Properties of some sets in bi-weak structure spaces

ILADA CHEENCHAN

A Thesis submitted in partial fulfillment of the requirements for
the degree of Master of Science in Mathematics
at Mahasarakham University
October 2019

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The examining committee has unanimously approved this thesis, submitted by Miss Ilada Cheenchan, as a partial fulfillment of the requirements for the Master of Science in Mathematics at Mahasarakham University.

Examining Committee

....	Chairman
(Asst. Prof. Supunnee Sompong, Ph.D.)	(Faculty graduate committee)
....	Committee
(Asst. Prof. Chokchai Viriyapong, Ph.D.)	(Advisor)
....	Committee
(Asst. Prof. Chawalit Boonpok, Ph.D.)	(Co-advisor)
....	Committee
(Asst. Prof. Jeeranunt Khampakdee, Ph.D.)	(Faculty graduate committee)
....	Committee
(Chalongchai Khanarong, Ph.D.)	(Faculty graduate committee)

Mahasarakham University has granted approval to accept this thesis as a partial fulfillment of the requirements for the Master of Science in Mathematics.

....	....
(Prof. Pairot Pramual, Ph.D.)	(Asst. Prof. Krit Chaimoon, Ph.D.)
Dean of the Faculty of Science	Dean of Graduate School

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Ilada Cheenchan

ชื่อเรื่อง สมบัติของบางเซตในปริภูมิสองโครงสร้างอ่อน
ผู้วิจัย นางสาวไอลดา จีนจัน
ปริญญา วิทยาศาสตรมหาบัณฑิต **สาขา** คณิตศาสตร์
กรรมการควบคุม ผู้ช่วยศาสตราจารย์ ดร. โชคชัย วิริยะพงษ์
ผู้ช่วยศาสตราจารย์ ดร. ชวลิต บุญปก
มหาวิทยาลัย มหาวิทยาลัยมหาสารคาม **ปีที่พิมพ์** 2562

บทคัดย่อ

ในการวิจัยนี้ ผู้วิจัยจะนำเสนอแนวคิดของเซตขอบ เซตภายนอกและเซตหนาแน่นในปริภูมิสองโครงสร้างอ่อน ได้แสดงให้เห็นสมบัติบางประการของเซตเหล่านี้ โดยเฉพาะอย่างยิ่งได้รับบางลักษณะเฉพาะของเซตปิดในปริภูมิสองโครงสร้างอ่อนโดยใช้เซตขอบหรือเซตภายนอก นอกจากนั้นเรายังศึกษาส่วนปิดคลุม bi-w-(Λ, θ) และภายใน bi-w-(Λ, θ) บนปริภูมิสองโครงสร้างอ่อน

คำสำคัญ : เซตขอบ, เซตหนาแน่น, เซตภายนอก, เซตปิด bi-w-(Λ, θ), เซตเปิด bi-w-(Λ, θ), ปริภูมิสองโครงสร้างอ่อน

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CANDIDATE Miss Ilada Cheenchan
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ADVISORS Asst. Prof. Chokchai Viriyapong, Ph.D.,
Asst. Prof. Chawalit Boonpok, Ph.D.
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ABSTRACT

In this research, the concepts of boundary sets, exterior sets and dense sets in bi-weak structure spaces are introduced. Some properties of their sets are obtained. In particular, some characterizations of closed sets in a bi-weak structure space using boundary sets or exterior sets are obtained. Moreover, we introduce the notions bi-w-(Λ, θ)-closure and bi-w-(Λ, θ)-interior on bi-weak structure spaces.

Keywords : boundary set, exterior set, dense set, bi-w-(Λ, θ)-closed sets, bi-w-(Λ, θ)-open sets, bi-weak structure space.

CONTENTS

CHAPTER 1

Introduction

1.1 Background

Topological space is the mathematical structure which consist of a set X, that we were interested with the structure on X called topology. The structure contains $\emptyset$ and X and also satisfies the two properties that an arbitrary union of its elements belongs to it and a finite intersection of its elements belongs to it. The members of topology are called open sets and the complements of open sets are called closed sets. Moreover, the closure operator and interior operator, which were defined by closed sets and open sets respectively, were two important operators on the topology. In 2018, Boonpok and others introduced closure and interior in another ways called (Λ, θ)-closure and (Λ, θ)-interior respectively, as well as defined (Λ, θ)-open set, $s(\Lambda, \theta)$-open set, $p(\Lambda, \theta)$-open set, $\alpha(\Lambda, \theta)$-open set, $\beta(\Lambda, \theta)$-open set and $b(\Lambda, \theta)$-open set, by using closure and interior that mentioned before to be determinant and studied the properties of that set.

Recently, mathematicians studied another structures beside topology such as minimal structure introduced by Popa and Noiri [10], and also introduced the idea about generalized topology and weak structure which was discovered by Császár [7], [8]. In addition, they also be studied on the space that has two structures. Kelly [9] introduced bitopological spaces, Boonpok [3] introduced the idea about bigeneralized topological spaces and Boonpok [2] also introduced biminimal structure spaces. Obviously, such structures were generalization of topology which be able to expand the results from topological space to another spaces. In the other word, that mean there are expansions for closed sets, open sets, closure, interior and others on topological spaces to spaces that were mentioned before. In 2011, Sompong [17] introduced about exterior sets on biminimal structure spaces and studied some fundamental properties and Sompong [15] introduced about boundary sets on biminimal structure spaces and studied some fundamental properties. And in 2012 Sompong [16] introduced dense sets and studied some fundamental properties of dense sets on biminimal structure spaces. Afterward, in 2013 Sompong [12] introduced the idea about some fundamental properties of dense sets on bigeneralized topological spaces. In the same year, Sompong and others [14] introduced the idea about some fundamental properties exterior sets on bigeneralized topological spaces and Sompong [13] also introduced the idea about boundary sets and studied some fundamental properties on bigeneralized topological spaces. In 2017, Puiwong and others [11] introduced new space, which consists of a nonempty set X and two weak structures on X. It is called a bi-weak structure space or briefly a bi-w space. Some properties of closed sets and open sets are studied in this space. Furthermore, some characterizations of weak separation axioms are obtained.

In conclusion, researcher was interested to expand the idea of dense sets, exterior sets and boundary sets on a bi-weak structure space and expand the idea about (Λ, θ) from topological space to a bi-weak structure space.

1.2 Objective of the research

The purposes of the research are:

1. To construct and investigate the properties of dense sets exterior sets and boundary sets on bi-weak structure spaces.

2. To construct and investigate the ptoperties of (Λ, θ)-closure and (Λ, θ)-interior operators on bi-weak structure spaces.

3. To construct and investigate some closed and open sets determined by (Λ, θ)-closure or (Λ, θ)-interior on bi-weak structure spaces.

1.3 Objective of the research

The research procedure of this thesis consists of the following steps:

1. Criticism and possible extension of the literature review.

2. Doing research to investigate the main results.

3. Applying the results from 1.3.1 and 1.3.2 to the main results.

1.4 Scope of the study

The scopes of the study are: studying some properties of dense sets, exterior sets and boundary sets on bi-weak structure spaces. And studying (Λ, θ)-closure and (Λ, θ)-interior operators in bi-weak structure spaces.

CHAPTER 2

Preliminaries

In this chapter, we will give some definitions, notations, dealing with some preliminaries and some useful results that will be duplicated in later chapter.

2.1 Topological spaces

The essential properties were distilled out and the concept of a collection of open sets, called a topology, evolved into the following definition:

Definition 2.1.1. [1] Let X be a set. A **topology** τ on X is a collection of subsets of X, each called an **open** set, such that

1. $\emptyset$ and X are open sets;
2. The intersection of finitely many open sets is an open set;
3. The union of any collection of open sets is an open set.

The set X together with a topology τ on X is called a **topological space**, denote by (X, τ).

Thus a collection of subsets of a set X is a topology on X if it includes the empty set and X, and if finite intersections and arbitrary unions of sets in the collection are also in the collection.

Theorem 2.1.2. [1] Let (X, τ) be a topological space. The following statements about the collection of closed sets in X hold:

1. $\emptyset$ and X are closed.
2. The intersection of any collection of closed sets is a closed set.
3. The union of finite many closed sets is a closed set.

Definition 2.1.3. [1] Let A be a subset of a topological space X. The **interior of** A, denoted $Int(A)$, is the union of all open sets contained in A. The **closure of** A, denote $Cl(A)$, is the intersection of all closed sets containing A.

Clealy, the interior of A is open and a subset of A, and the closure of A is closed and contain A. Thus we have the aforementioned set sandwich, with A caught between an open set and a closed set: $Int(A) \subseteq A \subseteq Cl(A)$.

The following properties follow readily from the definition of interior and closure.

Theorem 2.1.4. [1] Let (X, τ) be a topological space and A and B be subsets of X.

1. If U is an open set in X and $U \subseteq A$, then $U \subseteq Int(A)$.
2. If C is an closed set in X and $A \subseteq C$, then $Cl(A) \subseteq C$.
3. If $A \subseteq B$ then $Int(A) \subseteq Int(B)$.
4. If $A \subseteq B$ then $Cl(A) \subseteq Cl(B)$.
5. A is open if and only if $A = Int(A)$.
6. A is closed if and only if $A = Cl(A)$.

Theorem 2.1.5. [1] For sets A and B in a topological space X, the following statements hold:

1. $Int(X - A) = X - Cl(A)$.
2. $Cl(X - A) = X - Int(A)$.
3. $Int(A) \cup Int(B) \subseteq Int(A \cup B)$, and in general equality does not hold.
4. $Int(A) \cap Int(B) = Int(A \cap B)$.

Definition 2.1.6. [18] Let (X, τ) be a topological space and $A \subseteq X$. A point $x \in X$ is called a **θ-cluster point of** A if $A \cap Cl(U) \neq \emptyset$ for every open set U of X containing x. The set of all θ-cluster points of A is called **θ-closure** of A and is denoted by $Cl_\theta(A)$.

Definition 2.1.7. [18] A subset A of a topological space (X, τ) is called **θ-closed** if $A = Cl_\theta(A)$. The complement of a θ-closed set is said to be **θ-open**. The family of all θ-open sets in a topological space (X, τ) is denoted by $\theta(X, \tau)$.

Definition 2.1.8. [18] The union of all θ-open sets contained in A is called the **θ-interior of** A and is denoted by $Int_\theta(A)$.

Proposition 2.1.9. [18] $Cl_\theta(V) = Cl(V)$ for every open set V of X.

Proposition 2.1.10. [18] $Cl_\theta(B)$ is closed in (X, τ) for every subset B of X.

Definition 2.1.11. [6] Let A be a subset of a topological space (X, τ). A subset $\Lambda_\theta(A)$ is defined to be the set $\cap\{O \in \theta(X, \tau) | \ A \subseteq O\}$.

Lemma 2.1.12. [6] For subsets A, B, and A_i $(i \in I)$ of a topological space (X, τ), the following properties hold:

1. $A \subseteq \Lambda_\theta(A)$.
2. If $A \subseteq B$, then $\Lambda_\theta(A) \subseteq \Lambda_\theta(B)$.
3. $\Lambda_\theta(\Lambda_\theta(A)) = \Lambda_\theta(A)$.
4. $\Lambda_\theta(\cap\{A_i | i \in I\}) \subseteq \cap\{\Lambda_\theta(A_i) | i \in I\}$.
5. $\Lambda_\theta(\cup\{A_i | i \in I\}) = \cup\{\Lambda_\theta(A_i) | i \in I\}$.

Definition 2.1.13. [6] A subset A of a topological space (X, τ) is called a **Λ_θ-set** if $A = \Lambda_\theta(A)$.

Lemma 2.1.14. [6] For subsets A and A_i $(i \in I)$ of a topological space (X, τ), the following properties hold:

1. $\Lambda_\theta(A)$ is a Λ_θ-set.
2. If A is a θ-open, then A is a Λ_θ-set.
3. If A_i is a Λ_θ-set for each $i \in I$, then $\cap_{i\in I} A_i$ is a Λ_θ-set.
4. If A_i is a Λ_θ-set for each $i \in I$, then $\cup_{i\in I} A_i$ is a Λ_θ-set.

Definition 2.1.15. [6] Let A be a subset of a topological space (X, τ).

1. A is called a (Λ, θ)**-closed** set if $A = T \cap C$, where T is a Λ_θ-set and C is a θ-closed set. The complement of a (Λ, θ)-closed set is called (Λ, θ)**-open**. The collection of all (Λ, θ)-open (resp. (Λ, θ)-closed) sets in a topological space (X, τ) is denoted by $\Lambda_\theta O(X, \tau)$ (resp. $\Lambda_\theta C(X, \tau)$).

2. A point $x \in X$ is called a (Λ, θ)**-cluster point of** A if for every (Λ, θ)-open set U of X containing x, we have $A \cap U \neq \emptyset$. The set of all (Λ, θ)-cluster points of A is called the (Λ, θ)**-closure of** A and is denoted by $A^{(\Lambda,\theta)}$.

Lemma 2.1.16. [6] Let A and B be subsets of a topological space (X, τ). For the (Λ, θ)-closure, the following properties hold:

1. $A \subseteq A^{(\Lambda,\theta)}$.

2. $A^{(\Lambda,\theta)} = \cap\{F | A \subseteq F$ and F is (Λ, θ)-closed$\}$.

3. If $A \subseteq B$, then $A^{(\Lambda,\theta)} \subseteq B^{(\Lambda,\theta)}$.

4. $A^{(\Lambda,\theta)}$ is (Λ, θ)-closed.

Lemma 2.1.17. [5] Let A be a subset of a topological space (X, τ). Then the following properties hold:

1. If A is (Λ, θ)-closed, then $A = \Lambda_\theta(A) \cap Cl_\theta(A)$.

2. If A is θ-closed, then A is (Λ, θ)-closed.

3. If A_i is (Λ, θ)-closed for each $i \in I$, then $\cap_{i \in I} A_i$ is (Λ, θ)-closed.

Lemma 2.1.18. [4] For a subset A of a topological space (X, τ), $x \in A^{(\Lambda,\theta)}$ if and only if $U \cap A \neq \emptyset$ for every (Λ, θ)-open set U containing x.

Definition 2.1.19. [4] Let A be a subset of a topological space (X, τ). The union of all (Λ, θ)-open sets contained in A is called the (Λ, θ)**-interior of** A and is denoted by $A_{(\Lambda,\theta)}$.

Lemma 2.1.20. [4] Let A and B be subsets of a topological space (X, τ). For the (Λ, θ)-interior, the following properties hold:

1. $A_{(\Lambda,\theta)} \subseteq A$.

2. If $A \subseteq B$, then $A_{(\Lambda,\theta)} \subseteq B_{(\Lambda,\theta)}$.

3. A is (Λ,θ)-open if and only if $A_{(\Lambda,\theta)} = A$.

4. $A_{(\Lambda,\theta)}$ is (Λ,θ)-open.

Next we will recall the notions of $s(\Lambda,\theta)$-open, $p(\Lambda,\theta)$-open, $\alpha(\Lambda,\theta)$-open and $\beta(\Lambda,\theta)$-open sets.

Definition 2.1.21. [4] A subset A of a topological space (X,τ) is said to be:

1. $s(\Lambda,\theta)$**-open** if $A \subseteq [A_{(\Lambda,\theta)}]^{(\Lambda,\theta)}$;

2. $p(\Lambda,\theta)$**-open** if $A \subseteq [A^{(\Lambda,\theta)}]_{(\Lambda,\theta)}$;

3. $\alpha(\Lambda,\theta)$**-open** if $A \subseteq [[A_{(\Lambda,\theta)}]^{(\Lambda,\theta)}]_{(\Lambda,\theta)}$;

4. $\beta(\Lambda,\theta)$**-open** if $A \subseteq [[A^{(\Lambda,\theta)}]_{(\Lambda,\theta)}]^{(\Lambda,\theta)}$.

The family of all $s(\Lambda,\theta)$-open (resp. $p(\Lambda,\theta)$-open, $\alpha(\Lambda,\theta)$-open, $\beta(\Lambda,\theta)$-open) sets in a topological space X,τ is denoted by $s\Lambda_\theta O(X,\tau)$ (resp. $p\Lambda_\theta O(X,\tau)$, $\alpha\Lambda_\theta O(X,\tau)$, $\beta\Lambda_\theta O(X,\tau)$).

