to have *arity* $|\Sigma_j|$. We let $\left(f_{\Sigma_j}^A \right)_{j \in \Omega}$ be a function which assigns to every element of Ω and $|\Sigma_j|$ -ary operation $f_{\Sigma_j}^A$ defined on A^{Σ_j} . Then the pair $\mathcal{A}_\Sigma := \left(A; \left(f_{\Sigma_j}^A \right)_{j \in \Omega} \right)$ is called (indexed) *Generalized algebra* (indexed by the set Σ_j) for all $\Sigma_j \in \Sigma$; for all $j \in \Omega$. The set A is called the *base* or *carrier set* or *universe* of A and $\left(f_{\Sigma_j}^A \right)_{j \in \Omega}$ is called *the sequence of fundamental operations* of A , for each $j \in \Omega$ the index Σ_j is called *arity* of $\left(f_{\Sigma_j}^A \right)_{j \in \Omega}$. The sequence $\tau := \left(|\Sigma_j|_{j \in \Omega} \right)$ of all arities is called *type* of the generalized algebra. We use the name $Alg(\Sigma)$ for the calss of all algebras of a given type $\tau := \left(|\Sigma_j|_{j \in \Omega} \right)$.

2. Let $\mathcal{B}_\Sigma := \left(B; \left(f_{\Sigma_j}^B \right)_{j \in \Omega} \right)$ be a generalized sigma algebra of type $\tau := \left(|\Sigma_j|_{j \in \Omega} \right)$. Then an generalized sigma algebra $\mathcal{A}_\Sigma := \left(A; \left(f_{\Sigma_j}^A \right)_{j \in \Omega} \right)$ is called a generalized sigma subalgebra of $\mathcal{B}_\Sigma$, written as $\mathcal{A}_\Sigma \preceq \mathcal{B}_\Sigma$, if the following conditions are satisfied:

- (i) $\mathcal{A}_\Sigma := \left(A; \left(f_{\Sigma_j}^A \right)_{j \in \Omega} \right)$ is an algebra of type $\tau := \left(|\Sigma_j|_{j \in \Omega} \right)$.
- (ii) $A \subseteq B$.
- (iii) For all $j \in \Omega$, $f_{\Sigma_j}^B|_{A^{\Sigma_j}} = f_{\Sigma_j}^A$.

Then $\mathcal{A}_\Sigma \preceq \mathcal{B}_\Sigma$.

3. Let $\mathcal{A}_\Sigma = (A; (f_{\Sigma_j}^A)_{j \in \Omega})$ and $\mathcal{B}_\Sigma = (B; (f_{\Sigma_j}^B)_{j \in \Omega})$ be generalized sigma algebras of the same type. A mapping $h : A \rightarrow B$ is called a *generalized homomorphism* from $\mathcal{A}_\Sigma$ to $\mathcal{B}_\Sigma$ if for every $j \in \Omega$ and for every function $\psi \in A^{\Sigma_j}$, the following condition holds:

$$h \left(f_{\Sigma_j}^A(\psi) \right) = f_{\Sigma_j}^B(h \circ \psi)$$

where $h \circ \psi$ denotes the function composition. This composition results in a new function $h \circ \psi : \Sigma_j \rightarrow B$, which is a valid input for the operation $f_{\Sigma_j}^B$, and is defined by $(h \circ \psi)(\sigma) = h(\psi(\sigma))$ for all $\sigma \in \Sigma_j$.

4. If the function h is bijective, that is both one-to-one (injective) and "onto" (surjective), then the generalized homomorphism $h : \mathcal{A}_\Sigma \rightarrow \mathcal{B}_\Sigma$ is called a *generalized isomorphism* from $\mathcal{A}_\Sigma$ onto $\mathcal{B}_\Sigma$. An *injective generalized homomorphism* from $\mathcal{A}_\Sigma$ into $\mathcal{B}_\Sigma$ is also called a *generalized embedding* of $\mathcal{A}_\Sigma$ into $\mathcal{B}_\Sigma$.
5. A generalized homomorphism $h : \mathcal{A}_\Sigma \rightarrow \mathcal{A}_\Sigma$ of a generalized sigma algebra $\mathcal{A}_\Sigma$ into itself is called a *generalized endomorphism* of $\mathcal{A}_\Sigma$, and a *generalized isomorphism* $h : \mathcal{A}_\Sigma \rightarrow \mathcal{A}_\Sigma$ from $\mathcal{A}_\Sigma$ onto $\mathcal{A}_\Sigma$ is called a *generalized automorphism* of $\mathcal{A}_\Sigma$.
6. Let $\mathcal{A}_\Sigma = (A; (f_{\Sigma_j}^A)_{j \in \Omega})$ and $\mathcal{B}_\Sigma = (B; (f_{\Sigma_j}^B)_{j \in \Omega})$ be generalized sigma algebras of the same type, and let $h : A \rightarrow B$ be a generalized homomorphism. The kernel of a generalized homomorphism h , denoted by $\ker h$, is the binary relation on the base set A defined as follows:

$$\ker h := \{(a_1, a_2) \in A \times A \mid h(a_1) = h(a_2)\}.$$

7. Let A be a set, let θ be an equivalence relation on A , and let $f_{\Sigma_j}^A : A^{\Sigma_j} \rightarrow A$ be a generalized operation. We say that $f_{\Sigma_j}^A$ is *compatible* with θ if for all functions $\alpha, \beta \in A^{\Sigma_j}$, the following implication holds:

$$\text{If } (\alpha(\sigma), \beta(\sigma)) \in \theta \text{ for all } \sigma \in \Sigma_j, \text{ then } (f_{\Sigma_j}^A(\alpha), f_{\Sigma_j}^A(\beta)) \in \theta.$$

8. Let $\mathcal{A}_\Sigma := \left(A; \left(f_{\Sigma_j}^A \right)_{j \in \Omega} \right)$ be a generalized sigma algebra of type $\tau := \left(|\Sigma_j|_{j \in \Omega} \right)$. An equivalence relation θ on A is called a *generalized congruence relations* on $\mathcal{A}_\Sigma$ if all the fundamental operations $f_{\Sigma_j}^A$ are compatible with θ . We denote by $Con \mathcal{A}_\Sigma$ the set of all generalized congruence relations of the generalized sigma algebra $\mathcal{A}_\Sigma$. For every generalized sigma algebra $\mathcal{A}_\Sigma := \left(A; \left(f_{\Sigma_j}^A \right)_{j \in \Omega} \right)$, the trivial equivalence relations:

$$\Delta_A := \{(a, a) \mid a \in A\} \text{ and } \nabla_A = A \times A.$$

are congruence relations.

9. A generalized sigma algebra which has no generalized congruence relations except Δ_A and ∇_A is called *simple*.
10. Let $\mathcal{A}_\Sigma = (A; (f_{\Sigma_j}^A)_{j \in \Omega})$ be a generalized sigma algebra, and let θ be a generalized congruence relation on $\mathcal{A}_\Sigma$. The *quotient algebra* of $\mathcal{A}_\Sigma$ by θ , denoted $\mathcal{A}_\Sigma/\theta$, is a new algebra of the same type, defined as follows: The Carrier Set (Universe):

The carrier set of the quotient algebra is the set of all θ -equivalence classes of A , denoted by A/θ .

$$A/\theta := \{[a]_\theta \mid a \in A\}.$$

where $[a]_\theta = \{b \in A \mid (a, b) \in \theta\}$. is the equivalence class containing the element a . The Fundamental Operations :

For each operation index $j \in \Omega$, we define a new operation $f_{\Sigma_j}^{A/\theta}$ on the set A/θ . The domain of this new operation is the set of functions from Σ_j to the new carrier set, i.e., $(A/\theta)^{\Sigma_j}$.

Let $\Psi : \Sigma_j \rightarrow A/\theta$ be an arbitrary function in this domain. Each value $\Psi(\sigma)$ is an equivalence class. The operation $f_{\Sigma_j}^{A/\theta}$ is defined by the following rule:

$$f_{\Sigma_j}^{A/\theta}(\Psi) := [f_{\Sigma_j}^A(\psi)]_\theta.$$

where $\psi : \Sigma_j \rightarrow A$ is any function chosen such that it represents Ψ on a component-wise basis. That is, for every $\sigma \in \Sigma_j$, the element $\psi(\sigma)$ is a representative of the class $\Psi(\sigma)$, i.e., $[\psi(\sigma)]_\theta = \Psi(\sigma)$.