Definition 2.1.22. [4] The complement of a $s(\Lambda,\theta)$-open (resp. $p(\Lambda,\theta)$-open, $\alpha(\Lambda,\theta)$-open, $\beta(\Lambda,\theta)$-open) set is said to be $s(\Lambda,\theta)$**-closed** (resp. $p(\Lambda,\theta)$**-closed**, $\alpha(\Lambda,\theta)$**-closed**, $\beta(\Lambda,\theta)$**-closed**).

The family of all $s(\Lambda,\theta)$-closed (resp. $p(\Lambda,\theta)$-closed, $\alpha(\Lambda,\theta)$-closed, $\beta(\Lambda,\theta)$-closed) sets in a topological space (X,τ) is denoted by $s\Lambda_\theta C(X,\tau)$ (resp. $p\Lambda_\theta C(X,\tau)$, $\alpha\Lambda_\theta C(X,\tau)$, $\beta\Lambda_\theta C(X,\tau)$).

Proposition 2.1.23. [4] For a topological space (X,τ), the following properties hold:

1. $\Lambda_\theta O(X,\tau) \subseteq \alpha\Lambda_\theta O(X,\tau) \subseteq s\Lambda_\theta O(X,\tau) \subseteq \beta\Lambda_\theta O(X,\tau)$.

2. $\alpha\Lambda_\theta O(X,\tau) \subseteq p\Lambda_\theta O(X,\tau) \subseteq \beta\Lambda_\theta O(X,\tau)$.

3. $\alpha\Lambda_\theta O(X,\tau) = s\Lambda_\theta O(X,\tau) \cap p\Lambda_\theta O(X,\tau)$.

Definition 2.1.24. [4] A subset A of a topological space (X,τ) is said to be $r(\Lambda,\theta)$**-open** (resp. $r(\Lambda,\theta)$**-closed**) if $A = [A^{(\Lambda,\theta)}]_{(\Lambda,\theta)}$ (resp. $A = [A_{(\Lambda,\theta)}]^{(\Lambda,\theta)}$).

The family of all $r(\Lambda,\theta)$-open (resp. $r(\Lambda,\theta)$-closed) sets in a topological space (X,τ) is denoted by $r\Lambda_\theta O(X,\tau)$ (resp. $r\Lambda_\theta C(X,\tau)$).

Proposition 2.1.25. [4] For a subset A of a topological space (X,τ), the following properties hold:

1. A is $r(\Lambda,\theta)$-open if and only if $A = F_{(\Lambda,\theta)}$ for some (Λ,θ)-closed set F.
2. A is $r(\Lambda,\theta)$-closed if and only if $A = U^{(\Lambda,\theta)}$ for some (Λ,θ)-open set U.

Lemma 2.1.26. [4] For a subset A of a topological space (X,τ), the following properties hold:

1. $[X - A]_{(\Lambda,\theta)} = X - A^{(\Lambda,\theta)}$.
2. $[X - A]^{(\Lambda,\theta)} = X - A_{(\Lambda,\theta)}$.

Proposition 2.1.27. [4] For a subset A of a topological space (X,τ), the following properties hold:

1. A is $s(\Lambda,\theta)$-closed if and only if $[A^{(\Lambda,\theta)}]_{(\Lambda,\theta)} \subseteq A$.
2. A is $p(\Lambda,\theta)$-closed if and only if $[A_{(\Lambda,\theta)}]^{(\Lambda,\theta)} \subseteq A$.
3. A is $\alpha(\Lambda,\theta)$-closed if and only if $[[A^{(\Lambda,\theta)}]_{(\Lambda,\theta)}]^{(\Lambda,\theta)} \subseteq A$.
4. A is $\beta(\Lambda,\theta)$-closed if and only if $[[A_{(\Lambda,\theta)}]^{(\Lambda,\theta)}]_{(\Lambda,\theta)} \subseteq A$.

Proposition 2.1.28. [4] For a subset A of a topological space (X,τ), the following properties hold:

1. $[[[A^{(\Lambda,\theta)}]_{(\Lambda,\theta)}]^{(\Lambda,\theta)}]_{(\Lambda,\theta)} = [A^{(\Lambda,\theta)}]_{(\Lambda,\theta)}]$.
2. $[[[A_{(\Lambda,\theta)}]^{(\Lambda,\theta)}]_{(\Lambda,\theta)}]^{(\Lambda,\theta)} = [A_{(\Lambda,\theta)}]^{(\Lambda,\theta)}]$.

Proposition 2.1.29. [4] For a subset A of a topological space (X,τ), the following properties are equivalent:

1. A is $r(\Lambda,\theta)$-open.

2. A is (Λ, θ)-open and $s(\Lambda, \theta)$-closed.

3. A is $\alpha(\Lambda, \theta)$-open and $s(\Lambda, \theta)$-closed.

4. A is $p(\Lambda, \theta)$-open and $s(\Lambda, \theta)$-closed.

5. A is (Λ, θ)-open and $\beta(\Lambda, \theta)$-closed.

6. A is $\alpha(\Lambda, \theta)$-open and $\beta(\Lambda, \theta)$-closed.

Corollary 2.1.30. [4] For a subset A of a topological space (X, τ), the following properties are equivalent:

1. A is $r(\Lambda, \theta)$-closed.

2. A is (Λ, θ)-closed and $s(\Lambda, \theta)$-open.

3. A is $\alpha(\Lambda, \theta)$-closed and $s(\Lambda, \theta)$-open.

4. A is $p(\Lambda, \theta)$-closed and $s(\Lambda, \theta)$-open.

5. A is (Λ, θ)-closed and $\beta(\Lambda, \theta)$-open.

6. A is $\alpha(\Lambda, \theta)$-closed and $\beta(\Lambda, \theta)$-open.

2.2 Boundary sets, Exterior sets and Dense sets in bigeneralized topological spaces

Definition 2.2.1. [7] Let X be a nonempty set and $\mu \subseteq P(X)$. μ is called a **generalized topology**, briefly **GT**, on X if μ satisfies the following properties.

1. $\emptyset \in \mu$.

2. If $G_\gamma \in \mu$ for all $\gamma \in \Gamma$, then $\bigcup\limits_{\gamma \in \Gamma} G_\gamma \in \mu$.

In this case, (X, μ) is called a **generalized topological space**, briefly **GTS**. A is **μ-open** if $A \in \mu$ and A is **μ-closed** if $X - A \in \mu$.

Definition 2.2.2. [7] Let (X, μ) be a **GTS** and $A \subseteq X$.

1. $c_\mu(A) = \cap\{F | F \text{ is } \mu\text{-closed and } A \subseteq F\}$.

2. $i_\mu(A) = \cup\{G|G$ is μ-open and $G \subseteq A\}$.

Theorem 2.2.3. [7] Let (X, μ) be a generalized topological space. Then

1. $c_\mu(A) = X - i_\mu(X - A)$.

2. $i_\mu(A) = X - c_\mu(X - A)$.

Proposition 2.2.4. [7] Let (X, μ) be a generalized topological space and $A \subseteq X$. Then

1. $x \in i_\mu(A)$ if and only if there exists a μ-open set V such that $x \in V \subseteq A$.

2. $x \in c_\mu(A)$ if and only if $V \cap A \neq \emptyset$ for every μ-open set V such that $x \in V$.

Proposition 2.2.5. [7] Let (X, μ) be a generalized topological space. For subsets A and B of X, the following properties holds:

1. $c_\mu(X - A) = X - i_\mu(A)$ and $i_\mu(X - A) = X - c_\mu(A)$;

2. If $(X - A) \in \mu$, then $c_\mu(A) = A$ and if $A \in \mu$, then $i_\mu(A) = A$;

3. If $A \subseteq B$, then $c_\mu(A) \subseteq c_\mu(B)$ and $i_\mu(A) \subseteq i_\mu(B)$;

4. $A \subseteq c_\mu(A)$ and $i_\mu(A) \subseteq A$;

5. $c_\mu(c_\mu(A)) = c_\mu(A)$ and $i_\mu(i_\mu(A)) = i_\mu(A)$.

Next, we will recall the concept of bigeneralized topological spaces and properties $\mu_i\mu_j$-closed and $\mu_i\mu_j$-open sets in bigeneralized topological spaces.

Definition 2.2.6. [3] Let X be a nonempty set and μ_1, μ_2 be generalized toplogies on X. A triple (X, μ_1, μ_2) is called a **bigeneralized toplogical space** (briefly **BGTS**).

Let (X, μ_1, μ_2) be a bigeneralized topological space and A a subset of X. The closure of A and the interior of A with respect to μ_i are denote by $c_{\mu_i}(A)$ and $i_{\mu_i}(A)$, respectively, for $i = 1, 2$.

Next, let $i, j \in \{1, 2\}$ where $i \neq j$.

Definition 2.2.7. [3] A subset A of a bigeneralized topological space (X, μ_1, μ_2) is called $\mu_i\mu_j$**-closed** if $c_{\mu_i}(c_{\mu_j}(A)) = A$, The complement of $\mu_i\mu_j$-closed set is called $\mu_i\mu_j$**-open**.

Proposition 2.2.8. [3] Let (X, μ_1, μ_2) be a bigeneralized topological space and A subset of X. Then A is $\mu_i\mu_j$-closed if and only if A is both μ-closed in (X, μ_i) and (X, μ_j).

Proposition 2.2.9. [3] Let (X, μ_1, μ_2) be a bigeneralized topological space. If A and B are $\mu_i\mu_j$-closed, then $A \cap B$ is $\mu_i\mu_j$-closed.

Proposition 2.2.10. [3] Let (X, μ_1, μ_2) be a bigeneralized topological space. Then A is $\mu_i\mu_j$-open if and only if $A = i_{\mu_i}(i_{\mu_j}(A))$.

Proposition 2.2.11. [3] Let (X, μ_1, μ_2) be a bigeneralized topological space. If A and B are $\mu_i\mu_j$-open, then $A \cup B$ is $\mu_i\mu_j$-open.

Next, we will recall the concept and some fundamental properties of boundary set on bigeneralized topological spaces.

Definition 2.2.12. [13] Let (X, μ_1, μ_2) be a bigeneralized topological space, A be a subset of X and $x \in X$. We called x is (i,j)-**μ-boundary point of** A if $x \in c_{\mu_i}(c_{\mu_j}(A)) \cap c_{\mu_i}(c_{\mu_j}(X - A))$. We denote the set of all (i,j)-μ-boundary point of A by $\mu Bdr_{ij}(A)$.

From definition we have $\mu Bdr_{ij}(A) = c_{\mu_i}(c_{\mu_j}(A)) \cap c_{\mu_i}(c_{\mu_j}(X - A))$.

Lemma 2.2.13. [13] Let (X, μ_1, μ_2) be a bigeneralized topological space and A be a subset of X. Then $\mu Bdr_{ij}(A) = \mu Bdr_{ij}(X - A)$.

Theorem 2.2.14. [13] Let (X, μ_1, μ_2) be a bigeneralized topological space and A, B be a subset of X. We have the following statements;

1. $\mu Bdr_{ij}(A) = c_{\mu_i}(c_{\mu_j}(A)) - i_{\mu_i}(i_{\mu_j}(A))$;
2. $\mu Bdr_{ij}(A) \cap i_{\mu_i}(i_{\mu_j}(A)) = \emptyset$;
3. $\mu Bdr_{ij}(A) \cap i_{\mu_i}(i_{\mu_j}(X - A)) = \emptyset$;
4. $c_{\mu_i}(c_{\mu_j}(A)) = \mu Bdr_{ij}(A) \cup i_{\mu_i}(i_{\mu_j}(A))$;
5. $X = i_{\mu_i}(i_{\mu_j}(X - A)) \cup \mu Bdr_{ij}(A) \cup i_{\mu_i}(i_{\mu_j}(A))$ is a pairwise disjoint union.

Theorem 2.2.15. [13] Let (X, μ_1, μ_2) be a bigeneralized topological space and A be a subset of X. We have;

1. A is $\mu_i\mu_j$-closed if and only if $\mu Bdr_{ij}(A) \subseteq A$.
2. A is $\mu_i\mu_j$-open if and only if $\mu Bdr_{ij}(A) \subseteq X - A$.

Theorem 2.2.16. [13] Let (X, μ_1, μ_2) be a bigeneralized topological space and A be a subset of X. Then $\mu Bdr_{ij}(A) = \emptyset$ if and only if A is $\mu_i\mu_j$-closed and $\mu_i\mu_j$-open.

Next, we will recall the concept and some fundamental properties of exterior set on bigeneralized topological spaces.

Definition 2.2.17. [14] Let (X, μ_1, μ_2) be a bigeneralized topological space, A be a subset of X and $x \in X$. We called x is $\mu_i\mu_j$**-exterior point of** A if $x \in i_{\mu_i}(i_{\mu_j}(X - A))$. We denote the set of all $\mu_i\mu_j$-exterior point of A by $\mu Ext_{ij}(A)$.

From definition we have $\mu Ext_{ij}(A) = X - c_{\mu_i}(c_{\mu_j}(A))$.

Lemma 2.2.18. [14] Let (X, μ_1, μ_2) be a bigeneralized topological space and A be a subset of X. We have;

1. $\mu Ext_{ij}(A) \cap A = \emptyset$.
2. $\mu Ext_{ij}(X) = \emptyset$.

Theorem 2.2.19. [14] Let (X, μ_1, μ_2) be a bigeneralized topological space and A, B be two subsets of X. If $A \subseteq B$, then $\mu Ext_{ij}(B) \subseteq \mu Ext_{ij}(A)$.

Theorem 2.2.20. [14] Let (X, μ_1, μ_2) be a bigeneralized topological space and A be a subset of X. A is $\mu_i\mu_j$-closed if and only if $\mu Ext_{ij}(A) = X - A$.

Corollary 2.2.21. [14] Let (X, μ_1, μ_2) be a bigeneralized topological space and A be a subset of X. If A is $\mu_i\mu_j$-closed, then $\mu Ext_{ij}(X - \mu Ext_{ij}(A)) = \mu Ext_{ij}(A)$.

Theorem 2.2.22. [14] Let (X, μ_1, μ_2) be a bigeneralized topological space and A, B be two subsets of X. We have; If A and B are $\mu_i\mu_j$-closed, then $\mu Ext_{ij}(A) \cup \mu Ext_{ij}(B) = \mu Ext_{ij}(A \cap B)$.

Theorem 2.2.23. [14] Let (X, μ_1, μ_2) be a bigeneralized topological space and A be a subset of X. A is $\mu_i\mu_j$-open if and only if $\mu Ext_{ij}(X - A) = A$.

Corollary 2.2.24. [14] Let (X, μ_1, μ_2) be a bigeneralized topological space and A, B be subset of X. If A and B are $\mu_i\mu_j$-open, then $\mu Ext_{ij}(X - (A \cup B)) = A \cup B$.

Finally, we will recall the concept of dense sets on bigeneralized topological spaces and some fundamental of their properties.

Definition 2.2.25. [12] Let (X, μ_1, μ_2) be a bigeneralized topological spaces, A be a subset of X. A is called $\mu_i\mu_j$**-dense** set in X if $X = c_{\mu_i}(c_{\mu_j}(A))$.

Theorem 2.2.26. [12] Let (X, μ_1, μ_2) be a bigeneralized topological space and A be a subset of X. A is $\mu_i\mu_j$ -dense set in X if and only if $\mu Ext_{ij}(A) = \emptyset$.

Theorem 2.2.27. [12] Let (X, μ_1, μ_2) be a bigeneralized topological spaces and A be a subset of X. If A is $\mu_i\mu_j$-dense set in X then for any non-empty $\mu_i\mu_j$-closed subset F of X such that $A \subseteq F$, we have $F = X$.

Theorem 2.2.28. [12] Let (X, μ_1, μ_2) be a bigeneralized topological spaces and A be a subset of X. If for any non-empty $\mu_i\mu_j$-closed subset F of X such that $A \subseteq F$, then $F = X$ if and only if $G \cap A \neq \emptyset$ for any non-empty $\mu_i\mu_j$-open subset G of X.

Corollary 2.2.29. [12] Let (X, μ_1, μ_2) be a bigeneralized topological spaces and A be a subset of X. If A is $\mu_i\mu_j$-dense set in X, then $G \cap A \neq \emptyset$ for any non-empty $\mu_i\mu_j$ -open subset G of X.

Theorem 2.2.30. [12] Let (X, μ_1, μ_2) be a bigeneralized topological spaces and A be a subset of X, then $\mu Bdr_{ij}(A) = c_{\mu_i}(c_{\mu_j}(X - A))$ if and only if A is $\mu_i\mu_j$-dense set in X.

Theorem 2.2.31. [12] Let (X, μ_1, μ_2) be a bigeneralized topological spaces and A be a subset of X, then A is $\mu_i\mu_j$-open and $\mu_i\mu_j$-dense set in X if and only if $\mu Bdr_{ij}(A) = X - A$,

2.3 Boundary sets, Exterior sets and Dense sets in biminimal structure spaces

Definition 2.3.1. [10] Let X be a nonempty set and $P(X)$ the power set of X. A subfamily m of $P(X)$ is called a **minimal structure** (briefly **m-structure**) on X if $\emptyset \in m$ and $X \in m$

By (X, m), we denote a nonempty set X with an m-structure m on X and it is called an **m-space**. Each member of m is said to be **m-open** and the complement of an m-open set is said to be **m-closed**.

Definition 2.3.2. [10] Let X be a nonempty set and m an m-structure on X. For a subset A of X, the **m-closure of** A and the **m-interior of** A are defined as follows:

1. $c_m(A) = \cap\{F : A \subseteq F, X - F \in m\}$.

2. $c_m(A) = \cup\{U : U \subseteq A, U \in m\}$.

Lemma 2.3.3. [10] Let X be a nonempty set and m a minimal structure on X. For subset A and B of X, the following properties hold:

1. $c_m(X - A) = X - i_m(A)$ and $i_m(X - A) = X - c_m(A)$.

2. If $(X - A) \in m_X$, then $c_m(A) = A$ and if $A \in m_X$, then $i_m(A) = A$.

3. $c_m(\emptyset) = \emptyset, c_m(X) = X, i_m(\emptyset) = \emptyset$ and $i_m(X) = X$.

4. If $A \subseteq B$, then $c_m(A) \subseteq c_m(B)$ and $i_m(A) \subseteq i_m(B)$.

5. $A \subseteq c_m(A)$ and $i_m(A) \subseteq A$.

6. $c_m(c_m(A)) = c_m(A)$ and $i_m(i_m(A)) = i_m(A)$.

Lemma 2.3.4. [10] Let X be a nonempty set with a minimal structure m and A a subset of X. Then $x \in c_m(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m$ containing x.

Definition 2.3.5. [10] An m-structure m on a nonempty set X is said to have **property** B if the union of any family of subsets belong to m belong to m.

Lemma 2.3.6. [10] Let X be a nonempty set and m an m-structure on X sastisfying property B. For a subset A of X, the following properties hold:

1. $A \in m$ if and only if $i_m(A) = A$.

2. If A is m-closed if and only if $c_m(A) = A$.

3. $i_m(A) \in m$ and $c_m(A) \in m$-closed.

Next, we will recall the concept of biminimal structure spaces and some properties of m_1m_2-closed sets and m_1m_2-open sets in biminimal structure spaces.

Definition 2.3.7. [2] Let X be a nonempty set and m_1, m_2 be minimal structures on X. A triple (X, m_1, m_2) is called a **biminimal structure space** (briefly **bi-m space**).

Let (X, m_1, m_2) be a biminimal structure space and A be a subset of X. The m-closure and m-interior of A with respect to m_i are denote by $c_{m_i}(A)$ and $i_{m_i}(A)$, respectively, for $i = 1, 2$.

Next, let $i, j \in \{1, 2\}$ where $i \neq j$.

Definition 2.3.8. [2] A subset A of a biminimal structure space (X, m_1, m_2) is called m_im_j**-closed** if $c_{m_i}(c_{m_j}(A)) = A$. The complement of m_im_j-closed set is called m_im_j**-open**.

Proposition 2.3.9. [2] Let m_1 and m_2 be m-structures on X satisfying property B. Then A is a m_im_j-closed subset of a biminimal structure space (X, m_1, m_2) if and only if A is both m_i-closed and m_j-closed.

Proposition 2.3.10. [2] Let (X, m_1, m_2) be a biminimal structure space. If A and B are m_im_j-closed subsets of (X, m_1, m_2), then $A \cap B$ is m_im_j-closed.

Proposition 2.3.11. [2] Let (X, m_1, m_2) be a biminimal structure space. Then A is a m_im_j-open subset of (X, m_1, m_2) if and only if $A = i_{m_i}(i_{m_j}(A))$.

Proposition 2.3.12. [2] Let (X, m_1, m_2) be a biminimal structure space. If A and B are m_im_j-open subsets of (X, m_1, m_2), then $A \cup B$ is m_im_j-open.

Next, we will recall the concept and some fundamental properties of boundary set in biminimal structure space.

Definition 2.3.13. [15] Let (X, m_1, m_2) be a biminimal structure space, A be a subset of X and $x \in X$. We called x is (i,j)**-m-boundary point of** A if $x \in$

$c_{m_i}(c_{m_j}(A)) \cap c_{m_i}(c_{m_j}(X - A))$. We denote the set of all (i,j)-m-boundary point of A by $mBdr_{ij}(A)$.

From definition we have $mBdr_{ij}(A) = c_{m_i}(c_{m_j}(A)) \cap c_{m_i}(c_{m_j}(X - A))$.

Lemma 2.3.14. [15] Let (X, m_1, m_2) be a biminimal structure space and A be a subset of X, then $mBdr_{ij}(A) = mBdr_{ij}(X - A)$.

Theorem 2.3.15. [15] Let (X, m_1, m_2) be a biminimal structure space and A, B be a subset of X. We have the following statements;

1. $mBdr_{ij}(A) = c_{m_i}(c_{m_j}(A)) - i_{m_i}(i_{m_j}(A))$;
2. $mBdr_{ij}(A) \cap i_{m_i}(i_{m_j}(A)) = \emptyset$;
3. $mBdr_{ij}(A) \cap i_{m_i}(i_{m_j}(X - A)) = \emptyset$;
4. $c_{m_i}(c_{m_j}(A)) = mBdr_{ij}(A) \cup i_{m_i}(i_{m_j}(A))$;
5. $X = i_{m_i}(i_{m_j}(A)) \cup mBdr_{ij}(A) \cup i_{m_i}(i_{m_j}(X - A))$ is a pairwise disjoint union;
6. $c_{m_i}(c_{m_j}(A)) = mBdr_{ij}(A) \cup A$.

Theorem 2.3.16. [15] Let (X, m_1, m_2) be a biminimal structure space and A be a subset of X. We have;

1. A is m_im_j-closed if and only if $mBdr_{ij}(A) \subseteq A$.
2. A is m_im_j-open if and only if $mBdr_{ij}(A) \subseteq (X - A)$.

Theorem 2.3.17. [15] Let (X, m_1, m_2) be a biminimal structure space and A be a subset of X. Then $mBdr_{ij}(A) = \emptyset$ if and only if A is m_im_j-closed and m_im_j-open.

Next, we will recall the concept of dense sets in biminimal structure spaces and some fundamental of their properties.

Definition 2.3.18. [16] Let (X, m_1, m_2) be a biminimal structure space and A be a subset of X. A is called m_im_j**-dense** set in X if $X = c_{m_i}(c_{m_j}(A))$.

Theorem 2.3.19. [16] Let (X, m_1, m_2) be a biminimal structure space and A be a subset of X. If A is m_im_j-dense set in X then for any non-empty m_im_j-closed subset F of X such that $A \subseteq F$, we have $F = X$.

Theorem 2.3.20. [16] Let (X, m_1, m_2) be a biminimal structure space and A be a subset of X. If $i_{m_i}(i_{m_j}(X - A)) = \emptyset$. then for any non-empty m_im_j-closed subset F of X such that $A \subseteq F$, we have $F = X$.

Theorem 2.3.21. [16] Let (X, m_1, m_2) be a biminimal structure space and A be a subset of X. If A is m_im_j-dense set in X, then $G \cap A \neq \emptyset$ for any non-empty m_im_j-open subset G of X.

Theorem 2.3.22. [16] Let (X, m_1, m_2) be a biminimal structure space and A be a subset of X. If $i_{m_i}(i_{m_j}(X - A)) = \emptyset$. Then $G \cap A \neq \emptyset$ for any non-empty m_im_j-open subset G of X.

Theorem 2.3.23. [16] Let (X, m_1, m_2) be a biminimal structure space and A be a subset of X. If for any non-empty m_im_j-closed subset F of X such that $A \subseteq F$, then $F = X$ if and only if $G \cap A \neq \emptyset$ for any non-empty m_im_j-open subset G of X.

Theorem 2.3.24. [16] Let (X, m_1, m_2) be a biminimal structure space and A be a subset of X. $i_{m_i}(i_{m_j}(X - A)) = \emptyset$ if and only if A is m_im_j-dense set in X.

Finally we will recall the concept and some fundamental properties of exterior set in biminimal structure space.

Definition 2.3.25. [17] Let (X, m_1, m_2) be a biminimal structure space, A be a subset of X and $x \in X$. We called x is m_im_j**-exterior point of** A if $x \in i_{m_i}(i_{m_j}(X - A))$. We denote the set of all m_im_j-exterior point of A by $mExt_{ij}(A)$.

From definition we have $mExt_{ij}(A) = X - c_{m_i}(c_{m_j}(A))$.

Lemma 2.3.26. [17] Let (X, m_1, m_2) be a biminimal structure space and A be a subset of X. We have;

1. $mExt_{ij}(A) \cap A = \emptyset$.
2. $mExt_{ij}(\emptyset) = X$.
3. $mExt_{ij}(X) = \emptyset$.

Theorem 2.3.27. [17] Let (X, m_1, m_2) be a biminimal structure space and A, B be a subset of X. If $A \subseteq B$, then $mExt_{ij}(B) \subseteq mExt_{ij}(A)$.

Theorem 2.3.28. [17] Let (X, m_1, m_2) be a biminimal structure space and A be a subset of X. A is $m_i m_j$-closed if and only if $mExt_{ij}(A) = X - A$.

Corollary 2.3.29. [17] Let (X, m_1, m_2) be a biminimal structure space and A be a subset of X. A is $m_i m_j$-open if and only if $mExt_{ij}(X - A) = A$.

Theorem 2.3.30. [17] Let (X, m_1, m_2) be a biminimal structure space and A be a subset of X. If A is $m_i m_j$-closed, then $mExt_{ij}(X - mExt_{ij}(A)) = mExt_{ij}(A)$.

Theorem 2.3.31. [17] Let (X, m_1, m_2) be a biminimal structure space and A, B be two subsets of X. We have;

1. $mExt_{ij}(A) \cup mExt_{ij}(B) \subseteq mExt_{ij}(A \cap B)$.
2. If A and B are $m_i m_j$-closed, then $mExt_{ij}(A) \cup mExt_{ij}(B) = mExt_{ij}(A \cap B)$.

Theorem 2.3.32. [17] Let (X, m_1, m_2) be a biminimal structure space and A, B be two subsets of X. We have;

1. $mExt_{ij}(A \cup B) \subseteq mExt_{ij}(A) \cap mExt_{ij}(B)$.
2. If A and B are $m_i m_j$-open, then $mExt_{ij}(A \cup B) = mExt_{ij}(A) \cap mExt_{ij}(B)$.

2.4 Bi-weak structure spaces

Definition 2.4.1. [8] Let X be a nonempty set and $P(X)$ the power set of X. A subfamily w of $P(X)$ is called a **weak structure** (briefly WS) on X if $\varnothing \in w$.

By (X, w) we denote a nonempty set X with a WS w on X and it is called a w**-space**. The elements of w are called w-open sets and the complements are called w-closed sets.

Let w be a weak structure on X and $A \subseteq X$, the w-closure of A, denoted by $c_w(A)$ and w-interior of A denoted by $i_w(A)$. We define $c_w(A)$ as the intersection of all w-closed sets containing A and $i_w(A)$ as the union of all w-open subsets of A.

Theorem 2.4.2. [8] If w is a WS on X and $A, B \subseteq X$. Then

1. $A \subseteq c_w(A)$ and $i_w(A) \subseteq A$;
2. If $A \subseteq B$, then $c_w(A) \subseteq c_w(B)$ and $i_w(A) \subseteq i_w(B)$;

3. $c_w(c_w(A)) = c_w(A)$ and $i_w(i_w(A)) = i_w(A)$;

4. $c_w(X - A)) = X - i_w(A)$ and $i_w(X - A) = X - c_w(A)$;

5. $x \in i_w(A)$ if and only if there is a w-open set V such that $x \in V \subseteq A$;

6. $x \in c_w(A)$ if and only if $V \cap A \neq \emptyset$ for any w-open set V containing x;

7. If $A \in w$, then $A = i_w(A)$ and if A is w-closed, then $A = c_w(A)$.

Next we will recall the concept of bi-weak structure spaces and some fundamental properties of closed sets and open sets in bi-weak structure spaces.

Definition 2.4.3. [11] Let X be a nonempty set and w_1, w_2 be two weak structures on X. A triple (X, w_1, w_2) is called a **bi-weak structure space** (briefly **bi-w space**).

Let (X, w_1, w_2) be a bi-w space and A be a subset of X. The w-closure and w-interior of A with respect to w_j are denoted by $c_{w_j}(A)$ and $i_{w_j}(A)$, respectively, for $j \in \{1, 2\}$.

Definition 2.4.4. [11] A subset A of a bi-weak structure space (X, w_1, w_2) is called **closed** if $A = c_{w_1}(c_{w_2}(A))$. The complement of a closed set is called **open**.

Theorem 2.4.5. [11] Let (X, w_1, w_2) be a bi-w space and A be a subset of X. Then the following are equivalent:

1. A is closed;

2. $A = c_{w_1}(A)$ and $A = c_{w_2}(A)$;

3. $A = c_{w_1}(c_{w_2}(A))$.

Proposition 2.4.6. [11] Let (X, w_1, w_2) be a bi-w space and $A \subseteq X$. If A is both w_1-closed and w_2-closed, then A is a closed set in the bi-w space (X, w_1, w_2).

Proposition 2.4.7. [11] Let (X, w_1, w_2) be a bi-w space. If A_α is closed for all $\alpha \in \Lambda \neq \emptyset$, then $\bigcap\limits_{\alpha \in \Lambda} A_\alpha$ is closed.

Theorem 2.4.8. [11] Let (X, w_1, w_2) be a bi-w space and A be a subset of X. Then the following are equivalent:

1. A is open;
2. $A = i_{w_1}(i_{w_2}(A))$;
3. $A = i_{w_1}(A)$ and $A = i_{w_2}(A)$;
4. $A = i_{w_2}(i_{w_1}(A))$.

Proposition 2.4.9. [11] Let (X, w_1, w_2) be a bi-w space. If A_α is open for all $\alpha \in \Lambda \neq \emptyset$, then $\bigcup_{\alpha \in \Lambda} A_\alpha$ is open.

CHAPTER 3

Boundary sets, exterior sets and dense sets in bi-weak structure spaces

In this section, we introduce the concepts of boundary sets, exterior sets and dense sets in bi-weak structure space and study some fundamental properties. Next, let $i, j \in \{1, 2\}$ be such that $i \neq j$.

In this chapter, we shall call closed and open in a bi-w space that bi-w-closed and bi-w-open, respectively.

3.1 Boundary sets in bi-weak structure spaces

Definition 3.1.1. Let (X, w^1, w^2) be a bi-w space, A be a subset of X and $x \in X$. We called x is a $w_i w_j$*-boundary point* of A if $x \in c_{w^i}(c_{w^j}(A)) \cap c_{w^i}(c_{w^j}(X - A))$. We denote the set of all $w_i w_j$-boundary points of A by $wBdr_{ij}(A)$.

Remark 3.1.2. From the above definition, it is easy to verify that $wBdr_{ij}(A) = c_{w^i}(c_{w^j}(A)) \cap c_{w^i}(c_{w^j}(X - A))$.

Example 3.1.3. Let $X = \{1, 2, 3\}$. Define weak structures w^1 and w^2 on X as follows: $w^1 = \{\varnothing, \{1\}, \{2, 3\}\}$ and $w^2 = \{\varnothing, \{3\}, \{1, 2\}\}$. Hence $wBdr_{12}(\{1\}) = X$ and $wBdr_{21}(\{1\}) = \{1, 2\}$.