5.2 New theorem

1. Let $\mathcal{B}_\Sigma := \left(B; \left(f_{\Sigma_j}^B \right)_{j \in \Omega} \right)$ be a generalized sigma algebra of type $\tau := \left(|\Sigma_j|_{j \in \Omega} \right)$ and let $A \subseteq B$, then $\mathcal{A}_\Sigma := \left(A; \left(f_{\Sigma_j}^A \right)_{j \in \Omega} \right)$ is a generalized sigma subalgebra of $\mathcal{B}_\Sigma$ iff A is closed with respect to all the operations $f_{\Sigma_j}^B$ for all $j \in \Omega$; that is, if $f_{\Sigma_j}^B|_{A^{\Sigma_j}}(\sigma) \in A$ for all $\sigma \in A^{\Sigma_j}$ and for all $j \in \Omega$
2. Let $\mathcal{B}_\Sigma := \left(B; \left(f_{\Sigma_j}^B \right)_{j \in \Omega} \right)$ be a generalized sigma algebra of type τ . The non-empty intersection of any non-empty family of its generalized sigma subalgebras, $\{\mathcal{A}_{k,\Sigma} \mid k \in \Gamma\}$, is also a generalized sigma subalgebra of $\mathcal{B}_\Sigma$.
3. Let $\mathcal{A}_\Sigma, \mathcal{B}_\Sigma$ and $\mathcal{C}_\Sigma$ be generalized sigma algebra of type $\tau := \left(|\Sigma_j|_{j \in \Omega} \right)$. Then:
 - (i) If $(\mathcal{A}_\Sigma \preceq \mathcal{B}_\Sigma)$ and $(\mathcal{B}_\Sigma \preceq \mathcal{C}_\Sigma)$, then $\mathcal{A}_\Sigma \preceq \mathcal{C}_\Sigma$;
 - (ii) If $(A \subseteq B \subseteq C)$ and $(\mathcal{A}_\Sigma \preceq \mathcal{C}_\Sigma)$ and $(\mathcal{B}_\Sigma \preceq \mathcal{C}_\Sigma)$, then $\mathcal{A}_\Sigma \preceq \mathcal{B}_\Sigma$
4. Let $\mathcal{B}_\Sigma := \left(B; \left(f_{\Sigma_j}^B \right)_{j \in \Omega} \right)$ be a generalized sigma algebra of type τ . The non-empty intersection of any non-empty family of its generalized sigma subalgebras, $\{\mathcal{A}_{k,\Sigma} \mid k \in \Gamma\}$, is also a generalized sigma subalgebra of $\mathcal{B}_\Sigma$.
5. Let $\mathcal{A}_\Sigma, \mathcal{B}_\Sigma$, and $\mathcal{C}_\Sigma$ be generalized sigma algebras of the same type.
 - (i) The identity map on A , denoted $id_A : A \rightarrow A$, is a generalized homomorphism from $\mathcal{A}_\Sigma$ to $\mathcal{A}_\Sigma$ (i.e., a generalized endomorphism).
 - (ii) If $h : \mathcal{A}_\Sigma \rightarrow \mathcal{B}_\Sigma$ and $k : \mathcal{B}_\Sigma \rightarrow \mathcal{C}_\Sigma$ are generalized homomorphisms, then their composition $k \circ h : \mathcal{A}_\Sigma \rightarrow \mathcal{C}_\Sigma$ is also a generalized homomorphism.
6. Let $\mathcal{A}_\Sigma$ be a generalized sigma algebra. The set of all generalized endomorphisms of $\mathcal{A}_\Sigma$, denoted $\text{End}(\mathcal{A}_\Sigma)$, forms a *monoid* under the operation of function composition ($\circ$).

7. Let $h : \mathcal{A}_\Sigma \rightarrow \mathcal{B}_\Sigma$ be a generalized homomorphism between two generalized sigma algebras of the same type.

(i) If $\mathcal{A}' = (A', (f_{\Sigma_j}^A|_{A'})_{j \in \Omega})$ is a subalgebra of $\mathcal{A}_\Sigma$, then its image, the algebra $\mathcal{B}' = (h(A'), (f_{\Sigma_j}^B|_{h(A')})_{j \in \Omega})$, is a subalgebra of $\mathcal{B}_\Sigma$.

(ii) If $\mathcal{B}' = (B', (f_{\Sigma_j}^B|_{B'})_{j \in \Omega})$ is a subalgebra of $\mathcal{B}_\Sigma$, then its preimage, the set $A' = h^{-1}(B')$, is the carrier set of a subalgebra of $\mathcal{A}_\Sigma$.

8. If $h : \mathcal{A}_\Sigma \rightarrow \mathcal{B}_\Sigma$ is a generalized isomorphism between two generalized sigma algebras, then its inverse map $h^{-1} : B \rightarrow A$ is also an isomorphism from $\mathcal{B}_\Sigma$ to $\mathcal{A}_\Sigma$.