Example 3.1.4. Let $X = \mathbb{R}$. Define weak structures w^1 and w^2 on X as follows: $w^1 = \{\varnothing, \{1\}, \{2, 3\}\}$ and $w^2 = \{\varnothing, \{3\}, \{1, 2\}\}$. Hence $wBdr_{12}(\{1\}) = X$ and $wBdr_{21}(\{1\}) = X$.

Lemma 3.1.5. Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then $wBdr_{ij}(X - A) = wBdr_{ij}(A)$.

Proof. Since $wBdr_{ij}(X - A) = c_{w^i}(c_{w^j}(X - A)) \cap c_{w^i}(c_{w^j}(X - (X - A)))$
and $wBdr_{ij}(A) = c_{w^i}(c_{w^j}(A)) \cap c_{w^i}(c_{w^j}(X - A))$,
$wBdr_{ij}(X - A) = wBdr_{ij}(A)$. □

Theorem 3.1.6. Let (X, w^1, w^2) be a bi-w space and $A \subseteq X$. Then the following statements hold;

1. $wBdr_{ij}(A) = c_{w^i}(c_{w^j}(A)) - i_{w^i}(i_{w^j}(A))$;
2. $wBdr_{ij}(A) \cap i_{w^i}(i_{w^j}(A)) = \varnothing$;
3. $wBdr_{ij}(A) \cap i_{w^i}(i_{w^j}(X - A)) = \varnothing$;
4. $c_{w^i}(c_{w^j}(A)) = wBdr_{ij}(A) \cup i_{w^i}(i_{w^j}(A))$;
5. $X = i_{w^i}(i_{w^j}(A)) \cup wBdr_{ij}(A) \cup i_{w^i}(i_{w^j}(X - A))$ is a pairwise disjoint union;
6. $c_{w^i}(c_{w^j}(A)) = wBdr_{ij}(A) \cup A$.

Proof. 1.
$$\begin{aligned} wBdr_{ij}(A) &= c_{w^i}(c_{w^j}(A)) \cap c_{w^i}(c_{w^j}(X - A)) \\ &= c_{w^i}(c_{w^j}(A)) \cap c_{w^i}(X - i_{w^j}(A)) \\ &= c_{w^i}(c_{w^j}(A)) \cap (X - i_{w^i}(i_{w^j}(A))) \\ &= c_{w^i}(c_{w^j}(A)) - i_{w^i}(i_{w^j}(A)). \end{aligned}$$

2. From (1), we obtain that
$$\begin{aligned} wBdr_{ij}(A) \cap i_{w^i}(i_{w^j}(A)) &= [c_{w^i}(c_{w^j}(A)) - i_{w^i}(i_{w^j}(A))] \cap i_{w^i}(i_{w^j}(A)) \\ &= \varnothing. \end{aligned}$$

3.
$$\begin{aligned} wBdr_{ij}(A) \cap i_{w^i}(i_{w^j}(X - A)) &= wBdr_{ij}(X - A) \cap i_{w^i}(i_{w^j}(X - A)) \\ &= \varnothing. \end{aligned}$$

4.
$$\begin{aligned} wBdr_{ij}(A) \cup i_{w^i}(i_{w^j}(A)) &= [c_{w^i}(c_{w^j}(A)) - i_{w^i}(i_{w^j}(A))] \cup i_{w^i}(i_{w^j}(A)) \\ &= c_{w^i}(c_{w^j}(A)) \cup i_{w^i}(i_{w^j}(A)) \\ &= c_{w^i}(c_{w^j}(A)). \end{aligned}$$

5.
$$\begin{aligned} i_{w^i}(i_{w^j}(A)) \cup wBdr_{ij}(A) \cup i_{w^i}(i_{w^j}(X - A)) &= c_{w^i}(c_{w^j}(A)) \cup i_{w^i}(i_{w^j}(X - A)) \\ &= c_{w^i}(c_{w^j}(A)) \cup i_{w^i}(X - c_{w^j}(A)) \\ &= c_{w^i}(c_{w^j}(A)) \cup X - c_{w^i}(c_{w^j}(A)) \\ &= X. \end{aligned}$$

By (2) and (3), we have $wBdr_{ij}(A) \cap i_{w^i}(i_{w^j}(A)) = \varnothing$ and $wBdr_{ij}(A) \cap$

$i_{w^i}(i_{w^j}(X-A)) = \varnothing$.

Now, we will show that $i_{w^i}(i_{w^j}(A)) \cap i_{w^i}(i_{w^j}(X-A)) = \varnothing$.

Since $i_{w^i}(i_{w^j}(A)) \subseteq A$ and $i_{w^i}(i_{w^j}(X-A)) \subseteq X-A$,

we also have $i_{w^i}(i_{w^j}(A)) \cap i_{w^i}(i_{w^j}(X-A)) = \varnothing$.

Therefore $X = i_{w^i}(i_{w^j}(A)) \cup wBdr_{ij}(A) \cup i_{w^i}(i_{w^j}(X-A))$ is a pairwise disjoint union.

6. $$\begin{aligned} wBdr_{ij}(A) \cup A &= [c_{w^i}(c_{w^j}(A)) \cap c_{w^i}(c_{w^j}(X-A))] \cup A \\ &= [c_{w^i}(c_{w^j}(A)) \cup A] \cap [c_{w^i}(c_{w^j}(X-A)) \cup A] \\ &= c_{w^i}(c_{w^j}(A)) \cap [c_{w^i}(X - i_{w^j}(A)) \cup A] \\ &= c_{w^i}(c_{w^j}(A)) \cap [(X - i_{w^i}(i_{w^j}(A))) \cup A] \\ &= c_{w^i}(c_{w^j}(A)) \cap X \\ &= c_{w^i}(c_{w^j}(A)). \end{aligned}$$ □

Theorem 3.1.7. Let (X, w^1, w^2) be a bi-w space and $A \subseteq X$. Then

1. A is bi-w-closed if and only if $wBdr_{ij}(A) \subseteq A$.

2. A is bi-w-open if and only if $wBdr_{ij}(A) \subseteq X - A$.

Proof. 1. ($\Rightarrow$) Assume that A is bi-w-closed.

Thus $c_{w^i}(c_{w^j}(A)) = A$, and so

$$\begin{aligned} wBdr_{ij}(A) \cap (X-A) &= c_{w^i}(c_{w^j}(A)) \cap c_{w^i}(c_{w^j}(X-A)) \cap (X-A) \\ &= A \cap c_{w^i}(c_{w^j}(X-A)) \cap (X-A) \\ &= \varnothing. \end{aligned}$$

Therefore $wBdr_{ij}(A) \subseteq A$.

($\Leftarrow$) Assume that $wBdr_{ij}(A) \subseteq A$.

Thus $wBdr_{ij}(A) \cap (X-A) = \varnothing$, and also $c_{w^i}(c_{w^j}(A)) \cap c_{w^i}(c_{w^j}(X-A)) \cap (X-A) = \varnothing$.

Since $X - A \subseteq c_{w^i}(c_{w^j}(X-A))$, we have $c_{w^i}(c_{w^j}(A)) \cap (X-A) = \varnothing$.

Then $c_{w^i}(c_{w^j}(A)) \subseteq A$.

But $A \subseteq c_{w^i}(c_{w^j}(A))$.

Consequently $A = c_{w^i}(c_{w^j}(A))$.

Hence A is bi-w-closed.

2. $(\Rightarrow)$ Assume that A is bi-w-open.

   Thus $i_{w^i}(i_{w^j}(A)) = A$, and so $wBdr_{ij}(A) \cap A = c_{w^i}(c_{w^j}(A)) \cap c_{w^i}(c_{w^j}(X-A)) \cap A = c_{w^i}(c_{w^j}(A)) \cap (X - i_{w^i}(i_{w^j}(A))) \cap A = c_{w^i}(c_{w^j}(A)) \cap (X - A) \cap A = \varnothing$.

   Therefore $wBdr_{ij}(A) \subseteq X - A$.

   $(\Leftarrow)$ Assume that $wBdr_{ij}(A) \subseteq X - A$.

   Thus $wBdr_{ij}(A) \cap A = \varnothing$,

   and also $c_{w^i}(c_{w^j}(A)) \cap c_{w^i}(c_{w^j}(X - A)) \cap A = \varnothing$.

   Then $c_{w^i}(c_{w^j}(A)) \cap (X - i_{w^i}(i_{w^j}(A))) \cap A = \varnothing$.

   Since $A \subseteq c_{w^i}(c_{w^j}(A))$,

   we have $(X - i_{w^i}(i_{w^j}(A))) \cap A = \varnothing$.

   Thus $A \subseteq i_{w^i}(i_{w^j}(A))$.

   Clearly $i_{w^i}(i_{w^j}(A)) \subseteq A$.

   Consequently $A = i_{w^i}(i_{w^j}(A))$.

   Hence A is bi-w-open. □

Theorem 3.1.8. Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then $wBdr_{ij}(A) = \varnothing$ if and only if A is bi-w-closed and bi-w-open.

Proof. $(\Rightarrow)$ Assume that $wBdr_{ij}(A) = \varnothing$.

Thus we have $wBdr_{ij}(A) \subseteq A$ and $wBdr_{ij}(A) \subseteq X - A$.

By Theorem 3.1.6, we have A is bi-w-closed and bi-w-open.

$(\Leftarrow)$ Assume that A is bi-w-closed and bi-w-open.

By Theorem 3.1.6, we have $wBdr_{ij}(A) \subseteq A$ and $wBdr_{ij}(A) \subseteq X - A$.

Therefore $wBdr_{ij}(A) \subseteq A \cap (X - A) = \varnothing$.

Hence $wBdr_{ij}(A) = \varnothing$. □

3.2 Exterior sets in bi-weak structure spaces

Definition 3.2.1. Let (X, w^1, w^2) be a bi-w space, A be a subset of X and $x \in X$. We called x is a w^iw^j*-exterior point* of A if $x \in i_{w^i}(i_{w^j}(X - A))$. We denote the set of all w^iw^j-exterior points of A by $wExt_{ij}(A)$.

Remark 3.2.2. From the previous definition, it is easy to verify that $wExt_{ij}(A) = X - c_{w^i}(c_{w^j}(A))$.

Example 3.2.3. Let $X = \{1, 2, 3\}$ Define weak structures w^1 and w^2 on X as follows: $w^1 = \{\varnothing, \{1\}, \{2, 3\}\}$ and $w^2 = \{\varnothing, \{3\}, \{1, 2\}\}$. Hence $wExt_{12}(\{1\}) = X - c_{w^i}(c_{w^j}(\{1\})) = \varnothing$ and $wExt_{21}(\{1\}) = X - c_{w^i}(c_{w^j}(\{1\})) = \{3\}$.

Lemma 3.2.4. Let (X, w^1, w^2) be a bi-w space and $A \subseteq X$. Then

1. $wExt_{ij}(A) \cap A = \varnothing$.
2. $wExt_{ij}(X) = \varnothing$.

Proof. 1. Since $A \subseteq c_{w^i}(c_{w^j}(A))$, $(X - c_{w^i}(c_{w^j}(A))) \cap A \subseteq (X - A) \cap A = \varnothing$.
From $wExt_{ij}(A) = X - c_{w^i}(c_{w^j}(A))$.
Therefore $wExt_{ij}(A) \cap A = \varnothing$.
2. From (1), and $wExt_{ij}(X) \subseteq X$, we have $wExt_{ij}(X) = wExt_{ij}(X) \cap X = \varnothing$. □

Theorem 3.2.5. Let (X, w^1, w^2) be a bi-w space and A, B be two subsets of X. If $A \subseteq B$, then $wExt_{ij}(B) \subseteq wExt_{ij}(A)$.

Proof. Assume that $A \subseteq B$.
Thus $c_{w^i}(c_{w^j}(A)) \subseteq c_{w^i}(c_{w^j}(B))$ and so $X - c_{w^i}(c_{w^j}(B)) \subseteq X - c_{w^i}(c_{w^j}(A))$.
Hence $wExt_{ij}(B) \subseteq wExt_{ij}(A)$. □

Theorem 3.2.6. Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then A is bi-w-closed if and only if $wExt_{ij}(A) = X - A$.

Proof. $(\Rightarrow)$ Assume that A is bi-w-closed.
Then $A = c_{w^i}(c_{w^j}(A))$.
Therefore $wExt_{ij}(A) = X - c_{w^i}(c_{w^j}(A)) = X - A$.
$(\Leftarrow)$ Assume that $wExt_{ij}(A) = X - A$.
Thus $X - c_{w^i}(c_{w^j}(A)) = X - A$.
Consequently $c_{w^i}(c_{w^j}(A)) = A$.
Hence A is bi-w-closed. □

Corollary 3.2.7. Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then A is bi-w-open if and only if $wExt_{ij}(X - A) = A$.

Proof. ($\Rightarrow$) Assume that A is bi-w-open.

Thus $X - A$ is bi-w-closed.

By Theorem 3.2.6, $wExt_{ij}(X - A) = X - c_{w^i}(c_{w^j}(X - A)) = X - (X - A) = A$.

Therefore $wExt_{ij}(X - A) = A$.

($\Leftarrow$) Assume that $wExt_{ij}(X - A) = A$.

Then $wExt_{ij}(X - A) = X - (X - A)$.

By Theorem 3.2.6. $X - A$ is bi-w-closed.

Hence A is bi-w-open. □

Corollary 3.2.8. Let (X, w^1, w^2) be a bi-w space and A be a subset of X. If A is bi-w-closed, then $wExt_{ij}(X - wExt_{ij}(A)) = wExt_{ij}(A)$.

Proof. Assume that A is bi-w-closed.

By Theorem 3.2.6, $wExt_{ij}(A) = X - A$.

Hence $wExt_{ij}(X - wExt_{ij}(A)) = wExt_{ij}(A)$. □

Theorem 3.2.9. Let (X, w^1, w^2) be a bi-w space and A, B be two subsets of X. Then

1. $wExt_{ij}(A) \cup wExt_{ij}(B) \subseteq wExt_{ij}(A \cap B)$.
2. If A and B are bi-w-closed, then $wExt_{ij}(A) \cup wExt_{ij}(B) = wExt_{ij}(A \cap B)$.

Proof. 1. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by Theorem 3.2.5,
we have $wExt_{ij}(A) \subseteq wExt_{ij}(A \cap B)$ and $wExt_{ij}(B) \subseteq wExt_{ij}(A \cap B)$.
It follow that $wExt_{ij}(A) \cup wExt_{ij}(B) \subseteq wExt_{ij}(A \cap B)$.

2. Assume that A and B are bi-w-closed.
By Theorem 3.2.6, $wExt_{ij}(A) = X - A$ and $wExt_{ij}(B) = X - B$.
Moreover, $A \cap B$ is bi-w-closed. By Theorem 3.2.6,
Thus
$$\begin{aligned} wExt_{ij}(A \cap B) &= X - (A \cap B) \\ &= (X - A) \cup (X - B) \\ &= wExt_{ij}(A) \cup wExt_{ij}(B). \end{aligned}$$
□

Example 3.2.10. Let $X = \{1, 2, 3\}$. Define weak structures w^1 and w^2 on X as follows: $w^1 = \{\varnothing, \{1\}, \{2, 3\}\}$ and $w^2 = \{\varnothing, \{2\}, \{1, 3\}\}$. Hence $wExt_{12}(\{1\} \cap \{2\}) = X$, and $wExt_{12}(\{1\}) = X - c_{w^1}(c_{w^2}(\{1\})) = \varnothing$ and $wExt_{12}(\{2\}) = X - c_{w^1}(c_{w^2}(\{2\})) = \{1\}$. Therefore $wExt_{12}(\{1\}) \cup wExt_{12}(\{2\}) \neq wExt_{12}(\{1\} \cap \{2\})$.

Corollary 3.2.11. Let (X, w^1, w^2) be a bi-w space and A, B be two subsets of X. If A and B are bi-w-open, then $wExt_{ij}(X - (A \cup B)) = A \cup B$.

Proof. Since A and B are bi-w-open.
Then $A \cup B$ is bi-w-open.
By Corollary 3.2.7, we have $wExt_{ij}(X - (A \cup B)) = A \cup B$. □

3.3 Dense sets in bi-weak structure spaces

Definition 3.3.1. Let (X, w^1, w^2) be a bi-w space. A subset A of X is called a *$w^i w^j$-dense set* in X if $X = c_{w^i}(c_{w^j}(A))$.

Example 3.3.2. Let $X = \{1, 2, 3\}$. Define weak structures w^1 and w^2 on X as follows: $w^1 = \{\varnothing, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$ and $w^2 = \{\varnothing, \{1\}, \{3\}, \{2, 3\}\}$. Then $c_{w^1}(c_{w^2}(\{3\})) = X$ and $c_{w^2}(c_{w^1}(\{3\})) = \{2, 3\}$. Hence $\{3\}$ is a w^1w^2-dense set in X but $\{3\}$ is not w^2w^1-dense set in X.

Theorem 3.3.3. Let (X, w^1, w^2) be a bi-w space and A be a subset of X. If A is a $w^i w^j$-dense set in X, then for any nonempty bi-w-closed subset F of X, such that $A \subseteq F$, we have $F = X$.

Proof. Suppose that A is a $w^i w^j$-dense set in X and F is a bi-w-closed subset of X such that $A \subseteq F$.
Since A is a $w^i w^j$-dense set in X, $c_{w^i}(c_{w^j}(A)) = X$.
By assumption, F is a bi-w-closed set and $A \subseteq F$,
it follows that $X = c_{w^i}(c_{w^j}(A)) \subseteq c_{w^i}(c_{w^j}(F)) = F$.
Hence $F = X$. □

Remark 3.3.4. By the previous theorem, if A is a $w^i w^j$-dense set in X, then only F is a bi-w-closed set in X such that containing A. Moreover, it is not true if F is not bi-w-closed. We can be seen from the following example.

Example 3.3.5. Let $X = \{1, 2, 3\}$. Define weak structures w^1 and w^2 on X as follows: $w^1 = \{\varnothing, \{1\}, \{1, 3\}\}$ and $w^2 = \{\varnothing, \{1\}, \{2\}, \{1, 3\}\}$. Then $c_{w^1}(c_{w^2}(\{1\})) = X$. Hence $\{1\}$ is a w^1w^2-dense set in X. But $\{1\}$ is not bi-w-closed in X.

Theorem 3.3.6. Let (X, w^1, w^2) be a bi-w space and A be a subset of X. The following are equivalent.

1. If F is a non-empty bi-w-closed subset of X such that $A \subseteq F$, then $F = X$.

2. $G \cap A \neq \varnothing$ for any non-empty bi-w-open subset G of X.

Proof. $(1 \Rightarrow 2)$ Assume that if F is a non-empty bi-w-closed subset of X such that $A \subseteq F$, then $F = X$.
Suppose that $G \cap A = \varnothing$ for some non-empty bi-w-open subset G of X.
Thus $A \subseteq X - G$.
Since G is bi-w-open, $X - G$ is bi-w-closed.
By assumption, we have $X - G = X$.
Therefore $G = \varnothing$, this is a contradiction.
Hence $G \cap A \neq \varnothing$ for any non-empty bi-w-open subset G of X.
$(2 \Rightarrow 1)$ Assume that 2 holds and F is a non-empty bi-w-closed subset of X such that $A \subseteq F$.
Suppose that $F \neq X$.
Thus $X - F$ is a non-empty bi-w-open subset of X.
By assumption, we have $(X - F) \cap A \neq \varnothing$.
This is contradiction with $A \subseteq F$.
Therefore $F = X$. □

Corollary 3.3.7. Let (X, w^1, w^2) be a bi-w space and $A \subseteq X$. If A is a $w^i w^j$-dense set in X, then $G \cap A \neq \varnothing$ for any non-empty bi-w-open subset G of X.

Proof. It follows from Theorem 3.3.3 and Theorem 3.3.6. □

Theorem 3.3.8. Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then $i_{w^i}(i_{w^j}(X - A)) = \varnothing$ if and only if A is a $w^i w^j$-dense set in X.

Proof. $(\Rightarrow)$ Assume that $i_{w^i}(i_{w^j}(X - A)) = \varnothing$.
Thus $X - c_{w^i}(c_{w^j}(A)) = \varnothing$, it follows that $c_{w^i}(c_{w^j}(A)) = X$.
Therefore A is a $w^i w^j$-dense set in X.
$(\Leftarrow)$ Suppose that A is a $w^i w^j$-dense set in X.
Then we have $c_{w^i}(c_{w^j}(A)) = X$, and also $i_{w^i}(i_{w^j}(X - A)) = X - c_{w^i}(c_{w^j}(A)) = \varnothing$. □

Theorem 3.3.9. Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then A is a $w^i w^j$-dense set in X if and only if $wExt_{ij}(A) = \varnothing$.

Proof. $(\Rightarrow)$ Suppose that A is a $w^i w^j$-dense set in X.

Then we have $wExt_{ij}(A) = X - c_{w^i}(c_{w^j}(A)) = X - X = \varnothing$.

$(\Leftarrow)$ Assume that $wExt_{ij}(A) = \varnothing$.

Then $X - c_{w^i}(c_{w^j}(A)) = \varnothing$ it follows that $c_{w^i}(c_{w^j}(A)) = X$.

Therefore A is a $w^i w^j$-dense set in X. $\square$

CHAPTER 4

On bi-w-(Λ, θ)-closed and bi-w-(Λ, θ)-open sets in bi-weak structure spaces

In this section, we introduce the concepts of bi-w-(Λ, θ)-closed and bi-w-(Λ, θ)-open sets in bi-weak structure spaces and study some fundamental properties.

In this chapter, we shall call closed and open in a bi-w space that bi-w-closed and bi-w-open, respectively

4.1 On bi-w-(Λ, θ)-closed and bi-w-(Λ, θ)-open sets in bi-weak structure spaces

Definition 4.1.1. Let (X, w^1, w^2) be a bi-w space and $A \subseteq X$. The bi-w-closure of A is defined as follows:

$$\text{bi-}c^w(A) = \cap\{F \mid F \text{ is bi-}w\text{-closed and } A \subseteq F\}.$$

Example 4.1.2. Let $X = \{1, 2, 3\}$. Define weak structures w^1 and w^2 on X as follows: $w^1 = \{\varnothing, \{1\}, \{3\}, \{1, 2\}\}$ and $w^2 = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 3\}\}$. Hence bi-$c^w(\{1\}) = \{1, 2\}$.

Remark 4.1.3. From the above definition, we obtain that $A \subseteq$ bi-$c^w(A)$ for all $A \subseteq X$.

Theorem 4.1.4. Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then $x \in$ bi-$c^w(A)$ if and only if $A \cap U \neq \varnothing$ for all bi-w-open set U containing x.

Proof. ($\Rightarrow$) Assume that $x \in$ bi-$c^w(A)$.
Let U be a bi-w-open set containing x.
We will show that $A \cap U \neq \varnothing$.
Suppose that $A \cap U = \varnothing$.
Then $A \subseteq X - U$.
Since U is bi-w-open, $X - U$ is bi-w-closed.
Then bi-$c^w(A) \subseteq X - U$, and so $x \in X - U$.
This implies, $x \in U \cap (X - U) \neq \varnothing$.
It is a contradiction with the fact that $U \cap (X - U) = \varnothing$.

Hence $A \cap U \neq \varnothing$.

($\Leftarrow$) Assume that $x \notin$ bi-$c^w(A)$.

Then there is a bi-w-closed set F containing A such that $x \notin F$.

We shall show that there exists a bi-w-open set U such that $x \in U$ and $A \cap U = \varnothing$.

Choose $U = X - F$.

Then $U = X - F$ is bi-w-open.

Hence $A \cap U = A \cap (X - F) \subseteq F \cap (X - F) = \varnothing$ and $x \in X - F = U$. □

Theorem 4.1.5. Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then A is bi-w-closed if and only if $A =$ bi-$c^w(A)$.

Proof. ($\Rightarrow$) Assume that A is bi-w-closed.

Since bi-$c^w(A) = \cap\{F \mid F$ is bi-w-closed and $A \subseteq F\}$, $A \subseteq$ bi-$c^w(A)$.

Since A is bi-w-closed, $A \in \{F \mid F$ is bi-w-closed and $A \subseteq F\}$.

Hence $\cap\{F \mid F$ is bi-w-closed and $A \subseteq F\} \subseteq A$.

Then bi-$c^w(A) \subseteq A$.

Therefore $A =$ bi-$c^w(A)$.

($\Leftarrow$) Assume that $A =$ bi-$c^w(A)$.

Since bi-$c^w(A) = \cap\{F \mid F$ is bi-w-closed and $A \subseteq F\}$, by Proposition 2.4.7, A is bi-w-closed. □

Definition 4.1.6. Let (X, w^1, w^2) be a bi-w space and A be a subset of X and $x \in X$. Then $x \in$ bi-$c^w_\theta(A)$ if and only if $A \cap$ bi-$c^w(U) \neq \varnothing$ for all bi-w-open set U containing x.

Definition 4.1.7. Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then A is called *bi-w_θ-closed* if and only if $A =$ bi-$c^w_\theta(A)$. The complement of bi-w_θ-closed is called *bi-w_θ-open*.

Example 4.1.8. Let $X = \{1, 2, 3\}$. Define weak structures w^1 and w^2 on X as follows: $w^1 = \{\varnothing, \{1\}, \{3\}, \{2, 3\}\}$ and $w^2 = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 3\}\}$. Hence bi-$c^w_\theta(\{2, 3\}) = \{2, 3\}$.

Theorem 4.1.9. Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then bi-$c^w(A) \subseteq$ bi-$c^w_\theta(A)$.

Proof. Assume that $x \in$ bi-$c^w(A)$.

Then $A \cap U \neq \varnothing$ for all bi-w-open set U containing x.

From the fact that $B \subseteq$ bi-$c^w(B)$ for all $B \subseteq X$, we obtain that $A \cap$ bi-$c^w(U) \neq \varnothing$ for all bi-w-open set U containing x.

Hence $x \in$ bi-$c^w_\theta(A)$.

This implies bi-$c^w(A) \subseteq$ bi-$c^w_\theta(A)$. □

Example 4.1.10. Let $X = \{1, 2, 3\}$. Define weak structures w^1 and w^2 on X as follows: $w^1 = \{\varnothing, \{1\}, \{3\}, \{2, 3\}\}$ and $w^2 = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 3\}\}$.

Hence bi-$c^w(\{2\}) = \{2\}$, and bi-$c^w_\theta(\{2\}) = \{2, 3\}$.

Therefore bi-$c^w(\{2\}) \neq$ bi-$c^w_\theta(\{2\})$.

Corollary 4.1.11. Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then $A \subseteq$ bi-$c^w_\theta(A)$.

Proof. Since $A \subseteq$ bi-$c^w(A)$ and bi-$c^w(A) \subseteq$ bi-$c^w_\theta(A)$, $A \subseteq$ bi-$c^w_\theta(A)$. □

Lemma 4.1.12. Let (X, w^1, w^2) be a bi-w space and A, B be a subset of X. If $A \subseteq B$, then bi-$c^w_\theta(A) \subseteq$ bi-$c^w_\theta(B)$.

Proof. Assume that $x \notin$ bi-$c^w_\theta(B)$.

Then there exists a bi-w_θ-open set G containing x such that $B \cap$ bi-$c^w_\theta(G) = \varnothing$.

Since $A \subseteq B, A \cap$ bi-$c^w_\theta(G) = \varnothing$.

Thus $x \notin$ bi-$c^w_\theta(A)$.

This implies bi-$c^w_\theta(A) \subseteq$ bi-$c^w_\theta(B)$. □

Theorem 4.1.13. Let (X, w^1, w^2) be a bi-w space and $\{A_i \mid i \in J\}$ be a family of subsets of X. If A_i is bi-w_θ-closed for all $i \in J$, then $\bigcap_{i \in J} A_i$ is bi-w_θ-closed.

Proof. Assume that A_i is bi-w_θ-closed for all $i \in J$.

Clearly, $\bigcap_{i \in J} A_i \subseteq$ bi-$c^w_\theta(\bigcap_{i \in J} A_i)$.

We will show that bi-$c^w_\theta(\bigcap_{i \in J} A_i) \subseteq \bigcap_{i \in J} A_i$, let $A = \bigcap_{i \in J} A_i$

Since $A \subseteq A_i$ for all $i \in J$, bi-$c^w_\theta(A) \subseteq$ bi-$c^w_\theta(A_i) = A_i$ for all $i \in J$.

Thus bi-$c^w_\theta(A) \subseteq \bigcap_{i \in J} A_i$.

Hence $\bigcap_{i \in J} A_i =$ bi-$c^w_\theta(\bigcap_{i \in J} A_i)$.

Therefore $\bigcap_{i \in J} A_i$ is bi-w_θ-closed. □

Corollary 4.1.14. Let (X, w^1, w^2) be a bi-w space and $\{G_i \mid i \in J\}$ be a family of subsets of X. If G_i is bi-w_θ-open for all $i \in J$, then $\bigcup_{i\in J} G_i$ is bi-w_θ-open.

Proof. Assume that G_i is bi-w_θ-open for all $i \in J$.
Then $X - G_i$ is bi-w_θ-closed.
Thus $X - \bigcup_{i\in J} G_i = \bigcap_{i\in J}(X - G_i)$ is bi-w_θ-closed,
and so $\bigcup_{i\in J} G_i = X - (X - \bigcup_{i\in J} G_i)$ is bi-w_θ-open. □

Theorem 4.1.15. Let (X, w^1, w^2) be a bi-w space and A be a subset of X. If A is bi-w_θ-closed, then A is bi-w-closed.

Proof. Assume that A is bi-w_θ-closed.
Then $A =$ bi-$c^w_\theta(A)$.
Since bi-$c^w(A) \subseteq$ bi-$c^w_\theta(A)$, bi-$c^w(A) \subseteq A$.
Hence $A =$ bi-$c^w(A)$.
Therefore A is bi-w-closed. □

Definition 4.1.16. Let (X, w^1, w^2) be a bi-w space and A be a subset of X. A subset bi-w-$\Lambda_\theta(A)$ is defined by

$$\text{bi-}w\text{-}\Lambda_\theta(A) = \begin{cases} X, & \text{if} \quad \text{Bi-}w_\theta O(A) = \varnothing; \\ \bigcap \text{Bi-}w_\theta O(A) & \text{if} \quad \text{Bi-}w_\theta O(A) \neq \varnothing; \end{cases}$$

where Bi-$w_\theta O(A) = \{G : G$ is bi-w_θ-open and $A \subseteq G\}$.

Example 4.1.17. Let $X = \{1, 2, 3\}$. Define weak structures w^1 and w^2 on X as follows: $w^1 = \{\varnothing, \{1\}, \{2\}, \{1, 2\}, X\}$ and $w^2 = \{\varnothing, \{1\}, \{2\}, \{1, 2\}, X\}$. Hence bi-$w$-$\Lambda_\theta(\{2\}) = \{2, 3\}$.

Lemma 4.1.18. Let (X, w^1, w^2) be a bi-w space and $A \subseteq X$ be a subset of X. Then $A \subseteq$ bi-w-$\Lambda_\theta(A)$.

Proof. If Bi-$w_\theta O(A) = \varnothing$, $A \subseteq X =$bi-w-$\Lambda_\theta(A)$.
Assume that Bi-$w_\theta O(A) \neq \varnothing$.
Since $A \subseteq G$ for all $G \in$ Bi-$w_\theta O(A)$, $A \subseteq \bigcap$Bi-$w_\theta O(A) =$ bi-w-$\Lambda_\theta(A)$. □

Lemma 4.1.19. Let (X, w^1, w^2) be a bi-w space and $G \subseteq X$ be a subset of X. If G is a bi-w_θ-open set then bi-w-$\Lambda_\theta(G) = G$.