9. Let $h : \mathcal{A}_\Sigma \rightarrow \mathcal{B}_\Sigma$ be a generalized homomorphism. Then the quotient algebra $\mathcal{A}_\Sigma / \ker h$ is isomorphic to the image algebra $h(\mathcal{A}_\Sigma)$. Specifically, there exists a unique isomorphism

$$f : \mathcal{A}_\Sigma / \ker h \rightarrow h(\mathcal{A}_\Sigma)$$

such that the following diagram commutes, i.e., $h|_{\text{im}(h)} = f \circ \text{nat}(\ker h)$, where $\text{nat}(\ker h)$ is the natural generalized homomorphism and $h|_{\text{im}(h)}$ is the generalized homomorphism h with its codomain restricted to its image.

10. If a generalized homomorphism $h : \mathcal{A}_\Sigma \rightarrow \mathcal{B}_\Sigma$ is surjective, then

$$\mathcal{A}_\Sigma / \ker h \cong \mathcal{B}_\Sigma$$

.

11. Let $\mathcal{A}_\Sigma = (A; (f_{\Sigma_j}^A)_{j \in \Omega})$ be a generalized sigma algebra. Let $\{\theta_i\}_{i \in I}$ be any non-empty family of generalized congruence relations on $\mathcal{A}_\Sigma$, indexed by the set I . Then their intersection,

$$\theta := \bigcap_{i \in I} \theta_i$$

is also a generalized congruence relation on $\mathcal{A}_\Sigma$.

12. Let $\mathcal{A}_\Sigma = (A; (f_{\Sigma_j}^A)_{j \in \Omega})$ be a generalized sigma algebra and let θ be a generalized congruence relation on it. The mapping $\text{nat}_\theta : A \rightarrow A/\theta$, defined by

$$\text{nat}_\theta(a) := [a]_\theta.$$

for all $a \in A$, is a surjective generalized homomorphism from $\mathcal{A}_\Sigma$ onto the quotient algebra $\mathcal{A}_\Sigma/\theta$. This map is called the *natural generalized homomorphism* (or canonical generalized homomorphism) associated with θ .

13. Let $h : \mathcal{A}_\Sigma \rightarrow \mathcal{B}_\Sigma$, and $g : \mathcal{A}_\Sigma \rightarrow \mathcal{C}_\Sigma$, be generalized homomorphisms between generalized sigma algebras of the same type, and assume that g is surjective. A generalized homomorphism $f : \mathcal{C}_\Sigma \rightarrow \mathcal{B}_\Sigma$, satisfying the condition $f \circ g = h$ exists if and only if $\ker g \subseteq \ker h$.

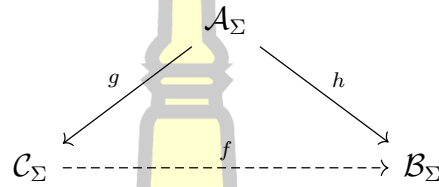


Figure 5.1: The Generalized Homomorphism Commute Diagram

Furthermore, if such a generalized homomorphism f exists, it satisfies the following:

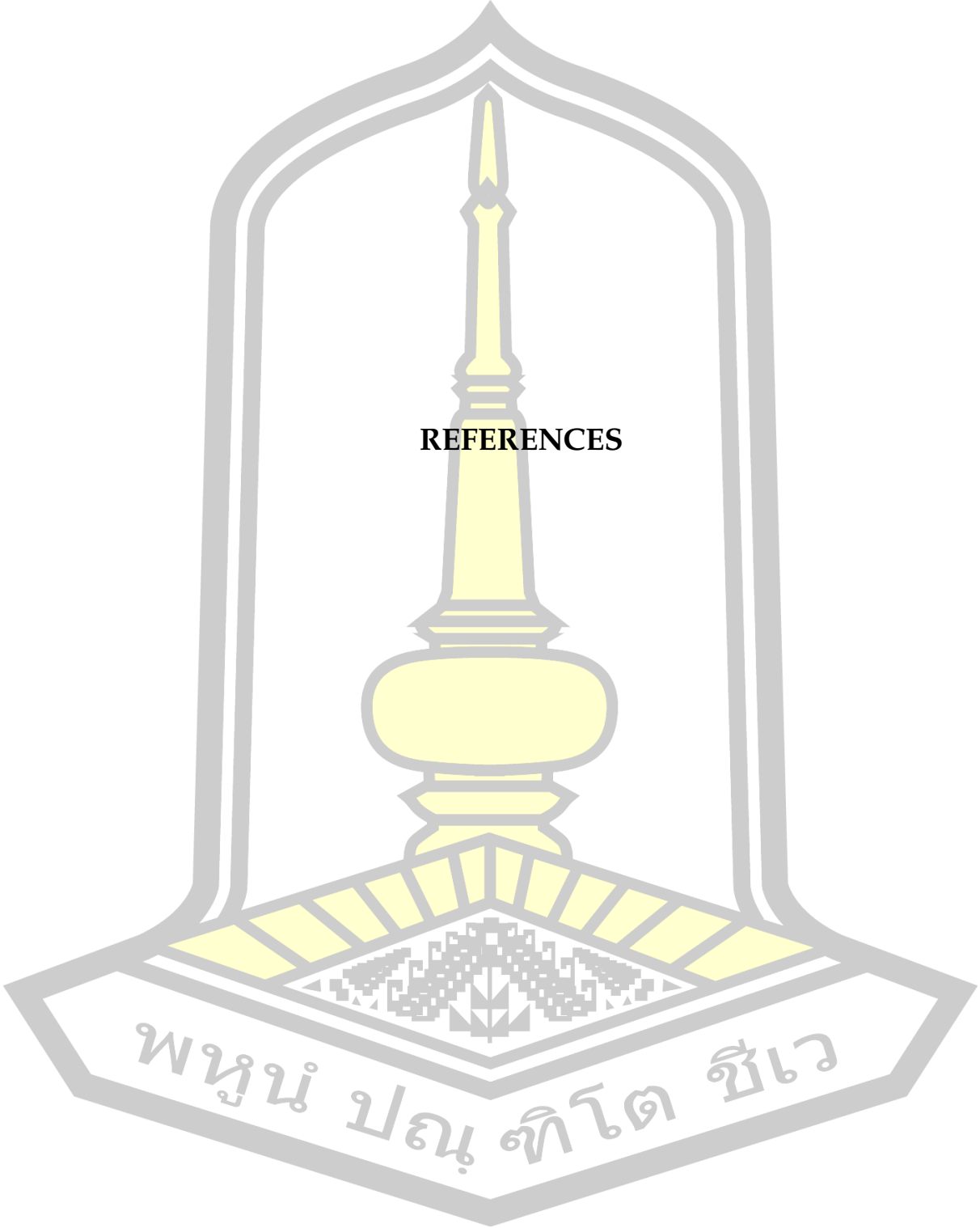
- (i) f is unique.
 - (ii) f is injective if and only if $\ker g = \ker h$.
 - (iii) f is surjective if and only if h is surjective.
14. If $h : \mathcal{A}_\Sigma \rightarrow \mathcal{B}_\Sigma$, is a generalized homomorphism, then the quotient algebra of the domain by a kernel is generalized isomorphic to the image of a generalized homomorphism.

$$\mathcal{A}_\Sigma / \ker h \cong h(\mathcal{A}_\Sigma).$$

15. Let $\mathcal{A}_\Sigma$ be a generalized sigma algebra, let $\mathcal{S}$ be a subalgebra of $\mathcal{A}_\Sigma$ with carrier set S , and let θ be a congruence on $\mathcal{A}_\Sigma$. Then:

- (i) The set $S \vee \theta := \{a \in A \mid \exists s \in S, (a, s) \in \theta\}$, is the carrier set of a subalgebra of $\mathcal{A}_\Sigma$, denoted $\mathcal{S} \vee \theta$.
- (ii) The relation $\theta_S := \theta \cap (S \times S)$ is a congruence on the subalgebra $\mathcal{S}$.
- (iii) The corresponding quotient algebras are isomorphic:

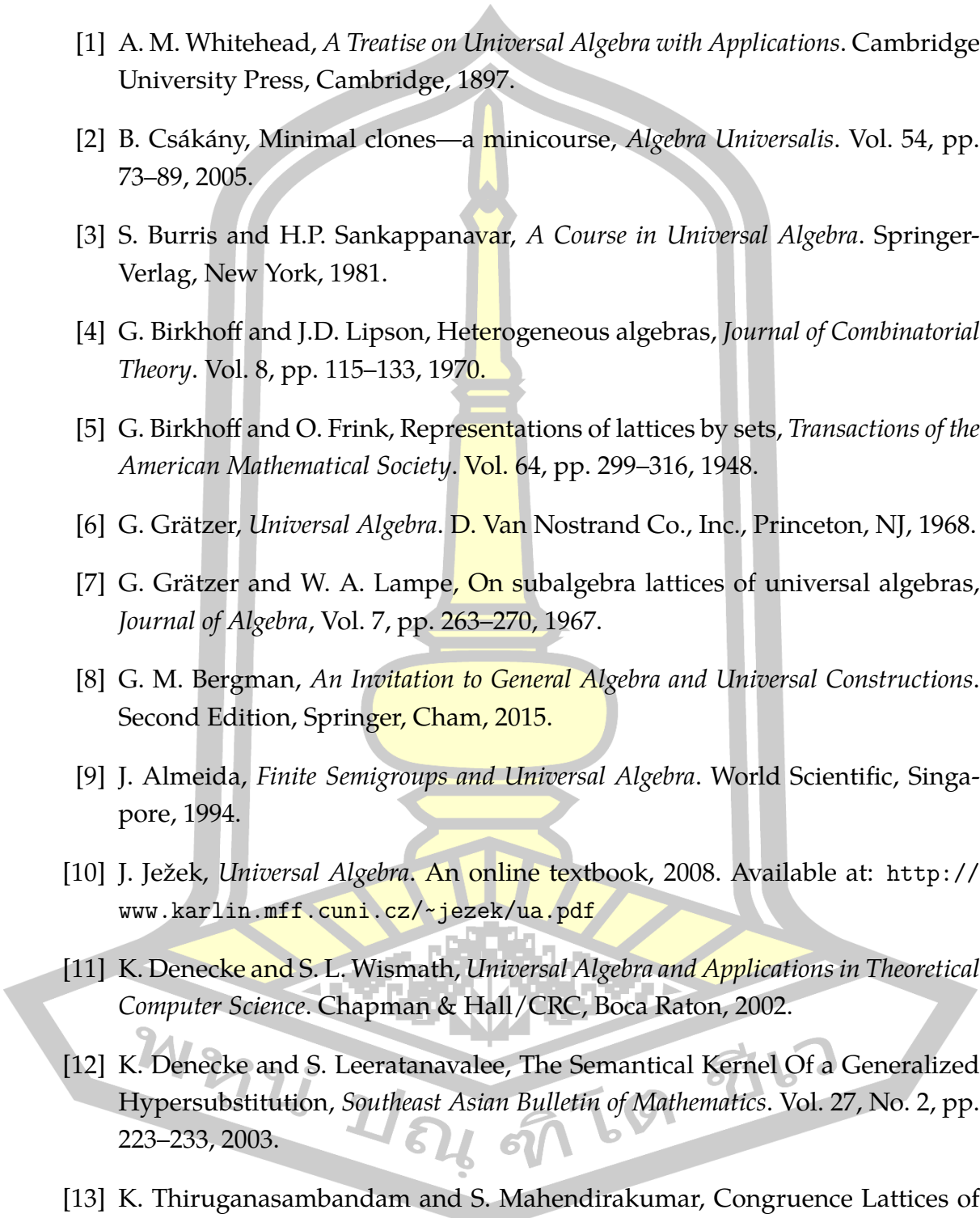
$$(\mathcal{S} \vee \theta) / \theta \cong \mathcal{S} / \theta_S.$$