Proof. Assume that G is bi-w_θ-open.
Clearly, $G \subseteq$ bi-w-$\Lambda_\theta(G)$.
Since G is bi-w_θ-open, $G \in$ Bi-$w_\theta O(G)$, and so bi-w-$\Lambda_\theta(G) \subseteq G$. □

Lemma 4.1.20. For subset A, B and $A_i (i \in J)$ of a bi-w space (X, w^1, w^2), the following properties hold :

1. If $A \subseteq B$, then bi-w-$\Lambda_\theta(A) \subseteq$ bi-w-$\Lambda_\theta(B)$;
2. bi-w-$\Lambda_\theta($ bi-w-$\Lambda_\theta(A)) =$ bi-w-$\Lambda_\theta(A)$;
3. bi-w-$\Lambda_\theta(\cap\{A_i \mid i \in J\}) \subseteq \cap\{$bi-$w$-$\Lambda_\theta(A_i) \mid i \in J\}$;
4. bi-w-$\Lambda_\theta(\cup\{A_i \mid i \in J\}) = \cup\{$bi-$w$-$\Lambda_\theta(A_i) \mid i \in J\}$.

Proof. 1. Assume that $A \subseteq B$.
If Bi-$w_\theta O(B) = \varnothing$, bi-$w$-$\Lambda_\theta(B) = X$, and so bi-$w$-$\Lambda_\theta(A) \subseteq$ bi-w-$\Lambda_\theta(B)$.
Assume that Bi-$w_\theta O(B) \neq \varnothing$.
Then there exists a bi-w_θ-open set G containing x such that $B \subseteq G$.
Since $B \subseteq G$ and $A \subseteq B$, $A \subseteq G$, and so $G \in$ Bi-$w_\theta O(A)$.
Hence Bi-$w_\theta O(A) \neq \varnothing$.
Moreover, Bi-$w_\theta O(B) \subseteq$ Bi-$w_\theta O(A)$.
Hence $\bigcap$Bi-$w_\theta O(A) \subseteq \bigcap$Bi-$w_\theta O(B)$.
Therefore bi-w-$\Lambda_\theta(A) \subseteq$ bi-w-$\Lambda_\theta(B)$.

2. ($\subseteq$) Since bi-w-$\Lambda_\theta(A) \subseteq G$ for all $G \in$ Bi-$w_\theta O(A)$, bi-w-$\Lambda_\theta($bi-w-$\Lambda_\theta(A)) \subseteq$ bi-w-$\Lambda_\theta(G)$ for all $G \in$ Bi-$w_\theta O(A)$.
This implies bi-w-$\Lambda_\theta($bi-w-$\Lambda_\theta(A)) \subseteq G$ for all $G \in$ Bi-$w_\theta O(A)$.
Hence bi-w-$\Lambda_\theta($ bi-w-$\Lambda_\theta(A)) \subseteq$ bi-w-$\Lambda_\theta(A)$.
($\supseteq$) From Lemma 4.1.18, we have bi-w-$\Lambda_\theta(A) \subseteq$ bi-w-$\Lambda_\theta($ bi-w-$\Lambda_\theta(A))$.
Then bi-w-$\Lambda_\theta($ bi-w-$\Lambda_\theta(A)) =$ bi-w-$\Lambda_\theta(A)$.

3. Let $A = \bigcap_{i \in J} \{A_i \mid i \in J\}$.

   Since $A \subseteq A_i$ for all $i \in J$, bi-w-$\Lambda_\theta(A) \subseteq$bi-$w$-$\Lambda_\theta(A_i)$ for all $i \in J$.

   Hence bi-w-$\Lambda_\theta(A) \subseteq \bigcap_{i \in J} \{$bi-$w$-$\Lambda_\theta(A_i) \mid i \in J\}$.

4. $(\subseteq)$ Assume that $x \notin \bigcup_{i \in J} \{$bi-$w$-$\Lambda_\theta(A_i) \mid i \in J\}$.

   Then $x \notin$bi-w-$\Lambda_\theta(A_i)$ for all $i \in J$.

   Thus for all $i \in J$, there is a bi-w_θ-open set G_i such that $A \subseteq G_i$ and $x \notin G_i$.

   Hence $x \notin \bigcup_{i \in J} G_i$ is bi-w_θ-open.

   Since $\bigcup_{i \in J} A_i \subseteq \bigcup_{i \in J} G_i$, $x \notin$bi-w-$\Lambda_\theta(\cup\{A_i \mid i \in J\})$.

   $(\supseteq)$ It is clear that $\bigcup\{$bi-w-$\Lambda_\theta(A_i) \mid i \in J\} \subseteq$ bi-w-$\Lambda_\theta(\cup\{A_i \mid i \in J\})$. □

Definition 4.1.21. A subset A of a bi-w space (X, w^1, w^2) is called a *bi-w-Λ_θ-set* if $A =$bi-w-$\Lambda_\theta(A)$.

Lemma 4.1.22. For subset A and $A_i (i \in I)$ of a bi-w space (X, w^1, w^2), the following properties hold :

1. bi-w-$\Lambda_\theta(A)$ is a bi-w-Λ_θ-set;

2. If A is bi-w_θ-open, then A is a bi-w-Λ_θ-set;

3. If A_i is a bi-w-Λ_θ-set for each $i \in J$, then $\bigcap_{i \in J} A_i$ is a bi-w-Λ_θ-set;

4. If A_i is a bi-w-Λ_θ-set for each $i \in J$, then $\bigcup_{i \in J} A_i$ is a bi-w-Λ_θ-set.

Proof. 1. By Lemma 4.1.20 (2), we have bi-w-Λ_θ(bi-w-$\Lambda_\theta(A)) =$ bi-w-$\Lambda_\theta(A)$.

   Then bi-w-$\Lambda_\theta(A)$ is a bi-w-Λ_θ-set.

2. It follow from Lemma 4.1.19.

3. Assume that A_i is a bi-w-Λ_θ-set for all $i \in J$.

   Then $A_i =$ bi-w-$\Lambda_\theta(A_i)$ for all $i \in J$.

   Let $A = \bigcap_{i \in J} A_i$.

   Since $A \subseteq A_i$ for all $i \in J$, bi-w-$\Lambda_\theta(A) \subseteq$ bi-w-$\Lambda_\theta(A_i) = A_i$ for all $i \in J$.

   Thus bi-w-$\Lambda_\theta(A) \subseteq \bigcap_{i \in J} A_i$, i.e., bi-$w$-$\Lambda_\theta(\bigcap_{i \in J} A_i) \subseteq \bigcap_{i \in J} A_i$.

   It is clear that $\bigcap_{i \in J} A_i \subseteq$ bi-w-$\Lambda_\theta(\bigcap_{i \in J} A_i)$.

   Hence $\bigcap_{i \in J} A_i =$ bi-w-$\Lambda_\theta(\bigcap_{i \in J} A_i)$.

   Therefore $\bigcap_{i \in J} A_i$ is a bi-w-Λ_θ-set.

4. Assume that A_i is a bi-w-Λ_θ-set for all $i \in J$.

   Then $A_i =$ bi-w-$\Lambda_\theta(A_i)$ for all $i \in J$.

   By Lemma 4.1.20 (4), we have bi-w-$\Lambda_\theta(\bigcup_{i\in J} A_i) = \bigcup_{i\in J}$bi-$w$-$\Lambda_\theta(A_i) = \bigcup_{i\in J} A_i$. □

Definition 4.1.23. Let A be a subset of a bi-w space (X, w^1, w^2).

1. A is called a *bi-w-(Λ,θ)-closed* set if $A = T \cap C$, where T is a bi-w-Λ_θ-set and C is a bi-w_θ-closed set. The complement of a bi-w-(Λ,θ)-closed set is called *bi-w-(Λ,θ)-open* set. The collection of all bi-w-(Λ,θ)-open (resp. bi-w-(Λ,θ)-closed) set in a bi-w-space (X, w^1, w^2) is denoted by bi-w-$\Lambda_\theta O(X, w^1, w^2)$ (resp. bi-w-$\Lambda_\theta C(X, w^1, w^2)$).

2. A point $x \in X$ is called a *bi-w-(Λ,θ)-cluster point* of A if for every bi-w-(Λ,θ)-open set U of X containing x, we have $A \cap U \neq \varnothing$. The set of all bi-$w$-$(\Lambda,\theta)$-cluster points of A is called the *bi-w-(Λ,θ)-closure* of A and is denoted by bi-$c^w_{(\Lambda,\theta)}(A)$.

Remark 4.1.24. For a subset A of a bi-w space (X, w^1, w^2), $x \in$ bi-$c^w_{(\Lambda,\theta)}(A)$ if and only if $U \cap A \neq \varnothing$ for every bi-w-(Λ,θ)-open set U containing x.

Lemma 4.1.25. Let A and B be subset of a bi-w space (X, w^1, w^2). For the bi-w-(Λ,θ)-closure, the following properties hold :

1. $A \subseteq$ bi-$c^w_{(\Lambda,\theta)}(A)$;

2. bi-$c^w_{(\Lambda,\theta)}(A) = \bigcap\{F \mid A \subseteq F$ and F is bi-w-(Λ,θ)-closed $\}$;

3. If $A \subseteq B$, then bi-$c^w_{(\Lambda,\theta)}(A) \subseteq$ bi-$c^w_{(\Lambda,\theta)}(B)$;

4. If A_i is bi-w-(Λ,θ)-closed for each $i \in J$, then $\bigcap_{i\in j} A_i$ is bi-w-(Λ,θ)-closed;

5. bi-$c^w_{(\Lambda,\theta)}(A)$ is bi-w-(Λ,θ)-closed.

Proof. 1. Assume that $x \in A$.

Then $A \cap U \neq \varnothing$ for all bi-w-(Λ,θ)-open set such that U containing x.

Thus $x \in$ bi-$c^w_{(\Lambda,\theta)}(A)$.

2. ($\subseteq$) Assume that $x \notin \cap\{F \mid F$ is bi-w-(Λ, θ)-closed and $A \subseteq F\}$.

   Then there exists a bi-w-(Λ, θ)-closed set F such that $A \subseteq F$ and $x \notin F$.

   Thus $X - F$ is a bi-w-(Λ, θ)-open and $x \in X - F$.

   Moreover, $A \cap (X - F) = \varnothing$.

   Hence $x \notin$ bi-$c^w_{(\Lambda,\theta)}(A)$.

   ($\supseteq$) Assume that $x \notin$ bi-$c^w_{(\Lambda,\theta)}(A)$.

   Then $A \cap U = \varnothing$ for some bi-w-(Λ, θ)-open set U containing x.

   Thus $X - U$ is bi-w-(Λ, θ)-closed and $x \notin X - U$.

   Moreover, $A \subseteq X - U$.

   Therefore $x \notin \cap\{F \mid F$ is bi-w-(Λ, θ)-closed and $A \subseteq F\}$.

3. Assume that $A \subseteq B$.

   Suppose $x \notin$ bi-$c^w_{(\Lambda,\theta)}(B)$.

   Then there exists a bi-w_θ-open set G containing x such that $G \cap B = \varnothing$.

   Since $A \subseteq B$, $G \cap A = \varnothing$, $x \notin$ bi-$c^w_{(\Lambda,\theta)}(A)$.

   This implies bi-$c^w_{(\Lambda,\theta)}(A) \subseteq$ bi-$c^w_{(\Lambda,\theta)}(B)$.

4. Assume that A_i is a bi-w-(Λ, θ)-closed set for all $i \in J$.

   We will show that $\bigcap_{i \in J} A_i$ is bi-w-(Λ, θ)-closed.

   For each $i \in J$, there exist a bi-w-Λ_θ-set T_i and a bi-w_θ-closed set C_i such that $A_i = T_i \cap C_i$.

   Then $\bigcap_{i \in J} T_i$ is a bi-w-Λ_θ-set and $\bigcap_{i \in J} C_i$ is bi-w_θ-closed.

   Moreover, $\bigcap_{i \in J} A_i = \bigcap_{i \in J}(T_i \cap C_i) = (\bigcap_{i \in J} T_i) \cap (\bigcap_{i \in J} C_i)$.

   Then $\bigcap_{i \in J} A_i$ is bi-w-(Λ, θ)-closed.

5. It follows from (2) and (4). $\square$

Lemma 4.1.26. Let A be a subset of a bi-w space (X, w^1, w^2). Then A is bi-w-(Λ, θ)-closed if and only if bi-$c^w_{(\Lambda,\theta)}(A) = A$.

Proof. ($\Rightarrow$) Assume that A is bi-w-(Λ, θ)-closed.

Since bi-$c^w_{(\Lambda,\theta)}(A) = \cap\{F \mid F$ is bi-w-(Λ, θ)-closed and $A \subseteq F\}$, $A \subseteq$ bi-$c^w_{(\Lambda,\theta)}(A)$.

Since A is bi-w-(Λ, θ)-closed, $A \in \{F \mid F$ is bi-w-(Λ, θ)-closed and $A \subseteq F\}$.

Hence $\cap\{F \mid F$ is bi-w-(Λ, θ)-closed and $A \subseteq F\} \subseteq A$.

Then bi-$c^{w}_{(\Lambda,\theta)}(A) \subseteq A$.

Therefore bi-$c^{w}_{(\Lambda,\theta)}(A) = A$.

($\Leftarrow$) Assume that bi-$c^{w}_{(\Lambda,\theta)}(A) = A$.

Since bi-$c^{w}_{(\Lambda,\theta)}(A)$ is bi-w-(Λ,θ)-closed, A is bi-w-(Λ,θ)-closed. □

Definition 4.1.27. Let A be a subset of a bi-w space (X, w^1, w^2). The union of all bi-w-(Λ,θ)-open sets contained in A is called the *bi-w-(Λ,θ)-interior* of A and is denoted by bi-$i^{w}_{(\Lambda,\theta)}(A)$.

Lemma 4.1.28. Let A and B be subsets of a bi-w space (X, w^1, w^2). For the bi-w-(Λ,θ)-interior, the following properties hold:

1. bi-$i^{w}_{(\Lambda,\theta)}(A) \subseteq A$;
2. If $A \subseteq B$, then bi-$i^{w}_{(\Lambda,\theta)}(A) \subseteq$ bi-$i^{w}_{(\Lambda,\theta)}(B)$;
3. If A_i is a bi-w-(Λ,θ)-open set for all $i \in J$, then $\bigcup_{i\in j} A_i$ is a bi-w-(Λ,θ)-open set.

Proof. 1. Let $x \in$ bi-$i^{w}_{(\Lambda,\theta)}(A)$.

Then there exists a bi-w-(Λ,θ)-open set O such that $x \in O \subseteq A$.

Thus $x \in A$.

Hence bi-$i^{w}_{(\Lambda,\theta)}(A) \subseteq A$.

2. Assume that $A \subseteq B$.

Let $x \in$ bi-$i^{w}_{(\Lambda,\theta)}(A)$.

Then there exists a bi-w-(Λ,θ)-open set O such that $x \in O \subseteq A$.

Since $A \subseteq B$, $x \in O \subseteq B$.

Hence $x \in$ bi-$i^{w}_{(\Lambda,\theta)}(B)$.

3. Assume that A_i is a bi-w-(Λ,θ)-open set for all $i \in J$.

Then $X - A_i$ is a bi-w-(Λ,θ)-closed set for all $i \in J$.

We will show that $\bigcup_{i\in j} A_i$ is a bi-w-(Λ,θ)-open set.

Thus $\bigcap_{i\in j}(X - A_i) = X - \bigcup_{i\in j} A_i$ is a bi-w-(Λ,θ)-closed set.

Hence $\bigcup_{i\in j} A_i$ is a bi-w-(Λ,θ)-open set. □

Lemma 4.1.29. For a subset A of a bi-w space (X, w^1, w^2), the following properties hold;

1. bi-$i^{w}_{(\Lambda,\theta)}(X - A) = X -$ bi-$c^{w}_{(\Lambda,\theta)}(A)$.

2. bi-$c^{w}_{(\Lambda,\theta)}(X - A) = X -$ bi-$i^{w}_{(\Lambda,\theta)}(A)$.

Proof. 1. ($\subseteq$) Let $x \in$ bi-$i^{w}_{(\Lambda,\theta)}(X - A)$.

Then there exists a bi-w-(Λ, θ)-open set V containing x such that $V \subseteq X - A$ and so $V \cap A = \varnothing$.

By Remark 4.1.23, we have $x \notin$ bi-$c^{w}_{(\Lambda,\theta)}(X - A)$.

Hence $x \in X -$ bi-$c^{w}_{(\Lambda,\theta)}(X - A)$.

Therefore, bi-$i^{w}_{(\Lambda,\theta)}(X - A) \subseteq X -$ bi-$c^{w}_{(\Lambda,\theta)}(A)$.