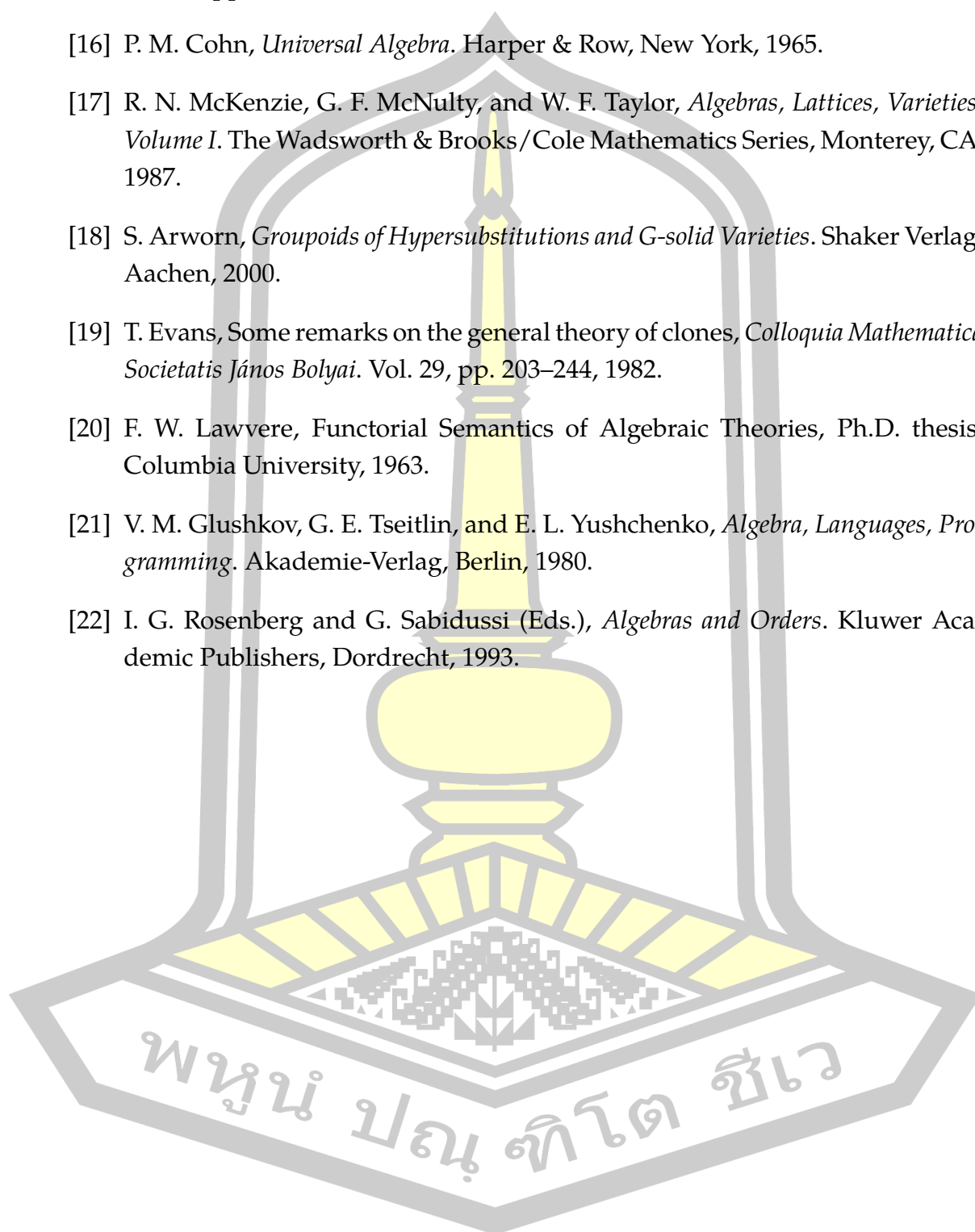
REFERENCES

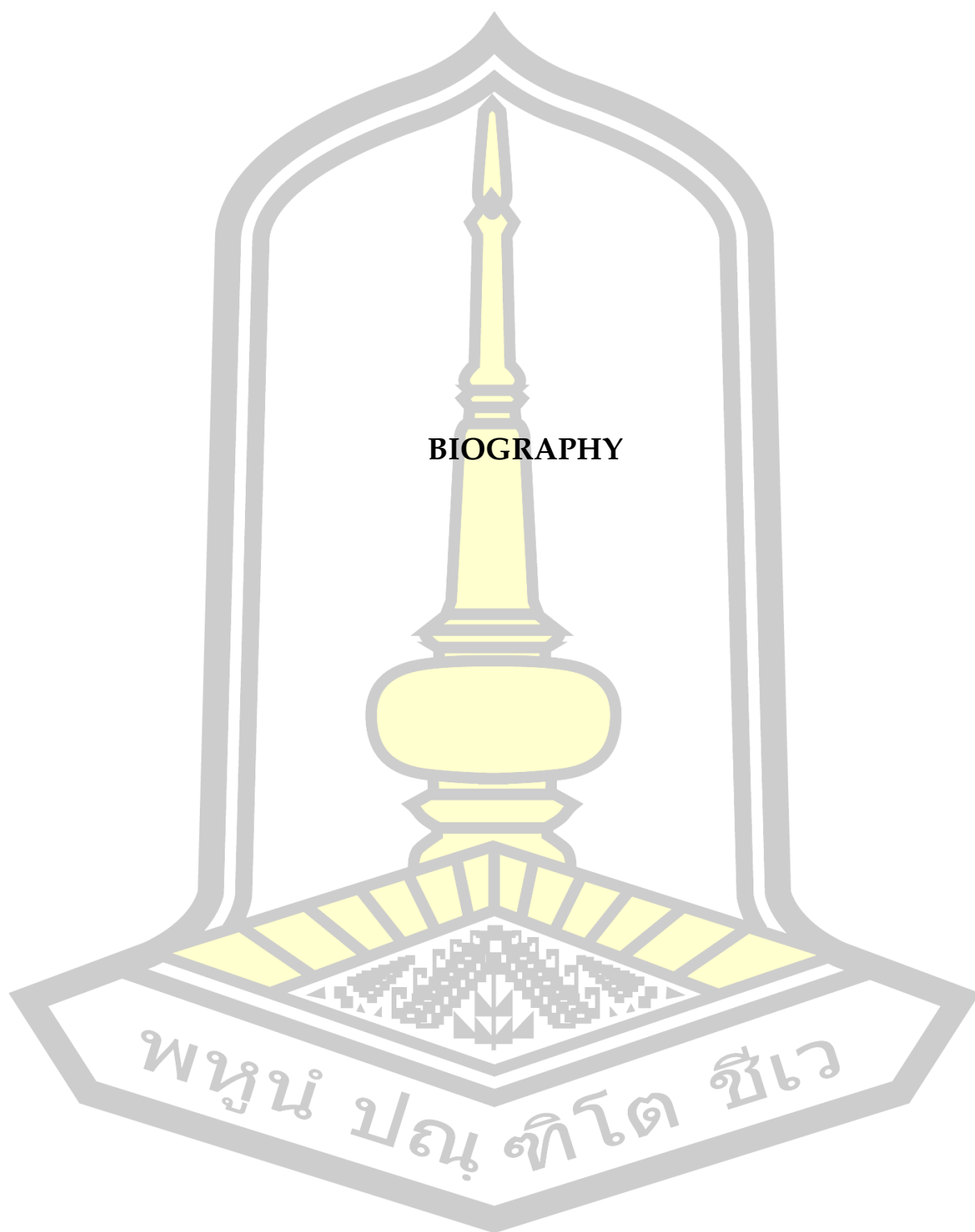
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BIBLIOGRAPHY

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- [1] A. M. Whitehead, *A Treatise on Universal Algebra with Applications*. Cambridge University Press, Cambridge, 1897.
- [2] B. Csákány, Minimal clones—a minicourse, *Algebra Universalis*. Vol. 54, pp. 73–89, 2005.
- [3] S. Burris and H.P. Sankappanavar, *A Course in Universal Algebra*. Springer-Verlag, New York, 1981.
- [4] G. Birkhoff and J.D. Lipson, Heterogeneous algebras, *Journal of Combinatorial Theory*. Vol. 8, pp. 115–133, 1970.
- [5] G. Birkhoff and O. Frink, Representations of lattices by sets, *Transactions of the American Mathematical Society*. Vol. 64, pp. 299–316, 1948.
- [6] G. Grätzer, *Universal Algebra*. D. Van Nostrand Co., Inc., Princeton, NJ, 1968.
- [7] G. Grätzer and W. A. Lampe, On subalgebra lattices of universal algebras, *Journal of Algebra*, Vol. 7, pp. 263–270, 1967.
- [8] G. M. Bergman, *An Invitation to General Algebra and Universal Constructions*. Second Edition, Springer, Cham, 2015.
- [9] J. Almeida, *Finite Semigroups and Universal Algebra*. World Scientific, Singapore, 1994.
- [10] J. Ježek, *Universal Algebra*. An online textbook, 2008. Available at: <http://www.karlin.mff.cuni.cz/~jezek/ua.pdf>
- [11] K. Denecke and S. L. Wismath, *Universal Algebra and Applications in Theoretical Computer Science*. Chapman & Hall/CRC, Boca Raton, 2002.
- [12] K. Denecke and S. Leeratanavalee, The Semantical Kernel Of a Generalized Hypersubstitution, *Southeast Asian Bulletin of Mathematics*. Vol. 27, No. 2, pp. 223–233, 2003.
- [13] K. Thiruganasambandam and S. Mahendirakumar, Congruence Lattices of Isoform Lattices, *IOSR Journal of Mathematics*. Vol. 6, Issue 1, pp. 62–73, 2013.
- [14] M. Bodirsky, *Graph Homomorphisms and Universal Algebra Course Notes*. TU Dresden, Germany, 2023.

- [15] P. J. Higgins, Algebras with a scheme of operators, *Mathematische Nachrichten*. Vol. 27, pp. 115–132, 1963.
- [16] P. M. Cohn, *Universal Algebra*. Harper & Row, New York, 1965.
- [17] R. N. McKenzie, G. F. McNulty, and W. F. Taylor, *Algebras, Lattices, Varieties, Volume I*. The Wadsworth & Brooks/Cole Mathematics Series, Monterey, CA, 1987.
- [18] S. Arworn, *Groupoids of Hypersubstitutions and G-solid Varieties*. Shaker Verlag, Aachen, 2000.
- [19] T. Evans, Some remarks on the general theory of clones, *Colloquia Mathematica Societatis János Bolyai*. Vol. 29, pp. 203–244, 1982.
- [20] F. W. Lawvere, Functorial Semantics of Algebraic Theories, Ph.D. thesis, Columbia University, 1963.
- [21] V. M. Glushkov, G. E. Tseitlin, and E. L. Yushchenko, *Algebra, Languages, Programming*. Akademie-Verlag, Berlin, 1980.
- [22] I. G. Rosenberg and G. Sabidussi (Eds.), *Algebras and Orders*. Kluwer Academic Publishers, Dordrecht, 1993.





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