($\supseteq$) Let $x \in X -$ bi-$c^{w}_{(\Lambda,\theta)}(A)$.

Then $x \notin$ bi-$c^{w}_{(\Lambda,\theta)}(A)$, and so there exists a bi-w-(Λ, θ)-open set V containing x such that $V \cap A = \varnothing$.

Thus $V \subseteq X - A$ and so, $x \in$ bi-$i^{w}_{(\Lambda,\theta)}(X - A)$.

This implies that $X -$ bi-$c^{w}_{(\Lambda,\theta)}(A) \subseteq$ bi-$i^{w}_{(\Lambda,\theta)}(X - A)$.

Consequently, we obtain bi-$i^{w}_{(\Lambda,\theta)}(X - A) = X -$ bi-$c^{w}_{(\Lambda,\theta)}(A)$.

2. It follows from (1). □

Lemma 4.1.30. Let A be a subset of a bi-w space (X, w^1, w^2). For the bi-w-(Λ, θ)-interior, the following properties hold:

1. A is bi-w-(Λ, θ)-open if and only if bi-$i^{w}_{(\Lambda,\theta)}(A) = A$;

2. bi-$i^{w}_{(\Lambda,\theta)}(A)$ is bi-w-(Λ, θ)-open.

Proof. 1. ($\Rightarrow$) Assume that A is bi-w-(Λ, θ)-open.

Then $X - A$ is bi-w-(Λ, θ)-closed.

Thus bi-$c^{w}_{(\Lambda,\theta)}(X - A) = (X - A)$.

Since bi-$c^{w}_{(\Lambda,\theta)}(X - A) = X -$ bi-$i^{w}_{(\Lambda,\theta)}(A) = X - A$, and so bi-$i^{w}_{(\Lambda,\theta)}(A) = (A)$.

($\Leftarrow$) Assume that bi-$i^{w}_{(\Lambda,\theta)}(A) = A$.

By Lemma 4.1.28. (3), A is bi-w-(Λ, θ)-open.

2. By Lemma 4.1.28. (3), bi-$i^{w}_{(\Lambda,\theta)}(A)$ is bi-w-(Λ, θ)-open. □

Lemma 4.1.31. For a subset A of a bi-w space (X, w^1, w^2), the following properties hold;

1. $\text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(A)) = \text{bi-}c^w_{(\Lambda,\theta)}(A)$.

2. $\text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(A)) = \text{bi-}i^w_{(\Lambda,\theta)}(A)$.

Proof. 1. Since $\text{bi-}c^w_{(\Lambda,\theta)}(A)$ is a bi-w-(Λ,θ)-closed, $\text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(A)) = \text{bi-}c^w_{(\Lambda,\theta)}(A)$.

2. Since $\text{bi-}i^w_{(\Lambda,\theta)}(A)$ is a bi-w-(Λ,θ)-open, $\text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(A)) = \text{bi-}i^w_{(\Lambda,\theta)}(A)$. □

Proposition 4.1.32. For a subset A of a bi-w space (X, w^1, w^2), the following properties hold;

1. $\text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(A)))) = \text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(A))$.

2. $\text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(A)))) = \text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(A))$.

Proof. 1. $(\subseteq)$ Since $\text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(A))) \subseteq \text{bi-}c^w_{(\Lambda,\theta)}(A)$, we obtain that

$$\text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(A)))) \subseteq \text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(A)).$$

$(\supseteq)$ Since $\text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(A)) \subseteq \text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(A)))$, we have

$$\begin{aligned}\text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(A)) &= \text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(A)))\\ &\subseteq \text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(A)))).\end{aligned}$$

Consequently, we obtain

$$\text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(A)) \subseteq \text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(A)))).$$

Thus $\text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(A)))) = \text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(A))$.

2. $(\subseteq)$ Since $\text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(A))) \subseteq \text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(A))$, we have

$$\text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(A)))) \subseteq \text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(A)).$$

$(\supseteq)$ Since $\text{bi-}i^w_{(\Lambda,\theta)}(A) \subseteq \text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(A))$, then

$$\text{bi-}i^w_{(\Lambda,\theta)}(A) = \text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(A)) \subseteq \text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(A))),$$

we have

$$\text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(A)) \subseteq \text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(A)))).$$

Consequently, we obtain

$$\text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(A)) \subseteq \text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(A)))).$$

Thus $\text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(A)))) = \text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(A))$. □

Definition 4.1.33. A subset A of a bi-w space (X, w^1, w^2) is said to be:

1. *bi-w-$s(\Lambda,\theta)$-open* if $A \subseteq \text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(A))$.
2. *bi-w-$p(\Lambda,\theta)$-open* if $A \subseteq \text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(A))$.

The family of all bi-w-$s(\Lambda,\theta)$-open (resp. bi-w-$p(\Lambda,\theta)$-open) sets in a bi-w-space (X, w^1, w^2) is denoted by bi-w-$s\Lambda_\theta O(X,x)$ (resp. bi-w-$p\Lambda_\theta O(X,x)$).

Definition 4.1.34. The complement of a bi-w-$s(\Lambda,\theta)$-open (resp. bi-w-$p(\Lambda,\theta)$-open) set is said to be bi-w-$s(\Lambda,\theta)$-closed (resp. bi-w-$p(\Lambda,\theta)$-closed) set.

The family of all bi-w-$s(\Lambda,\theta)$-closed (resp. bi-w-$p(\Lambda,\theta)$-closed) sets in a bi-w space (X, w^1, w^2) is denoted by bi-w-$s\Lambda_\theta C(X,x)$ (resp. bi-w-$p\Lambda_\theta C(X,x)$).

Proposition 4.1.35. In a bi-w space (X, w^1, w^2), the following properties hold;

1. If A_i is bi-w-$s(\Lambda,\theta)$-open for all $i \in J$, then $\bigcup_{i\in J} A_i$ is bi-w-$s(\Lambda,\theta)$-open.
2. If A_i is bi-w-$p(\Lambda,\theta)$-open for all $i \in J$, then $\bigcup_{i\in J} A_i$ is bi-w-$p(\Lambda,\theta)$-open.
3. If A_i is bi-w-$s(\Lambda,\theta)$-closed for all $i \in J$, then $\bigcap_{i\in J} A_i$ is bi-w-$s(\Lambda,\theta)$-closed.
4. If A_i is bi-w-$p(\Lambda,\theta)$-closed for all $i \in J$, then $\bigcap_{i\in J} A_i$ is bi-w-$p(\Lambda,\theta)$-closed.

Proof. 1. Assume that A_i is bi-w-$s(\Lambda,\theta)$-open for all $i \in J$.

Then $A_i \subseteq \text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(A_i))$ for all $i \in J$.

We will show that $\bigcup_{i\in J} A_i \subseteq \text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(\bigcup_{i\in J} A_i))$.

Since $\text{bi-}i^w_{(\Lambda,\theta)}(A_i) \subseteq \text{bi-}i^w_{(\Lambda,\theta)}(\bigcup_{i\in J} A_i)$ for all $i \in J$,

$A_i \subseteq \text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(A_i)) \subseteq \text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(\bigcup_{i\in J} A_i))$ for all $i \in J$.

Hence $\bigcup_{i\in J} A_i \subseteq$ bi-$c^w_{(\Lambda,\theta)}$(bi-$i^w_{(\Lambda,\theta)}(\bigcup_{i\in J} A_i))$.
Therefore $\bigcup_{i\in J} A_i$ is bi-w-$s(\Lambda,\theta)$-open.

2. Assume that A_i is bi-w-$p(\Lambda,\theta)$-open for all $i \in J$.

   Then $A_i \subseteq$ bi-$i^w_{(\Lambda,\theta)}$(bi-$c^w_{(\Lambda,\theta)}(A_i))$ for all $i \in J$.

   We will show that $\bigcup_{i\in J} A_i \subseteq$ bi-$i^w_{(\Lambda,\theta)}$(bi-$c^w_{(\Lambda,\theta)}(\bigcup_{i\in J} A_i))$.

   Since bi-$c^w_{(\Lambda,\theta)}(A_i) \subseteq$ bi-$c^w_{(\Lambda,\theta)}(\bigcup_{i\in J} A_i)$ for all $i \in J$,

   $A_i \subseteq$ bi-$i^w_{(\Lambda,\theta)}$(bi-$c^w_{(\Lambda,\theta)}(A_i)) \subseteq$ bi-$i^w_{(\Lambda,\theta)}$(bi-$c^w_{(\Lambda,\theta)}(\bigcup_{i\in J} A_i))$ for all $i \in J$.

   Hence $\bigcup_{i\in J} A_i \subseteq$ bi-$i^w_{(\Lambda,\theta)}$(bi-$c^w_{(\Lambda,\theta)}(\bigcup_{i\in J} A_i))$.

   Therefore $\bigcup_{i\in J} A_i$ is bi-w-$p(\Lambda,\theta)$-open.

3. Assume that A_i is bi-w-$s(\Lambda,\theta)$-closed for all $i \in J$.

   Then $X - A_i$ is bi-w-$s(\Lambda,\theta)$-open.

   Thus $X - \bigcap_{i\in J} A_i = \bigcup_{i\in J}(X - A_i)$ is bi-w-$s(\Lambda,\theta)$-open,

   and so $\bigcap_{i\in J} A_i = X - (X - \bigcap_{i\in J} A_i)$ is bi-w-$s(\Lambda,\theta)$-closed.

4. Assume that A_i is bi-w-$p(\Lambda,\theta)$-closed for all $i \in J$.

   Then $X - A_i$ is bi-w-$p(\Lambda,\theta)$-open.

   Thus $X - \bigcap_{i\in J} A_i = \bigcup_{i\in J}(X - A_i)$ is bi-w-$p(\Lambda,\theta)$-open,

   and so $\bigcap_{i\in J} A_i = X - (X - \bigcap_{i\in J} A_i)$ is bi-w-$p(\Lambda,\theta)$-closed. □

Proposition 4.1.36. For a subset A of a bi-w space (X, w^1, w^2), the following properties hold;

1. A is bi-w-$s(\Lambda,\theta)$-closed if and only if bi-$i^w_{(\Lambda,\theta)}$(bi-$c^w_{(\Lambda,\theta)}(A)) \subseteq A$.

2. A is bi-w-$p(\Lambda,\theta)$-closed if and only if bi-$c^w_{(\Lambda,\theta)}$(bi-$i^w_{(\Lambda,\theta)}(A)) \subseteq A$.

Proof. 1. $(\Rightarrow)$ Suppose that A is bi-w-$s(\Lambda,\theta)$-closed.

Then $X - A$ is bi-w-$s(\Lambda,\theta)$-open and so $X - A \subseteq$ bi-$c^w_{(\Lambda,\theta)}$(bi-$i^w_{(\Lambda,\theta)}(X - A))$.

By Lemma 4.1.29, $X - A \subseteq$ bi-$c^w_{(\Lambda,\theta)}$(bi-$i^w_{(\Lambda,\theta)}(X - A))$

$= $ bi-$c^w_{(\Lambda,\theta)}(X -$ bi-$c^w_{(\Lambda,\theta)}(A))$

$= X - ($bi-$i^w_{(\Lambda,\theta)}$(bi-$c^w_{(\Lambda,\theta)}(A)))$.

Consequently, we obtain bi-$i^w_{(\Lambda,\theta)}$(bi-$c^w_{(\Lambda,\theta)}(A)) \subseteq A$.

$(\Leftarrow)$ Suppose that bi-$i^w_{(\Lambda,\theta)}$(bi-$c^w_{(\Lambda,\theta)}(A)) \subseteq A$.

Then $X - A \subseteq X -$ bi-$i^w_{(\Lambda,\theta)}$(bi-$c^w_{(\Lambda,\theta)}(A))$ and by Lemma 4.1.29, we obtain

$$\begin{aligned} X - A &\subseteq X - \text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(A)) \\ &= \text{bi-}c^w_{(\Lambda,\theta)}(X - \text{bi-}c^w_{(\Lambda,\theta)}(A)) \\ &= \text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(X - A)). \end{aligned}$$

This implies that $X - A$ is bi-w-$s(\Lambda, \theta)$-open and so A is bi-w-$s(\Lambda, \theta)$-closed.

2. ($\Rightarrow$) Assume that A is bi-w-$p(\Lambda, \theta)$-closed.
   Then $X - A$ is bi-w-$p(\Lambda, \theta)$-open and so $X - A \subseteq \text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(X - A))$.
   By Lemma 4.1.29, $X-A \subseteq \text{bi-}i^w_{(\Lambda,\theta)}(X-\text{bi-}i^w_{(\Lambda,\theta)}(A)) = X-(\text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(A)))$.
   Consequently, we obtain $\text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(A)) \subseteq A$.
   ($\Leftarrow$) Assume that $\text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(A)) \subseteq A$.
   Then $X - A \subseteq X - \text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(A))$ and by Lemma 4.1.29, we obtain

$$\begin{aligned} X - A &\subseteq X - \text{bi-}c^w_{(\Lambda,\theta)}(\text{bi-}i^w_{(\Lambda,\theta)}(A)) \\ &= \text{bi-}i^w_{(\Lambda,\theta)}(X - \text{bi-}i^w_{(\Lambda,\theta)}(A)) \\ &= \text{bi-}i^w_{(\Lambda,\theta)}(\text{bi-}c^w_{(\Lambda,\theta)}(X - A)). \end{aligned}$$

This implies that $X-A$ is bi-w-$p(\Lambda, \theta)$-open and so A is bi-w-$p(\Lambda, \theta)$-closed. □

CHAPTER 5

Conclusions

The aim of this thesis is to introduce the results of properties of some sets in bi-weak structure spaces. And we study some properties of boundary sets, exterior sets and dense sets in bi-weak structure spaces are introduced. Some properties of their sets are obtained. In particular, some characterizations of closed sets in bi-weak structure spaces using boundary sets or exterior sets are obtained. Moreover, we introduce the notions bi-w-(Λ, θ)-closure and bi-w-(Λ, θ)-interior on bi-weak structure spaces. The results are follows:

1) Let (X, w^1, w^2) be a bi-w space, A be a subset of X and $x \in X$. We called x is a w_iw_j-*boundary point* of A if $x \in c_{w^i}(c_{w^j}(A)) \cap c_{w^i}(c_{w^j}(X - A))$. We denote the set of all w_iw_j-boundary points of A by $wBdr_{ij}(A)$.

   From the above definition, the following theorems are derived:

   1.1) Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then $wBdr_{ij}(X - A) = wBdr_{ij}(A)$.

   1.2) Let (X, w^1, w^2) be a bi-w space and $A \subseteq X$. Then the following statements hold;

   1.2.1) $wBdr_{ij}(A) = c_{w^i}(c_{w^j}(A)) - i_{w^i}(i_{w^j}(A))$;

   1.2.2) $wBdr_{ij}(A) \cap i_{w^i}(i_{w^j}(A)) = \varnothing$;

   1.2.3) $wBdr_{ij}(A) \cap i_{w^i}(i_{w^j}(X - A)) = \varnothing$;

   1.2.4) $c_{w^i}(c_{w^j}(A)) = wBdr_{ij}(A) \cup i_{w^i}(i_{w^j}(A))$;

   1.2.5) $X = i_{w^i}(i_{w^j}(A)) \cup wBdr_{ij}(A) \cup i_{w^i}(i_{w^j}(X - A))$ is a pairwise disjoint union;

   1.2.6) $c_{w^i}(c_{w^j}(A)) = wBdr_{ij}(A) \cup A$.

   1.3) Let (X, w^1, w^2) be a bi-w space and $A \subseteq X$. Then

   1.3.1) A is bi-w-closed if and only if $wBdr_{ij}(A) \subseteq A$.

   1.3.2) A is bi-w-open if and only if $wBdr_{ij}(X - A) \subseteq (X - A)$.

1.4) Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then $wBdr_{ij}(A) = \varnothing$ if and only if A is bi-w-closed and bi-w-open.

2) Let (X, w^1, w^2) be a bi-w space, A be a subset of X and $x \in X$. We called x is a w^iw^j*-exterior point* of A if $x \in i_{w^i}(i_{w^j}(X - A))$. We denote the set of all w^iw^j-exterior points of A by $wExt_{ij}(A)$.

From the above definition, the following theorems are derived:

2.1) Let (X, w^1, w^2) be a bi-w space and $A \subseteq X$. Then

2.1.1) $wExt_{ij}(A) \cap A = \varnothing$.

2.1.2) $wExt_{ij}(X) = \varnothing$.

2.2) Let (X, w^1, w^2) be a bi-w space and A, B be two subsets of X. If $A \subseteq B$, then $wExt_{ij}(B) \subseteq wExt_{ij}(A)$.

2.3) Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then A is bi-w-closed if and only if $wExt_{ij}(A) = X - A$.

2.4) Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then A is bi-w-open if and only if $wExt_{ij}(X - A) = A$.

2.5) Let (X, w^1, w^2) be a bi-w space and A be a subset of X. If A is bi-w-closed, then $wExt_{ij}(X - wExt_{ij}(A)) = wExt_{ij}(A)$.

2.6) Let (X, w^1, w^2) be a bi-w space and A, B be two subsets of X. Then

2.6.1) $wExt_{ij}(A) \cup wExt_{ij}(B) \subseteq wExt_{ij}(A \cap B)$.

2.6.2) If A and B are bi-w-closed, then $wExt_{ij}(A) \cup wExt_{ij}(B) = wExt_{ij}(A \cap B)$.

2.7) Let (X, w^1, w^2) be a bi-w space and A, B be two subsets of X. If A and B are bi-w-open, then $wExt_{ij}(X - (A \cup B)) = A \cup B$.

3) Let (X, w^1, w^2) be a bi-w space. A subset A of X is called a w^iw^j*-dense set* in X if $X = c_{w^i}(c_{w^j}(A))$.

From the above definition, the following theorems are derived:

3.1) Let (X, w^1, w^2) be a bi-w space and A be a subset of X. If A is a $w^i w^j$-dense set in X, then for any non-empty bi-w-closed subset F of X, such that $A \subseteq F$, we have $F = X$.

3.2) Let (X, w^1, w^2) be a bi-w space and A be a subset of X. The following are equivalent.

3.2.1) If F is non-empty bi-w-closed subset of X such that $A \subseteq F$, then $F = X$.

3.2.2) $G \cap A \neq \varnothing$ for any non-empty bi-w-open subset G of X.

3.3) Let (X, w^1, w^2) be a bi-w space and $A \subseteq X$. If A is a $w^i w^j$-dense set in X, then $G \cap A \neq \varnothing$ for any non-empty bi-w-open subset G of X.

3.4) Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then $i_{w^i}(i_{w^j}(X - A)) = \varnothing$ if and only if A is a $w^i w^j$-dense set in X.

3.5) Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then A is a $w^i w^j$-dense set in X if and only if $wExt_{ij}(A) = \varnothing$.

4) Let (X, w^1, w^2) be a bi-w space and $A \subseteq X$. The bi-w-closure of A is defined as follows: bi-$c^w(A) = \cap\{F \mid F$ is bi-w-closed and $A \subseteq F\}$.

From the above definition, the following theorems are derived:

4.1) From the above definition, we obtain that $A \subseteq$ bi-$c^w(A)$ for all $A \subseteq X$.

4.2) Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then $x \in$ bi-$c^w(A)$ if and only if $A \cap U \neq \varnothing$ for all bi-w-open set U containing x.

4.3) Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then A is a bi-w-closed if and only if $A =$ bi-$c^w(A)$.

5) Let (X, w^1, w^2) be a bi-w space and A be a subset of X and x. Then $x \in$ bi-$c^w_\theta(A)$ if and only if $A \cap$ bi-$c^w(U) \neq \varnothing$ for all bi-w-open set U containing x.

6) Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then A is called *bi-w_θ-closed* if and only if $A =$ bi-$c^w_\theta(A)$. The complement of bi-w_θ-closed is called bi-w_θ-open.

From the above definition, the following theorems are derived:

6.1) Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then bi-$c^w(A) \subseteq$ bi-$c^w_\theta(A)$.

6.2) Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then $A \subseteq$ bi-$c^w_\theta(A)$.

6.3) Let (X, w^1, w^2) be a bi-w space and A be a subset of X. If $A \subseteq B$, then bi-$c^w_\theta(A) \subseteq$ bi-$c^w_\theta(B)$.

6.4) Let (X, w^1, w^2) be a bi-w space and $\{A_i \mid i \in J\}$ be a family of subsets of X. If A is bi-w_θ-closed for all $i \in J$, then $\bigcap_{i\in J} A_i$ is bi-w_θ-closed.

6.5) Let (X, w^1, w^2) be a bi-w space and A be a subset of X. If G_i is a bi-w_θ-open set for all $i \in J$, then $\bigcup_{i\in J} G_i$ is a bi-w_θ-open set.

6.6) Let (X, w^1, w^2) be a bi-w space and A be a subset of X. If A is bi-w_θ-closed, then A is bi-w-closed.

7) Let (X, w^1, w^2) be a bi-w space and A be a subset of X. A subset bi-w-$\Lambda_\theta(A)$ is defined by

$$\text{bi-}w\text{-}\Lambda_\theta(A) = \begin{cases} X, & \text{if} \quad \text{Bi-}w_\theta O(A) = \varnothing; \\ \bigcap \text{Bi-}w_\theta O(A) & \text{if} \quad \text{Bi-}w_\theta O(A) \neq \varnothing; \end{cases}$$

where Bi-$w_\theta O(A) = \{G : G$ is bi-w_θ-open and $A \subseteq G\}$.

From the above definition, the following theorems are derived:

7.1) Let (X, w^1, w^2) be a bi-w space and $A \subseteq X$ be a subset of X. Then $A \subseteq$ bi-w-$\Lambda_\theta(A)$.

7.2) Let (X, w^1, w^2) be a bi-w space and $G \subseteq X$ be a subset of X. If G is a bi-w_θ-open set then bi-w-$\Lambda_\theta(G) = G$.

7.3) For subset A, B and $A_i (i \in J)$ of a bi-w space (X, w^1, w^2), the following properties hold :

7.3.1) If $A \subseteq B$, then bi-w-$\Lambda_\theta(A) \subseteq$ bi-w-$\Lambda_\theta(B)$;

7.3.2) bi-w-$\Lambda_\theta($ bi-w-$\Lambda_\theta(A)) =$ bi-w-$\Lambda_\theta(A)$;

7.3.3) bi-w-$\Lambda_\theta(\cap\{A_i \mid i \in I\}) \subseteq \cap\{$bi-$w$-$\Lambda_\theta(A_i) \mid i \in I\}$;

7.3.4) bi-w-$\Lambda_\theta(\cup\{A_i \mid i \in I\}) = \cup\{$bi-$w$-$\Lambda_\theta(A_i) \mid i \in I\}$.

8) A subset A of a bi-w space (X, w^1, w^2) is called a bi-w-Λ_θ-set if $A =$bi-w-$\Lambda_\theta(A)$.

From the above definition, the following theorems are derived:

8.1) For subset A and $A_i (i \in I)$ of a bi-w space (X, w^1, w^2), the following properties hold :

8.1.1) bi-w-$\Lambda_\theta(A)$ is a bi-w-Λ_θ-set;

8.1.2) If A is bi-w_θ-open, then A is a bi-w-Λ_θ-set;

8.1.3) If A_i is a bi-w-Λ_θ-set for each $i \in J$, then $\bigcap_{i\in J} A_i$ is a bi-w-Λ_θ-set;

8.1.4) If A_i is a bi-w-Λ_θ-set for each $i \in J$, then $\bigcup_{i\in J} A_i$ is a bi-w-Λ_θ-set.

9) Let A be a subset of a bi-w space (X, w^1, w^2).

i A is called a *bi-w-(Λ,θ)-closed* set if $A = T \cap C$, where T is a bi-w-Λ_θ-set and C is a bi-w_θ-closed set. The complement of a bi-w-(Λ,θ)-closed set is called a *bi-w-(Λ,θ)-open* set. The collection of all bi-w-(Λ,θ)-open (resp. bi-w-(Λ,θ)-closed) sets in a bi-w space (X, w^1, w^2) is denoted by bi-w-$\Lambda_\theta O(X, w^1, w^2)$ (resp. bi-w-$\Lambda_\theta C(X, w^1, w^2)$).

ii A point $x \in X$ is called a *bi-w-(Λ,θ)-cluster point* of A if for every bi-w-(Λ,θ)-open set U of X containing x, we have $A \cap U \neq \varnothing$. The set of all bi-$w$-$(\Lambda,\theta)$-cluster points of A is called the *bi-w-(Λ,θ)-closure* of A and is denoted by bi-$c^w_{(\Lambda,\theta)}(A)$.

From the above definitions, the following theorems are derived:

9.1) For a subset A of a bi-w space (X, w^1, w^2), $x \in$ bi-$c^w_{\Lambda,\theta}(A)$ if and only if $U \cap A \neq \varnothing$ for every bi-w-(Λ,θ)-open set U containing x.

9.2) Let A and B be subset of a bi-w space (X, w^1, w^2). For the bi-w-(Λ,θ)-closure, the following properties hold :

9.2.1) $A \subseteq$ bi-$c^w_{(\Lambda,\theta)}(A)$;

9.2.2) bi-$c^w_{(\Lambda,\theta)}(A) = \bigcap\{F \mid A \subseteq F$ and F is bi-w-(Λ,θ)-closed $\}$;

9.2.3) If $A \subseteq B$, then bi-$c^w_{(\Lambda,\theta)}(A) \subseteq$ bi-$c^w_{(\Lambda,\theta)}(B)$;

9.2.4) If A_i is bi-w-(Λ,θ)-closed for each $i \in J$, then $\bigcap_{i\in j} A_i$ is bi-w-(Λ,θ)-closed;

9.2.5) bi-$c^w_{(\Lambda,\theta)}(A)$ is bi-w-(Λ,θ)-closed.

9.3) Let A be subset of a bi-w space (X, w^1, w^2). Then A is bi-w-(Λ,θ)-closed if and only if bi-$c^w_{(\Lambda,\theta)}(A) = A$.

10) Let A be a subset of a bi-w space (X, w^1, w^2). The union of all bi-w-(Λ,θ)-open sets contained in A is called the *bi-w-(Λ,θ)-interior* of A and is denoted by bi-$i^w_{(\Lambda,\theta)}(A)$.

From the above definition, the following theorems are derived:

10.1) Let A and B be subsets of a bi-w space (X, w^1, w^2). For the bi-w-(Λ,θ)-interior, the following properties hold:

10.1.1) bi-$i^w_{(\Lambda,\theta)}(A) \subseteq A$;

10.1.2) If $A \subseteq B$, then bi-$i^w_{(\Lambda,\theta)}(A) \subseteq$ bi-$i^w_{(\Lambda,\theta)}(B)$;

10.1.3) If A_i is a bi-w-(Λ,θ)-open set for all $i \in J$, then $\bigcup_{i\in j} A_i$ is a bi-w-(Λ,θ)-open set.

10.2) For a subset A of a bi-w space (X, w^1, w^2), the following properties hold;

10.2.1) bi-$i^w_{(\Lambda,\theta)}(X - A) = X -$ bi-$c^w_{(\Lambda,\theta)}(A)$.

10.2.2) bi-$c^w_{(\Lambda,\theta)}(X - A) = X -$ bi-$i^w_{(\Lambda,\theta)}(A)$.

10.3) Let A be a subset of a bi-w space (X, w^1, w^2). For the bi-w-(Λ,θ)-interior, the following properties hold:

10.3.1) A is bi-w-(Λ,θ)-open if and only if bi-$i^w_{(\Lambda,\theta)}(A) = A$;

10.3.2) bi-$i^w_{(\Lambda,\theta)}(A)$ is bi-w-(Λ,θ)-open.

10.4) For a subset A of a bi-w space (X, w^1, w^2), the following properties hold;

10.4.1) bi-$c^w_{(\Lambda,\theta)}($bi-$c^w_{(\Lambda,\theta)}(A)) =$ bi-$c^w_{(\Lambda,\theta)}(A)$.

10.4.2) bi-$i^w_{(\Lambda,\theta)}($bi-$i^w_{(\Lambda,\theta)}(A)) =$ bi-$i^w_{(\Lambda,\theta)}(A)$.

10.5) For a subset A of a bi-w space (X, w^1, w^2), the following properties hold;

10.5.1) bi-$i^w_{(\Lambda,\theta)}($bi-$c^w_{(\Lambda,\theta)}($bi-$i^w_{(\Lambda,\theta)}($bi-$c^w_{(\Lambda,\theta)}(A)))) =$ bi-$i^w_{(\Lambda,\theta)}($bi-$c^w_{(\Lambda,\theta)}(A))$.

10.5.2) bi-$c^w_{(\Lambda,\theta)}$(bi-$i^w_{(\Lambda,\theta)}$(bi-$c^w_{(\Lambda,\theta)}$(bi-$i^w_{(\Lambda,\theta)}(A)))) $=$ bi-$c^w_{(\Lambda,\theta)}$(bi-$i^w_{(\Lambda,\theta)}(A)$).

11) A subset A of a bi-w space (X, w^1, w^2) is said to be:

i *bi-w-$s(\Lambda,\theta)$-open* if $A \subseteq$ bi-$c^w_{(\Lambda,\theta)}$(bi-$i^w_{(\Lambda,\theta)}(A)$).

ii *bi-w-$p(\Lambda,\theta)$-open* if $A \subseteq$ bi-$i^w_{(\Lambda,\theta)}$(bi-$c^w_{(\Lambda,\theta)}(A)$).

The family of all bi-w-$s(\Lambda,\theta)$-open (resp. bi-w-$p(\Lambda,\theta)$-open) sets in a bi-w-space (X, w^1, w^2) is denoted by bi-w-$s\Lambda_\theta O(X,x)$ (resp. bi-w-$p\Lambda_\theta O(X,x)$). The complement of bi-w-$s(\Lambda,\theta)$-open (resp. bi-w-$p(\Lambda,\theta)$-open) sets is sad to be bi-w-$s(\Lambda,\theta)$-closed (resp. bi-w-$p(\Lambda,\theta)$-closed) sets. The family of all bi-w-$s(\Lambda,\theta)$-closed (resp. bi-w-$p(\Lambda,\theta)$-closed) sets in a bi-w-space (X, w^1, w^2) is denoted by bi-w-$s\Lambda_\theta C(X,x)$ (resp. bi-w-$p\Lambda_\theta C(X,x)$).

From the above definitions, the following theorems are derived:

11.1) In a bi-w space (X, w^1, w^2), the following properties hold;

11.1.1) If A_i is bi-w-$s(\Lambda,\theta)$-open for all $i \in J$, then $\bigcup_{i \in j} A_i$ is bi-w-$s(\Lambda,\theta)$-open.

11.1.2) If A_i is bi-w-$p(\Lambda,\theta)$-open for all $i \in J$, then $\bigcup_{i \in j} A_i$ is bi-w-$s(\Lambda,\theta)$-open.

11.1.3) If A_i is bi-w-$s(\Lambda,\theta)$-closed for all $i \in J$, then $\bigcap_{i \in J} A_i$ is bi-w-$s(\Lambda,\theta)$-closed.

11.1.4) If A_i is bi-w-$p(\Lambda,\theta)$-closed for all $i \in J$, then $\bigcap_{i \in J} A_i$ is bi-w-$p(\Lambda,\theta)$-closed.

11.2) For a subset A of a bi-w space (X, w^1, w^2), the following properties hold;

11.2.1) A is bi-w-$s(\Lambda,\theta)$-closed if and only if bi-$i^w_{(\Lambda,\theta)}$(bi-$c^w_{(\Lambda,\theta)}(A)$) $\subseteq A$.

11.2.2) A is bi-w-$p(\Lambda,\theta)$-closed if and only if bi-$c^w_{(\Lambda,\theta)}$(bi-$i^w_{(\Lambda,\theta)}(A)$) $\subseteq A$.

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BIOGRAPHY

BIOGRAPHY

Name	Miss Ilada Cheenchan
Date of brith	July 19, 1993
Place of birth	Mukdahan Province, Thailand
Institution attended	
2012	High School in Nasokewittayakarn School, Mukdahan, Thailand
2016	Bachelor of Science and technology in Mathematics, Sakonnakhon Rajabhat University, Thailand
2019	Master of Science in Mathematics, Mahasarakham University, Thailand
Contact address	
	130, Nasoke sub-district, Mueang Mukdahan district, Mukdahan provice 49000, Thailand
	59010283014@msu.ac.th